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ON EXISTENCE OF OSCILLATORY SOLUTIONS TO HIGHER ORDER LINEAR HYPERBOLIC EQUATIONS

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In [13,14] on the basis of methods of oscillation theory for ordinary differential equations (see [1–12,15–19]) and references quoted therein) oscillatory properties of solutions to higher order linear hyperbolic equations are studied. In the present paper, which is sequel to [13,14], in the half strip $D = \mathbb{R}_+ \times [0, b]$ the hyperbolic equation

$$\frac{\partial^{m+n} u}{\partial x^m \partial y^n} = p_0(x, y)u + p(x)\frac{\partial^n u}{\partial y^n} \tag{1}$$

is considered, for which the problem on existence of oscillatory solutions satisfying the boundary conditions

$$\frac{\partial^k u(x, y)}{\partial y^k}\bigg|_{y=ib} = 0 \quad (k = 0, \dots, n_i - 1;\ i = 0, 1) \tag{2}$$

is studied. Here $m \geq 2$ and $n \geq 2$, $n_0 \in \Big\{1, \dots, \Big[n/2\Big]\Big\}$, $n_1 = n - n_0$,

$$p_0 \in L^2_{loc}(D), \quad p \in L^2_{loc}(\mathbb{R}_+).$$

The following notation will be used in the sequel.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$.

$[z]$ is the integral part of $z \in \mathbb{R}$.

$H^k([0, b])$ is the space of functions $z \in L^2([0, T])$ having the generalized derivatives $z^{(i)} \in L^2([0, T])$ $(i = 1, \dots, k)$.

$L^2_{loc}(D)$ is the space of locally square integrable measurable functions $z : D \to \mathbb{R}$.

$H^{k,l}_{loc}(D)$ is the space of functions $z \in L^2_{loc}(D)$, having the generalized derivatives $\frac{\partial^{i+j} z}{\partial x^i \partial y^j} \in L^2_{loc}(D)$ $(i = 0, \dots, m;\ j = 0, \dots, n)$.

By a *solution of equation* (1) we understand a function $u \in H^{m,n}_{loc}(D)$ satisfying equation (1) almost everywhere in D.

A solution $u \in H^{m,n}_{loc}(D)$ of equation (1) satisfying conditions (2) will be called a *solution of problem* (1), (2).

Definition 1. A function $u \in H^{m,0}_{loc}(D_b)$ will be called a *generalized solution of equation* (1) if it satisfies the integral equality

$$(-1)^n \int\limits_0^{+\infty} \int\limits_0^b \frac{\partial^m u(s, t)}{\partial s^m} \, \frac{\partial^n v(s, t)}{\partial t^n} \, ds \, dt$$

$$= \int\limits_0^{+\infty} \int\limits_0^b \Big(p_0(s, t)u(s, t)v(s, t) + (-1)^n p_2(s)u(s, t)\frac{\partial^n v(s, t)}{\partial t^n} \Big) \, ds \, dt$$

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for every smooth compactly supported in $(0,+\infty)\times(0,b)$ function v.

A generalized solution $u\in H^{m,n_1}_{loc}(D)$ of equation (1) satisfying conditions (2) will be called a *generalized solution of problem* (1), (2).

Definition 2. A nontrivial solution u of problem (1),(2) is said to be *oscillatory* if for every $x_0>0$ it changes its sign in the half strip $(x_0,+\infty)\times(0,\omega)$. Otherwise, u is said to be *nonoscillatory.*

Definition 3. Problem (1),(2) has *property A* if every nontrivial generalized solution of this problem for m even is oscillatory and for m odd is either oscillatory or satisfies the condition

$$\lim_{x\to+\infty}\frac{\partial^j u(x,y)}{\partial x^j}=0 \quad \text{for} \quad 0<y<b \quad (j=0,\dots,m-1). \tag{3}$$

Definition 4. Problem (1),(2) has *property B* if every nonoscillatory generalized solution of this problem for m even satisfies either (3) or the condition

$$\lim_{x\to+\infty}\left|\frac{\partial^j u(x,y)}{\partial x^j}\right|=+\infty \quad \text{for} \quad 0<y<b \quad (j=0,\dots,m-1), \tag{4}$$

and for m odd satisfies (4).

Lemma 1. *Let $x_0\in\mathbb{R}_+$, $n^*\in\{1,\dots,n\}$, $\varphi_j\in H^{n^*}([0,b])$ $(j=0,\dots,\ m-1)$, and let for every $j\in\{0,\dots,m-1\}$ the equalities*

$$\varphi_j^{(k)}(0)=0 \quad (k=0,\dots,n_0-1), \quad \varphi_j^{(k)}(b)=0 \quad (k=0,\dots,n_1-1) \tag{5}$$

hold. Then problem (1), (2) *has a unique generalized solution $u\in H^{m,n^*}_{loc}(D)$ satisfying the initial conditions*

$$\left.\frac{\partial^j u(x,y)}{\partial x^j}\right|_{x=x_0}=\varphi_j(y) \quad \textit{for} \quad 0\le y\le b \quad (j=0,\dots,m-1). \tag{6}$$

Proof. Let u be a generalized solution of problem (1),(2). Then almost for every $x\in\mathbb{R}_+$ and for every $y\in[0,b]$ the equality

$$\frac{\partial^m u(x,y)}{\partial x^m}=p(x)u(x,y)+\int_0^b g(y,t)p_0(x,t)u(x,t)\,dt \tag{7}$$

holds, where g is a Green's function of the boundary value problem

$$z^{(n)}=0; \quad z^{(k)}(0)=0 \ \ (k=0,\dots,n_0-1), \quad z^{(k)}(b)=0 \ \ (k=0,\dots,n_1-1).$$

By conditions (6), we have

$$u(x,y)=\sum_{j=0}^{m-1}c_j(x,x_0)\varphi_j(y)+\int_{x_0}^{x}\int_0^b c_{m-1}(x,s)g(y,t)p_0(s,t)u(s,t)\,ds\,dt, \tag{8}$$

where for every $s\in[0,+\infty)$ the function $c_j(\cdot,s)$ is a solution of the Cauchy problem for the ordinary differential equation

$$\frac{d^m z}{dx^m}=p(x)z; \quad z^{(j)}(s)=\delta_{ij} \ \ (j=0,\dots,m-1)$$

and δ_{ij} is Kronecker's symbol.

In view of (5) it is clear that a continuous solution $u:D\to\mathbb{R}$ of equation (8) belongs to $H^{m,n^*}_{loc}(D)$ and is a generalized solution of problem (1),(2),(6). Consequently integral equation (8) is equivalent to problem (1),(2),(6). On the other hand the unique solvability of equation (8) can be easily proved applying the method of successive approximations. □

As it was noted above an arbitrary generalized solution of problem (1),(2) is a solution of integro–differential equation (7). Therefore Lemmas 2.5 and 2.9 from [13] imply

Lemma 2. *Let*

$$(-1)^{m+n_1} p_0(x,y) \geq 0, \quad (-1)^m p(x) \geq 0 \quad \textit{for} \quad (x,y) \in D,$$

and u be a nonoscillatory generalized solution of problem (1), (2) *satisfying condition* (3). *Then*

$$(-1)^j \frac{\partial^j u(x,y)}{\partial x^j}\, u(x,y) \geq 0 \quad \textit{for} \quad (x,y) \in D \quad (j = 0, \dots, m). \tag{9}$$

Theorem 1. *Let*

$$(-1)^{n_1} p_0(x,y) \leq 0, \quad p(x) \geq 0 \quad \textit{for} \quad (x,y) \in D$$

and problem (1), (2) *have property A. Then this problem has an infinite dimensional set of oscillatory solutions.*

Proof. Let $x_0 \in \mathbb{R}$ and $\varphi_j \in H^n([0,b])$ $(j = 0, \dots, m-1)$ be arbitrary functions for every $j \in \{0, \dots, m-1\}$ satisfying conditions (5). Moreover, let

$$\max\left\{ \sum_{k=0}^{m-1} |\varphi_k(y)| : 0 \leq y \leq b \right\} > 0, \tag{10}$$

if m is even, and

$$(-1)^{j_0} \varphi_{j_0}(y_0)\varphi(y_0) < 0 \tag{11}$$

for some $y_0 \in (0,b)$ and $j_0 \in \{1, \dots, m-1\}$ if m is odd.

By Lemma 1 problem (1),(2) has a unique solution u satisfying initial conditions (6). If m is even, then according to Definition 3 and condition (10) u is oscillatory. In view of arbitrariness of φ_j $(j = 0, \dots, m-1)$, to complete the proof of the theorem it is sufficient to show that u is oscillatory for m odd as well. Assume the contrary that u is a nonoscillatory solution. Then by Definition 3 u satisfies condition (3). On the other hand, by Lemma 2, u satisfies inequalities (9). But this is impossible, since by (6) and (11)

$$(-1)^{j_0} \frac{\partial^{j_0} u(x,y_0)}{\partial x^{j_0}}\, u(x,y_0) \bigg|_{x=x_0} < 0.$$

The obtained contradiction proves the theorem . □

Corollary 1. *Let*

$$(-1)^{n_1} p_0(x,y) \leq q(x), \quad p(x) \leq 0 \quad \textit{for} \quad (x,y) \in D, \tag{12}$$

where $q : \mathbb{R}_+ \to (-\infty, 0]$ is a locally summable function such that

$$\liminf_{x \to +\infty} \left(x \int_x^{+\infty} s^{m-2}(p(s) + q(s))\, ds + x^{-1} \int_0^x s^m (p(s) + q(s))\, ds \right) = -\infty. \tag{13}$$

Then problem (1), (2) *has an infinite dimensional set of oscillatory solutions.*

Proof. By Corollary 1.1 from [13], conditions (12) and (13) guarantee that problem (1),(2) has property A. Hence by Theorem 1, it follows that problem (1),(2) has an infinite dimensional set of oscillatory solutions. $\square$

Remark 1. Condition (13) holds if

$$\int_0^{+\infty} s^{m-1-\varepsilon}\big(p(s)+q(s)\big)\,ds=-\infty$$

for some $\varepsilon\in(0,+\infty)$ (see [12], Proof of Corollary 1.8).

Theorem 2. *Let $n=2n_0$, $m\geq 5$, m_0 be the integral part of $m/2$, $m-m_0$ be odd,*

$$(-1)^{n_0}p_0(x,y)\geq 0,\quad p(x)\geq 0\quad for\quad (x,y)\in D, \tag{14}$$

$$\int_{y_1}^{y_2}\int_0^{+\infty} s^{2m-2}|p_0(s,t)|\,ds\,dt=+\infty\quad for\ every\quad 0<y_1<y_2<b, \tag{15}$$

and let problem (1), (2) *have property B. Then this problem has an infinite dimensional set of oscillatory generalized solutions. Moreover, if $p_0\in H^{0,n}_{loc}(D)$ and*

$$(-1)^{n_0+k}\frac{\partial^{2k}p_0(x,y)}{\partial y^{2k}}\geq 0\quad for\quad (x,y)\in D\quad (k=0,\dots,n_0), \tag{16}$$

then problem (1), (2) *an infinite dimensional set of oscillatory solutions.*

Proof. Let $\psi_j\in H^n([0,b])$ $(j=0,\dots,m_0-1)$ be arbitrary functions such that

$$\psi_j^{(k)}(0)=0\ \ (k=0,\dots,n_0-1),\ \ \psi_j^{(k)}(b)=0\ \ (k=0,\dots,n_0-1),\quad j=(0,\dots,m_0-1),$$

$$\max\Big\{\sum_{k=0}^{m_0-1}|\psi_k(y)|:0\leq y\leq b\Big\}>0.$$

Moreover, if m is even let

$$(-1)^{j_0}\psi_{j_0}(y_0)\psi(y_0)<0 \tag{17}$$

for some $y_0\in(0,b)$ and $j_0\in\{1,\dots,m_0-1\}$.

From (14) it is clear that

$$(-1)^{m-m_0+n-n_0-1}p_0(x,y)\geq 0,\quad (-1)^{m-m_0-1}p(x)\geq 0\quad \text{for}\quad (x,y)\in D,$$

since $n=2n_0$ and $m-m_0$ is an odd number. However, by Theorem 1.1 from [14], this inequality and parity of n guarantee the existence of a unique generalized solution u of problem (1),(2) satisfying the conditions

$$\frac{\partial^j u(x,y)}{\partial x^j}\bigg|_{x=0}=\psi_j(y)\quad \text{for}\quad 0\leq y\leq b\quad (j=0,\dots,m_0-1), \tag{18}$$

$$\int_0^b\int_0^{+\infty}\Big(\Big|\frac{\partial^{m_0+n_0}u(s,t)}{\partial s^{m_0}\partial t^{n_0}}\Big|^2+|p_0(s,t)|u^2(s,t)\Big)\,ds\,dt<+\infty. \tag{19}$$

Moreover, if (16) holds, then $u\in H^{m,n}_{loc}(D)$ and

$$\int_0^b\int_0^{+\infty}\Big|\frac{\partial^{m_0+n}u(s,t)}{\partial s^{m_0}\partial t^{n}}\Big|^2\,ds\,dt<+\infty.$$

In view of arbitrariness of ψ_j $(j = 0, \dots, m_0 - 1)$, to prove the theorem it is sufficient to show that u is oscillatory.

Assume the contrary that u is a nonoscillatory solution. Then by Definition 4 either u satisfies condition (4), or m is even and u satisfies condition (3).

Assume first that u satisfies conditions (4). Let $y_0 \in (0, b)$ be an arbitrary fixed number. Then without loss of generality we may assume that

$$u(x, y) \geq 0 \quad \text{for} \quad x \geq x_0,\ y \geq y_0, \tag{20}$$

$$\frac{\partial^j u(x, y_0)}{\partial x^j}\bigg|_{x=x_0} > j! \quad (j = 0, \dots, m-1). \tag{21}$$

Set

$$\varphi_j(y) = \frac{\partial^j u(x, y)}{\partial x^j}\bigg|_{x=x_0} = 0 \quad (j = 0, \dots, m-1).$$

Then in view of continuity of φ_j $(j = 0, \dots, m-1)$ and inequalities (21) there exist $y_1 \in (0, y_0)$ and $y_2 \in (y_0, b)$ such that

$$\varphi_j(y) > j! \quad \text{for} \quad y_1 \leq y \leq y_2 \quad (j = 0, \dots, m-1). \tag{22}$$

As it was mentioned above u admits representation (8). On the other hand, it is well known that

$$(-1)^{n_0} g(y, t) > 0 \quad \text{for} \quad 0 < y,\ t < b \tag{23}$$

(see, e.g., [5], Theorem 9.4).

By virtue of (14),(19),(21) and (22) from (8) we find that

$$u(x, y) > (x - x_0)^{m-1} \quad \text{for} \quad x \geq x_0,\ y_1 \leq y \leq y_2.$$

This estimate together with (18) yields

$$\int_{y_1}^{y_2} \int_0^{+\infty} (s - x_0)^{2m-2} |p_0(s, t)|\, ds\, dt < +\infty.$$

But this contradicts to condition (15). Consequently, u does not satisfy conditions (4).

Assume now that m is even and u satisfies conditions (3). Then by Lemma 2 inequality (9) holds. But this contradicts to inequalities (17) and (18). The obtained contradiction proves the theorem. □

Corollary 2. *Let $n = 2n_0$, $m \geq 5$, m_0 be the integral part of $m/2$, $m - m_0$ be odd and*

$$(-1)^{n_0} p_0(x, y) \geq q(x), \quad p(x) \geq 0 \quad \textit{for} \quad (x, y) \in D,$$

where $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally summable function. Moreover, let either

$$\limsup_{x \to +\infty} \left(x \int_x^{+\infty} s^{m-1} q(s)\, ds + x^{-1} \int_0^x s^m q(s)\, ds \right) = +\infty,$$

or

$$\int_0^{+\infty} s^{m-1} q(s)\, ds = +\infty$$

and

$$\limsup_{x \to +\infty} \left(x \int_x^{+\infty} s^{m-1} p(s)\, ds + x^{-1} \int_0^x s^m p(s)\, ds \right) = +\infty.$$

Then problem (1), (2) *has an infinite dimensional set of oscillatory solutions.*

Proof. By virtue of Corollary 1.2 from [14], the conditions of Corollary 2 guarantee the existence of property B to problem (1),(2). Consequently, if all the conditions of Corollary 2 hold, then all the conditions of Theorem 2 hold also. □

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