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ON A WEIGHTED BOUNDARY VALUE PROBLEM FOR A SYSTEM OF SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $-\infty < a < b < +\infty$, $0 \le \alpha$, $\beta \le 1$, $n \ge 1$, and f be a continuous operator acting from the space of continuously differentiable vector functions $u :]a,b[\to R^n$ satisfying the condition

$$\sup\Big\{(t-a)^{\alpha-1}(b-t)^{\beta-1}\|u(t)\| + (t-a)^{\alpha}(b-t)^{\beta}\|u'(t)\| : \ a<t<b\Big\} < +\infty$$

to the space of n-dimensional, summable with the weight $(t-a)^{\alpha}(b-t)^{\beta}$, vector functions. Consider the system of functional differential equations

$$u''(t) = f(u)(t) \tag{1}$$

with the boundary conditions

$$\begin{gathered}\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0,\\ \sup\Big\{(t-a)^{\alpha-1}(b-t)^{\beta-1}\|u(t)\| + (t-a)^{\alpha}(b-t)^{\beta}\|u'(t)\| : a<t<b\Big\} < +\infty.\end{gathered} \tag{2}$$

In case $n = 1$ or $f(u)(t) = f_0(t, u(t), u'(t))$, where $f_0 :]a,b[\times R^n \to R^n$ is a vector function satisfying the local Carathéodory conditions, boundary value problems of the type (1), (2) are investigated in full detail (see [1–9, 11–20] and the references therein). Below we give optimal sufficient conditions for the solvability and the unique solvability of the problem (1), (2) which generalize the results of [19].

Throughout the paper the following notation will be used.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$.

R^n is the space of n-dimensional vector columns $x = (x_i)_{i=1}^n$ with the components $x_i \in R$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

If $x = (x_i)_{i=1}^n$, then $|x| = (|x_i|)_{i=1}^n$.

$R_+^n = \{(x_i)_{i=1}^n \in R^n : \ x_i \in R_+ \ (i = 1, \dots, n)\}$.

The inequality between vectors is understood componentwise, i.e, if $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n \in R^n$, then

$$x \le y \Longleftrightarrow x_i \le y_i \ \ (i = 1, \dots, n).$$

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$R^{n\times n}$ is the space of $n\times n$-matrices $X=(x_{ik})_{i,k=1}^n$ with the components $x_{ik}\in R$ $(i,k=1,\dots,n)$ and the norm

$$\|X\|=\sum_{i,k=1}^n |x_k|.$$

$R_+^{n\times n}=\{X=(x_{ik})_{i,k=1}^n:\ x_{ik}\in R_+\ (i,k=1,\dots,n)\}$.

$r(X)$ is the spectral radius of a matrix $X\in R^{n\times n}$.

$C^1_{\alpha,\beta}(]a,b[\,;R^n)$ is the space of continuously differentiable functions $u:]a,b[\to R^n$ such that the norm

$$\|u\|_{C^1_{\alpha,\beta}}=\sup\Big\{(t-a)^{\alpha-1}(b-t)^{\beta-1}\|u(t)\|+(t-a)^{\alpha}(b-t)^{\beta}\|u'(t)\|:\ \ a<t<b\Big\}$$

is finite.

If $u=(u_i)_{i=1}^n\in C^1_{\alpha,\beta}(]a,b[\,;R^n)$, then

$$\nu_{0,\alpha,\beta}(u_i)=\sup\Big\{(t-a)^{\alpha-1}(b-t)^{\beta-1}|u_i(t)|:\ \ a<t<b\Big\},$$

$$\nu_{1,\alpha,\beta}(u_i)=\sup\left\{\frac{(t-a)^{\alpha}(b-t)^{\beta}}{b-a}\,|u_i'(t)|:\ \ a<t<b\right\},$$

$$\nu_{\alpha,\beta}(u_i)=\max\Big\{\nu_{0,\alpha,\beta}(u_i),\nu_{1,\alpha,\beta}(u_i)\Big\},\quad \nu_{\alpha,\beta}(u)=\Big(\nu_{\alpha,\beta}(u_i)\Big)_{i=1}^n.$$

$L_{\alpha,\beta}(]a,b[\,;R^n)$ is the space of vector functions $v:]a,b[\to R^n$ with summable with the weight $(t-a)^{\alpha}(b-t)^{\beta}$ components and the norm

$$\|v\|_{L_{\alpha,\beta}}=\int_a^b (t-a)^{\alpha}(b-t)^{\beta}\|v(t)\|\,dt.$$

$L_{\alpha,\beta}(]a,b[\,;R_+^{n\times n})$ is the set of matrix functions $H:]a,b[\to R_+^{n\times n}$ with summable with the weight $(t-a)^{\alpha}(b-t)^{\beta}$ nonnegative components.

$M_{\alpha,\beta}(]a,b[\times R_+;R_+^n)$ is the set of vector functions $h:]a,b[\times R_+\to R_+^n$ summable in the first argument with the weight $(t-a)^{\alpha}(b-t)^{\beta}$ and nondecreasing in the second argument.

In what follows it will be assumed that the operator $f:\ C^1_{\alpha,\beta}(]a,b[\,;R^n)\to L_{\alpha,\beta}(]a,b[\,;R^n)$ is continuous and

$$\sup\Big\{\Big\|f(u)(\cdot)\Big\|:\ \ \|u\|_{C^1_{\alpha,\beta}}\le\rho\Big\}\in L_{\alpha,\beta}(]a,b[\,;R)\quad\text{for}\quad 0<\rho<+\infty.$$

A vector function $u:]a,b[\to R^n$ is called a solution of the problem (1), (2) if:

(i) u is continuously differentiable and u' is locally absolutely continuous in $]a,b[$;

(ii) u satisfies the boundary conditions (2);

(iii) u satisfies the system (1) almost everywhere on $]a,b[$.

Analogously to Theorem 1 of [10] it can be proved the following

Theorem 1. *Let there exist a positive number ρ_0 such that for any $\lambda\in]0,1[$ an arbitrary solution of the differential system*

$$\frac{du(t)}{dt}=\lambda f(u)(t)$$

satisfying the boundary conditions (2) *admits the estimation*

$$\|u\|_{C^1_{\alpha,\beta}}\le\rho_0.$$

Then the problem (1), (2) *is solvable.*

Corollary 1. *Let there exist* $H \in L_{\alpha,\beta}(]a,b[\,; R_+^{n\times n})$ *and* $h \in M_{\alpha,\beta}(]a,b[\times R_+; R_+^n)$ *such that*

$$\Big|f(u)(t)\Big| \le H(t)\nu_{\alpha,\beta}(u) + h\Big(t; \|u\|_{C^1_{\alpha,\beta}}\Big) \quad \text{for} \quad t \in]a,b[\,, \quad u \in C^1_{\alpha,\beta}(]a,b[\,; R^n). \quad (3)$$

Moreover, let

$$\lim_{\rho\to+\infty}\Big(\frac{1}{\rho}\int_a^b \|h(t,\rho)\|\,dt\Big) = 0 \quad (4)$$

and the system of differential inequalities

$$|u''(t)| \le H(t)\nu_{\alpha,\beta}(u) \quad (5)$$

under the boundary conditions (2) *have only the trivial solution. Then the problem* (1), (2) *is solvable.*

The problem (5), (2) has only the trivial solution if

$$r\Big(\int_a^b (t-a)^{\alpha}(b-t)^{\alpha}H(t)\,dt\Big) < b-a, \quad (6)$$

or

$$H(t) \le (t-a)^{-\alpha}(b-t)^{-\beta}H_0, \quad H_0 \in R_+^{n\times n}$$

and

$$r(H_0) < \min\{2-\alpha, 2-\beta\}. \quad (7)$$

Therefore from Corollary 1 it follows

Corollary 2. *Let there exist* $H \in L_{\alpha,\beta}(]a,b[\,; R_+^{n\times n})$ *and* $h \in M_{\alpha,\beta}(]a,b[\times R_+; R_+^n)$ *such that the conditions* (3), (4) *and* (6) *are fulfilled. Then the problem* (1), (2) *is solvable.*

Corollary 3. *Let there exist* $H_0 \in R_+^{n\times n}$ *and* $h \in M_{\alpha,\beta}(]a,b[\times R_+; R_+^n)$ *such that*

$$\Big|f(u)(t)\Big| \le (t-a)^{-\alpha}(b-t)^{-\beta}H_0\nu_{\alpha,\beta}(u) + h\Big(t; \|u\|_{C^1_{\alpha,\beta}}\Big)$$
$$\text{for} \quad t \in]a,b[\,, \quad u \in C^1_{\alpha,\beta}(]a,b[\,; R^n)$$

and the conditions (4) *and* (7) *are fulfilled. Then the problem* (1), (2) *is solvable.*

Theorem 2. *Let there exist* $H \in L_{\alpha,\beta}(]a,b[\,; R_+^{n\times n})$ *such that*

$$\Big|f(u)(t) - f(v)(t)\Big| \le H(t)\nu_{\alpha,\beta}(u-v) \quad \text{for} \quad t \in]a,b[\,, \quad u, v \in C^1_{\alpha,\beta}(]a,b[\,; R^n) \quad (8)$$

and the problem (5), (2) *has only the trivial solution. Then the problem* (1), (2) *has one and only one solution.*

Corollary 4. *Let there exist* $H \in L_{\alpha,\beta}(]a,b[\,; R_+^{n\times n})$ *such that the conditions* (6) *and* (8) *are fulfilled. Then the problem* (1), (2) *has one and only one solution.*

Corollary 5. *Let there exist* $H_0 \in R_+^{n\times n}$ *such that*

$$\Big|f(u)(t) - f(v)(t)\Big| \le (t-a)^{-\alpha}(b-t)^{-\beta}H_0\nu_{\alpha,\beta}(u-v)$$
$$\text{for} \quad t \in]a,b[\,, \quad u, v \in C^1_{\alpha,\beta}(]a,b[\,; R^n).$$

Then the problem (1), (2) *has one and only one solution.*

A particular case of (1) is the differential system with deviating arguments

$$u''(t) = f_0\Big(t, u(\tau_1(t)), u'(\tau_2(t))\Big). \tag{9}$$

This system will be considered under the following assumptions:

(i) f_0 maps $I \times R^{2n}$ into R^n, where $I \subset [a,b]$ and $\operatorname{mes} I = b-a$;

(ii) $f_0(t,\cdot,\cdot) : R^{2n} \to R^n$ is continuous for every $t \in I$, and $f(\cdot,x,y) : I \to R^n$ is measurable for every x and $y \in R^n$;

(iii) $\tau_i : I \to]a,b[$ $(i=1,2)$ are measurable functions.

The propositions below on the solvability and the unique solvability of the problem (9), (2) follow from Corollaries 2–5.

Corollary 6. *Let there exist* $H_i \in L_{\alpha,\beta}(]a,b[;R_+^{n\times n})$ $(i=1,2)$ *and* $h \in M_{\alpha,\beta}(]a,b[\times R_+;R_+^n)$ *such that*

$$\Big|f_0\Big(t, (\tau_1(t)-a)^{1-\alpha}(b-\tau_1(t))^{1-\beta}x, (\tau_2(t)-a)^{-\alpha}(b-\tau_2(t))^{-\alpha}y\Big)\Big| \leq$$
$$\leq H_1(t)|x + |H_2(t)|y| +$$
$$+h\Big(t, (\tau_1(t)-a)^{1-\alpha}(b-\tau_1(t))^{1-\alpha}|x| + (\tau_2(t)-a)^{-\alpha}(b-\tau_2(t))^{-\beta}|y|\Big)$$
$$\text{for} \quad t \in I, \quad x,\, y \in R^n$$

and the condition (4) *hold. Moreover, let either*

$$r\Big(\int_a^b (t-a)^\alpha (b-t)^\beta \Big[H_1(t) + (b-a)H_2(t)\Big]\,dt\Big) < b-a \tag{10}$$

or

$$H_1(t) + (b-a)H_2(t) \leq (t-a)^{-\alpha}(b-t)^{-\alpha}H_0, \quad r(H_0) < 2. \tag{11}$$

Then the problem (9), (2) *is solvable.*

Corollary 7. *Let there exist* $H_i \in L_{\alpha,\beta}(]a,b[;R_+^{n\times n})$ $(i=1,2)$ *such that*

$$\Big|f_0\Big(t, (\tau_1(t)-a)^{1-\alpha}(b-\tau_1(t))^{1-\beta}x, (\tau_2(t)-a)^{-\alpha}(b-\tau_2(t))^{-\beta}y\Big) -$$
$$-f_0\Big(t, (\tau_1(t)-a)^{1-\alpha}(b-\tau_1(t))^{1-\beta}\overline{x}, (\tau_2(t)-a)^{-\alpha}(b-\tau_2(t))^{-\beta}\overline{y}\Big)\Big| \leq$$
$$\leq H_1(t)|x-\overline{x}| + H_2(t)|y-\overline{y}| \quad \text{for} \quad t \in]a,b[, \quad x,\, \overline{x},\, y,\, \overline{y} \in R^n.$$

Moreover, let either the condition (10) *or the condition* (11) *hold. Then the problem* (9), (2) *has one and only one solution.*

As an example, consider the differential system

$$u''(t) = (\tau_1(t)-a)^{\alpha-1}(b-\tau_1(t))^{\beta-1}F_1(t)|u(\tau_1(t))| +$$
$$+ (\tau_2(t)-a)^{\alpha}(b-\tau_2(t))^{\beta}F_2(t)|u'(\tau_2(t))| + q(t), \tag{12}$$

where $F_i : I \to R^{n\times n}$ $(i=1,2)$ are matrix functions with measurable bounded components, $\tau_i : I \to]a,b[$ $(i=1,2)$ are measurable functions, and $q \in L_{\alpha,\beta}(]a,b[;R^n)$.

From Corollary 7 it follows

Corollary 8. *Let there exist* $H_0 \in R_+^{n\times n}$ *satisfying the inequality* (7) *such that*

$$|F_1(t)| + (b-a)|F_2(t)| \leq H_0 \quad \text{for} \quad t \in I.$$

Then the problem (12), (2) *has one and only one solution.*

Suppose now that

$$\alpha = \beta = 0, \quad F_1(t) \equiv H_0, \quad F_2(t) = \Theta, \quad q(t) = \ell,$$

where Θ is the zero matrix, $H_0 \in R_+^{n\times n}$, and $\ell \in R^n$ is a vector with positive components. Let us show that if

$$r(H_0) \geq 2,$$

then the problem (12), (2) has no solution. Indeed, let u be an arbitrary solution of that problem. Then

$$u''(t) \geq \ell.$$

Thus

$$u(t) \leq -\frac{(t-a)(b-t)}{2}\,\ell \quad \text{for} \quad a < t < b.$$

Taking into account this inequality, we obtain

$$u''(t) \geq \frac{1}{2}\,H_0 + \ell$$

and

$$u(t) \leq -\Big(\frac{1}{2}\,H_0\ell + \ell\Big)\ell(t-a)(b-t) \quad \text{for} \quad a < t < b.$$

If we continue this process, then we will get

$$-u(t) \geq \sum_{k=0}^{+\infty} \left(\frac{1}{2}\,H_0\right)^k \ell(t-a)(b-t) \quad \text{for} \quad a < t < b,$$

which is impossible since $r(\frac{1}{2}\,H_0) \geq 1$.

The above example shows that the condition (7) in Corollaries 3 and 5–8 cannot be replaced by the condition

$$r(H_0) \leq \min\{2-\alpha, 2-\beta\}.$$

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References

1. M. Zh. Alves, On the solvability of the two-point boundary value problem for a singular nonlinear functional differential equation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **2**(1996), 12–19.

2. M. Alves, About a problem arising in chemical reactor theory. *Mem. Differential Equations Math. Phys.* **19**(2000), 133–141.

3. G. D. Gaprindashvili, On a boundary value problem for systems of nonlinear ordinary differential equations with singularities. (Russian) *Differentsial'nye Uravneniya* **20**(1984), No. 9, 1514–1523.

4. G. D. Gaprindashvili, On certain boundary value problems for systems of second order nonlinear ordinary differential equations. (Russian) *Trudy Inst. Prikl. Mat. im. I. N. Vekua* **31**(1988), 23–52.

5. I. T. Kiguradze, On some singular boundary value problems for nonlinear second order ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **4**(1968), No. 10, 1753–1773. English transl.: *Differ. Equations* **4**(1968), 901–910.

6. I. T. Kiguradze, On a singular two-point boundary value problem. (Russian) *Differentsial'nye Uravneniya* **5**(1969), No. 11, 2002–2016. English transl.: *Differ. Equations* **5**(1969), 1493–1504.

7. I. T. Kiguradze, On a singular boundary value problem. *J. Math. Anal. Appl.* **30**(1970), No. 3, 475–489.

8. I. T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

9. I. T. Kiguradze and A. G. Lomtatidze, On certain boundary value problems for second order linear ordinary differential equations with singularities. *J. Math. Anal. Appl.* **10**(1984), No. 2, 325–347.

10. I. T. Kiguradze and B. Půža, On boundary value problems for functional differential equations. *Mem. Differential Equations Math. Phys.* **12**(1997), 106–113.

11. I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for ordinary differential equations of second order. (Russian) *Sovrem. Probl. Mat. Noveishie Dostizhenija* **30**(1987), 105–201.

12. A. G. Lomtatidze, On one boundary value problem for linear differential equations with nonintegrable singularities. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **14**(1983), 136–144.

13. A. G. Lomtatidze, On one singular boundary value problem for linear equations of second order. *In: Boundary value problems. Perm, Perm Polytechnical Inst.*, 1984, 46–50.

14. A. G. Lomtatidze, On the solvability of boundary value problems for nonlinear ordinary differential equations of second order with singularities. (Russian) *Reports of the Extended Session of a Seminar of the I. N. Vekua Institute of Applied Mathematics, Vol.* 1, No. 3 (*Russian*) (*Tbilisi*, 1985), 85–92, *Tbiliss. Gos. Univ., Tbilisi*, 1985.

15. A. G. Lomtatidze, On one boundary value problem for nonlinear ordinary differential equations of second order with singularities. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 3, 416–426. English transl.: *Differ. Equations* **22**(1986), 301–310.

16. A. G. Lomtatidze, On positive solutions of singular boundary value problems for nonlinear ordinary differential equations of second order. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 6, 1092.

17. A. G. Lomtatidze, On positive solutions of boundary value problems for second order ordinary differential equations with singularities. (Russian) *Differentsial'nye Uravneniya* **23**(1987), No. 10, 1685–1692.

18. S. Mukhigulashvili, Two-point boundary value problems for second order functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 1–112.

19. A. S. Rabbimov and G. S. Tabidze, On two-point singular boundary value problems for systems of second order ordinary differential equations. *Trudy Inst. Prikl. Mat. im. I. N. Vekua* **31**(1988), 131–158.

20. N. I. Vasil'ev and A. I. Lomakina, On a two-point boundary value problem with a nonsummable singularity. *Differentsial'nye Uravneniya* **14**(1978), No. 2, 195–200.

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