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AN EXISTENCE THEOREM FOR A CLASS OF OPTIMAL PROBLEMS WITH DELAYED ARGUMENT

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1. Statement of the Problem. An Existence Theorem

Let $J=[a,b]$ be a finite closed interval; $O\subset\mathbb{R}^n$–be an open set; K_i, $i=0,1$, $U\subset\mathbb{R}^r$, $V\subset\mathbb{R}^p$ be compact sets; for each fixed $(x_1,x_2,u_1,u_2)\in O^2\times U^2$ let the function $f:J\times O^2\times U^2\to\mathbb{R}^n$ be measurable with respect to $t\in J$; for an arbitrary compact $K\subset O$ there exist measurable functions $m_K(t)$, $L_K(t)$, $t\in J$, such that

$$|f(t,x_1,x_2,u_1,u_2)|\le m_K(t),\quad \forall(t,x_1,x_2,u_1,u_2)\in J\times K^2\times U^2,$$

$$|f(t,x_1',x_2',u_1,u_2)-f(t,x_1'',x_2'',u_1,u_2)||\le L_K(t)\sum_{i=1}^{2}|x_i''-x_i'|,$$

$$\forall(t,x_1',x_2',x_1'',x_2'',u_1,u_2)\in J\times K^4\times U^4.$$

Further, let the functions $\tau(t)$, $\theta(t)$, $t\in J$, be absolutely continuous and satisfy the conditions: $\tau(t)\le t$, $\dot\tau(t)>0$, $\theta(t)\le t$, $\dot\theta(t)>0$; $\Omega=\Omega(J_0,V,m,L)$ be the set of piecewise continuous functions $v:J_0=[a_0,b_0]\to V$ satisfying the condition: for each function $v(\cdot)\in\Omega$ there exists a partition $a_0=\xi_0<\cdots<\xi_n=b_0$ such that the restriction of the function $v(t)$ satisfies the Lipschitz condition on the open interval (ξ_i,ξ_{i+1}), $i=0,\dots,m$, i.e., $|v(t')-v(t'')|\le L|t'-t''|$, $\forall t',t''\in(\xi_i,\xi_{i+1})$, $i=0,\dots,m$, where the numbers m and L do not depend on $v\in\Omega$; let $\Omega_0=\Omega([\tau(a),b],K_0,m_0,L_0)$, elements of this set will be denoted by $\varphi(\cdot)$; $\Omega_1=\Omega([\theta(a),b],U,m_1,L_1)$, its elements being denoted by $u(\cdot)$; let $q^i:j\times O^2\to\mathbb{R}^1$, $i=0,\dots,l$, be continuous functions.

Consider the problem:

$$\dot x(t)=f(t,x(t),x(\tau(t)),u(t),u(\theta(t))),\quad t\in[t_0,t_1]\subset J,\quad u(\cdot)\in\Omega_1 \tag{1}$$

$$x(t)=\varphi(t),\ \ t\in[\tau(t_0),t_0),\quad x(t_0)=x_0,\quad \varphi(\cdot)\in\Omega_0,\ \ x_0\in K_1, \tag{2}$$

$$q^i(t_0,t_1,x_0,x(t_1))=0,\ \ i=0,\dots,l, \tag{3}$$

$$q^0(t_0,t_1,x_0,x(t_1))\to\min. \tag{4}$$

Definition 1. The function $x(t)=x(t,z)\in O$, $t\in[\tau(t_0),t_1]$, is said to be a solution corresponding to the element $z=(t_0,t_1,x_0,\varphi(\cdot),u(\cdot))\in A=J^2\times K_1\times\Omega_0\times\Omega_1$, if on $[\tau(t_0),t_0]$ it satisfies the condition (2), while on the interval $[t_0,t_1]$ it is absolutely continuous and the pair $(u(\cdot),x(\cdot))$ almost everywhere (a.e.) on $[t_0,t_1]$ satisfies the equation (1).

Definition 2. The element $z\in A$ is said to be admissible if the corresponding solution $x(t)$ satisfies the condition (3).

The set of admissible elements will be denoted by Δ.

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Definition 3. The element $\tilde{z} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{\varphi}(\cdot), \tilde{u}(\cdot)) \in \Delta$ is said to be optimal if

$$\tilde{I} = I(\tilde{z}) = \inf_{z \in \Delta} I(z),$$

where

$$I(z) = q^0(t_0, t_1, x_0, x(t_1)), \quad x(t) = x(t, z).$$

Theorem 1. *Let the following conditions be valid:*
1) $\Delta \neq \emptyset$;
2) *there exists a compact set* $K_2 \subset O$ *such that*

$$x(t, z) \in K_2, \quad \forall z \in \Delta.$$

Then there exists an optimal element.

2. Auxiliary Lemmas

Lemma 1. *Let* $x_k(t) = x(t, z_k)$, $t \in [\tau(t_0^k), t_1^k]$, *be the solution corresponding to the element* $z_k \in A$; $t_0^k \to t_0$, $t_1^k \to t_1$ *as* $k \to \infty$, $t_0^k \geq t_0$, $t_1^k \leq t_1$; $K_i \subset O$, $i = 3, 4$, *be compact sets with* $K_3 \subset intK_4$ *and* $x_k(t) \in K_3$, $t \in [t_1^k, t_2^k]$. *Then for sufficiently large* k *the functional differential equation*

$$\dot{y}(t) = f(t, y(t), h(t_0^k, \varphi_k(\cdot), y_k(\cdot))(\tau(t)), u_k(t), u_k(\theta(t))), \tag{5}$$
$$y(t_0^k) = x_0^k,$$

where

$$h(t_0, \varphi(\cdot), y(\cdot))(t) = \begin{cases} \varphi(t), & t \in [\tau(a), t_0), \\ y(t), & t \in [t_o, b], \end{cases}$$

has a solution $y_k(t) = y(t, z_k) \in K_4$ *defined on* $[t_0, t_1]$, *and* $y_k(t) = x_k(t)$, $t \in [t_0^k, t_1^k]$.

The proof of this lemma can be carried out in the standard way (for example, see Theorem 2 in [1]), since (5) is an ordinary differential eqation for $t < t_0^k$, and is a differential equation with delayed argument for $t > t_0^k$.

Lemma 2. *Let* $v_k(\cdot) \in \Omega$, $k = 1, 2, \ldots$. *Then there exists a subsequence of the sequence* $\{v_k(\cdot)\}_{k=1}^{\infty}$ *such that it converges to some function* $v_o(\cdot) \in \Omega$ *for each* $t \in J$, *except for not more than* $(m+1)$ *points.*

Proof. By assumption the function $v_k(t)$, $t \in (\xi_i^k, \xi_{i+1}^k)$, satisfies Lipschitz condition. From this it imediately follows the existence of one-sided limits

$$\lim_{t \to \xi_i^k -} v_k(t) = v_{k_i}^-, \ i = 0, \ldots, q-1, \qquad \lim_{t \to \xi_i^k +} v_k(t) = v_{k_i}^+, \ i = 1, \ldots, q.$$

We set the function

$$\omega_{k_i}(t) = \begin{cases} v_{k_i}^-, & t \leq \xi_i^k, \\ v_k(t), & t \in (\xi_i^k, \xi_{i+1}^k), \\ v_{k_i}^+, & t \geq \xi_{i+1}^k, \end{cases}$$

$$\omega_k(t) = \sum_{i=0}^{m} \chi_{k_i}(t) \omega_{k_i}(t), \ t \in J_0, \ \omega_k(b_0) = \omega_k(b_0-),$$

where $\chi_{k_i}(t)$ is the characteristic function of the semi-open interval $E_{k_i} = [\xi_i^k, \xi_{i+1}^k)$.
Obviously, $\omega_k(\cdot) \in \Omega$ and

$$\omega_k(t) = v_k(t), \ t \in (\xi_i^k, \xi_{i+1}^k). \tag{6}$$

The sequence $\{\omega_{k_i}(t)\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous for each $i = 0,\dots,m$. Therefore, by virtue of Arzela-Ascoli's lemma, from $\{\omega_{k_i}(t)\}_{k=1}^{\infty}$ it can be picked out a uniformly convergent subsequence which again is denoted by $\{\omega_{k_i}(t)\}_{k=1}^{\infty}$.

Thus

$$\lim_{k\to\infty} \omega_{k_i}(t) = \omega_i(t) \text{ uniformly for } t \in J_0.$$

Without loss of generality we will assume that

$$\lim_{k\to\infty} \xi_i^k = \xi_i, \quad i = 1,\dots,q-1.$$

Conseqently we have

$$\lim_{k\to\infty} E_{k_i} = E_i, \quad \lim_{k\to\infty} \chi_{k_i}(t) = \chi_i(t), \quad t \in \mathbb{R},$$

where E_i is an interval and $\chi_i(t)$ is the charasteristic function of the interval E_i.

Therefore for each $t \in J_0$

$$\lim_{k\to\infty} \omega_k(t) = \omega(t) = \sum_{i=0}^{m} \chi_i(t)\omega_i(t),$$

besides $\omega(\cdot) \in \Omega$.

Taking into account (6), we can conclude that

$$\lim_{k\to\infty} v_k(t) = \omega(t) = v_0(t), \quad t \in (\xi_i, \xi_{i+1}), \quad i = 0,1,\dots,m. \quad \square$$

3. Proof of the Theorem

There exists a sequence $z_k = (t_0^k, t_1^k, x_0^k, \varphi_k(\cdot), u_k(\cdot)) \in \Delta$, $k = 1,2,\dots$, such that

$$I(z_k) \to \tilde{I}, \ t_0^k \to \tilde{t}_0, \ t_1^k \to \tilde{t}_1, \ x_0^k \to \tilde{x}_0 \quad \text{as } k \to \infty;$$
$$\lim_{k\to\infty} \varphi_k(t) = \tilde{\varphi}(t), \quad \text{a.e. on } [\tau(a), b], \tilde{\varphi}(\cdot) \in \Omega_0;$$
$$\lim_{k\to\infty} u_k(t) = \tilde{u}(t), \quad \text{a.e. on } [\theta(a), b], \tilde{u}(\cdot) \in \Omega_1$$

(see Lemma 2).

Consider the case where $t_0^k \geq \tilde{t}_0$, $t_1^k \leq \tilde{t}_1$. The remaining cases can be considered analogously.

Let $K_5 \in O$ be a compact set, $K_2 \in \operatorname{int} K_5$. For sufficiently large $k \geq k_0$ there exists the solution $y_k(t) \in K_5$ of the equation (5) defined on $[\tilde{t}_0, \tilde{t}_1]$ and $y_k(t) = x_k(t)$, $t \in [t_0^k, t_1^k]$, (see Lemma 1).

Obviously

$$h(t_0^k, \varphi_k(\cdot), y_k(\cdot))(t) \in K_6, \quad k \geq k_0, \quad t \in [\tau(\tilde{t}_0), \tilde{t}_1], \quad K_6 = K_5 \cup K_0,$$

therefore

$$|\dot{y}(t)| \leq m_{K_6}(t), \quad t \in [\tilde{t}_0, \tilde{t}_1].$$

Thus the sequence $\{y_k(\cdot)\}_{k=1}^{\infty}$ is uniformly bounded and equicontinouos. Without loss of generality we can assume that

$$\lim_{k\to\infty} y_k(t) = \tilde{y}(t) \quad \text{uniformly with } t \in [\tilde{t}_0, \tilde{t}_1].$$

Consequently,

$$\lim_{k\to\infty} f_k[t] = \tilde{f}[t], \quad \text{a.e. } t \in [\tilde{t}_0, \tilde{t}_1],$$

where

$$f_k[t] = f(t, y_k(t), h(t_0^k, \varphi_k(\cdot), y_k(\cdot))(\tau(t)), u_k(t), u_k(\theta(t))),$$
$$\tilde{f}[t] = f(t, y(t), h(\tilde{t}_0, \tilde{\varphi}(\cdot), \tilde{y}(\cdot))(\tau(t)), \tilde{u}(t), \tilde{u}(\theta(t))).$$

Further,

$$y_k(t) = x_0^k + \int_{t_0^k}^{t} \tilde{f}[s]ds + \alpha_k + \beta_k(t), \tag{7}$$

where

$$\alpha_k = \int_{\tilde{t}_0}^{t_0^k} f_k(t]dt, \quad \beta_k(t) = \int_{\tilde{t}_0}^{t} \Big[f_k[s] - \tilde{f}[s]\Big]ds.$$

Evidently

$$\lim_{k\to\infty} \alpha_k = 0, \quad |\beta_k(t)| \le \int_{\tilde{t}_0}^{\tilde{t}_1} |f_k[s] - \tilde{f}[s]|ds.$$

By virtue of Lebesgue's theorem on passage to limit under the integral sign we have

$$\lim_{k\to\infty} \beta_k(t) = 0 \quad \text{uniformly with} \quad t \in [\tilde{t}_0, \tilde{t}_1].$$

From (7) as $k \to \infty$ we get

$$\tilde{y}(t) = \tilde{x}_0 + \int_{\tilde{t}_0}^{t} \tilde{f}[s]ds.$$

It is easy to see that

$$\lim_{k\to\infty} y_k(t_1^k) = \tilde{y}(\tilde{t}_1),$$

therefore

$$q^i(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{y}(\tilde{t}_1)) = 0, \quad i = 1, \dots, l, \quad \tilde{I} = q^0(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{y}(\tilde{t}_1)).$$

Introduce the function

$$\tilde{x}(t) = \begin{cases} \tilde{\varphi}(t), & t \in [\tau(\tilde{t}_0), \tilde{t}_0), \\ \tilde{y}(t), & t \in [\tilde{t}_0, \tilde{t}_1). \end{cases}$$

Obviously $\tilde{z} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\cdot)) \in \Delta$ and $I(\tilde{z}) = \tilde{I}$.

Finally, note that the proved theorem is also valid in the case where the right-hand side of the equation (1) has the form

$$f(t, x(\tau_1(t)), \dots, x(\tau_s(t)), u(\theta_1(t)), \dots, u(\theta_\nu(t))),$$

where the functions $\tau_i(t)$, $i = 1, \dots, s$, $\theta_i(t)$, $i = 1, \dots, \nu$, are absolutely continuous and satisfy the conditions $\tau_i(t) \le t$, $\dot{\tau}_i(t) > 0$; $\theta_i(t) \le t$, $\dot{\theta}_i(t) > 0$.

If K_0, U are convex sets and the points of discontinuity of the functions from the set Ω_i, $i = 0, 1$, are fixed be forehand, then for the problem (1)–(4) necessary conditions of optimality are valid in the form given in [2]. In the class of measurable functions the problem of existence is studied in [3,4].

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