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CONTINUOUS DEPENDENCE AND DIFFERENTIABILITY OF SOLUTION WITH RESPECT TO INITIAL DATA AND RIGHT-HAND SIDE FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

Abstract. Non-linear differential equations with variable delay and quasi-linear neutral differential equations are considered in the case where at the initial moment of time the value of the initial function, generally speaking, does not coincide with the initial value of the trajectory (discontinuity at the initial moment). Theorems on continuity of solution of the Cauchy problem with respect to initial data and right-hand side are proved. The perturbations of the initial data, i.e., of the initial function and the initial values (the initial moment, the initial value of the trajectory) are small in the uniform and Euclidean norms, respectively. The pertrurbation of the right-hand side of the equation is small in the integral sense. Representation formulas of the differential of solution are obtained, when pertrurbations are small in the Euclidean topology. If the effect of discontinuouty at the initial moment influences upon the right-hand side of the equation, then, in contrast to earlier obtained formulas, representation formulas of the differential contain a new term.

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რეზიუმე. განხილულია ცვლადი დაგვიანების შემცველი არაწრფივი და ნეიტრალური ტიპის კვაზი – წრფივი დიფერენციალური განტოლებები, როცა საწყისი ფუნქციის მნიშვნელობა საწყის მომენტში, საზოგადოდ, არ ემთხვევა ტრაექტორიის მნიშვნელობას (წყვეტილობა საწყის მომენტში). დამტკიცებულია თეორემები კოშის ამოცანის ამონახსნის უწყვეტად დამოკიდებულების შესახებ საწყის მონაცემებსა და მარჯვენა მხარეზე. საწყისი მონაცემების – საწყისი ფუნქციისა და საწყისი მნიშვნელობების (საწყისი მომენტი, ტრაექტორიის საწყისი მნიშვნელობა) – შეშფოთებები, შესაბამისად, მცირეა თანაბარ და ევკლიდურ ნორმაში. მარჯვენა მხარის შეშფოთება მცირეა ინტეგრალური აზრით. მიღებულია ამონახსნის დიფერენციალის წარმოდგენის ფორმულები, როცა შეშფოთებები მცირეა ევკლიდურ ტოპოლოგიაში. თუ საწყის მომენტში წყვეტილობის ეფექტი გავლენას ახდენს განტოლების მარჯვენა მხარეზე, მაშინ დიფერენციალის წარმოდგენის ფორმულები, განსხვავებით ადრე მიღებული ფორმულებისაგან, შეიცავენ ახალ წევრს.

Introduction

In the present work two classes of ordinary differential equations with deviating argument are considered, namely, non-linear equations with variable delay and quasi-linear neutral equations. The question on continuity and differentiability of solution of the Cauchy problem with respect to initial data and right-hand side is investigated in the case where at the initial moment of time the value of the initial function, generaly speaking, does not coincide with the initial value of the trajectory (discontinuity at the initial moment).

The first chapter deals with delay differential equations. In §1 a theorem on continuous dependence of solution on perturbations is proved, which is an analogue of a theorem given in [11], [12]. The perturbations of the initial data, i.e., of the initial function and the initial values (the initial moment, the initial value of the trajectory) are small in the uniform and Euclidean norms, respectively. The petrurbation of the right-hand side of the equation is small in the integral sense.

Theorems on continuous dependence of solutions of the Cauchy problem and boundary value problems for various classes of ordinary differential equations and differential equations with deviating argument, when petrurbation of the right-hand side is small in the integral sense, were proved in [3], [4], [7], [8], [16], [17], [19-21], [23], [24], [26-29].

Differential equations with deviating argument, when pertrubations of the initial data and the right-hand side are small in the Euclidean norm were considered in [9], [13-15], [18], [22].

In §2 estimates of the increment of solutions are established with respect to small perturbations in the sense of the Euclidean topology. In §3 on the basis of these estimates representation formulas for the differential of solutions are obtained. If the effect of discontinuouty at the initial moment influences upon the right-hand side of the equation, then, in contrast to formulas given in [16], representation formulas for the differential contain a new term (see the formula (3.20)).

In the second chapter the above results are extended to neutral differential equations whose right-hand sides are linear with respect to the phase velocity.

Finally we note that the results obtained in this work play an important role in investigating optimal control problems with deviating argument.

CHAPTER I

CONTINUOUS DEPENDENCE AND DIFFERENTIABILITY OF SOLUTION OF DELAY DIFFERENTIAL EQUATIONS

1. Continuous Dependence of Solution

1.1. Preliminary Notes. Let X be a metric space, ρ be the distance function on X and let

$$F(\cdot,\mu):X\longrightarrow X \tag{1.1}$$

be a family of mappings depending on a parameter $\mu\in G$, where G is a topological space. The family (1.1) is said to be a uniform contraction if there exists a number $\alpha\in(0,1)$ not depending on μ and such that for any $\mu\in G$ the inequality

$$\rho(F(y_1,\mu),F(y_2,\mu))\le\alpha\rho(y_1,y_2)\quad\forall(y_1,y_2)\in X^2$$

is fulfilled.

Define the k-iteration of the mapping (1.1) by

$$F^k(y,\mu)=F(F^{k-1}(y,\mu),\mu),\quad k=1,2,\dots,\quad F^0(y,\mu)=y.$$

It is obvious that

$$F^k(\cdot,\mu):X\longrightarrow X,\quad\forall\mu\in G. \tag{1.2}$$

Theorem 1.1 ([25]). *Let X be a complete metric space. If some k-iteration (1.2) is a uniform contraction family, then for any $\mu\in G$ the mapping (1.1) has a unique fixed point y_μ, i.e. $F(y_\mu,\mu)=y_\mu$. Moreover, if the mapping*

$$F^k(y_{\tilde\mu},\cdot):G\longrightarrow X,\quad\tilde\mu\in G,$$

is continuous at the point $\tilde\mu$, then the mapping $y_\mu:G\longrightarrow X$ is also continuous at the point $\tilde\mu$.

Let R^n be the n-dimensional Euclidean space of the points

$$x=\begin{pmatrix}x^1\\ \vdots\\ x^n\end{pmatrix},\qquad |x|^2=\sum_{i=1}^n(x^i)^2;$$

$J=[a,b]$ be a finite interval; $O\subset R^n$ be an open set; $L_1(J,R_0^+)$ be the space of integrable functions $m:J\to R_0^+=[0,\infty)$.

We denote by $E(J\times O^2,R^n)$ the space of n-dimensional functions $f:J\times O^2\to R^n$ satisfying the conditions

1) for every $(x_1,x_2)\in O^2$ the function $f(\cdot,x_1,x_2):J\to R^n$ is measurable;

2) for any compact $K \subset O$[1] and any function $f \in E(J \times O^2, R^n)$, there exist functions $m_{f,K}(t)$, $L_{f,K}(t)$ from the space $L_1(J, R_0^+)$ such that

$$|f(t, x_1, x_2)| \leq m_{f,K}(t), \quad \forall (t, x_1, x_2) \in J \times K^2,$$

$$|f(t, x_1', x_2') - f(t, x_1'', x_2'')| \leq L_{f,K}(t) \sum_{i=1}^{2} |x_i' - x_i''|, \ \forall (t, x_1', x_2', x_1'', x_2'') \in J \times K^4.$$

From the conditions 1), 2) it easily follows the following

Lemma 1.1. *For every $f \in E(J \times O^2, R^n)$ the function*

$$H_f(t', t'', x_1, x_2) = \left| \int_{t'}^{t''} f(t, x_1, x_2) dt \right|$$

is continuous in $(t', t'', x_1, x_2) \in J^2 \times O^2$.

Let $\tau : R^1 \to R^1$ be an absolutely continuous function satisfying $\tau(t) \leq t$, $\dot{\tau}(t) > 0$; $\Delta(J_1, O)$ be the space of piecewise continuous functions[2] $\varphi : J_1 = [\tau(a), b] \to O$ satisfying the condition $cl\varphi(J_1) \subset O$, $\varphi(J_1) = \{\varphi(t) : t \in J_1\}$, $\|\varphi\| = \sup\{|\varphi(t)| : t \in J_1\}$.

Lemma 1.2. *Let $\psi(t) \in K$, $t \in J$, be a continuous function and $\varphi \in \Delta(J_1, O)$, $\varphi(t) \in K$, $t \in J_1$. Let $t_i \in (a, b)$, $i = 1, \ldots, l$, be points of discontinuity of the function $\phi(t) = (\psi(t), \varphi(\tau(t)))$, $t \in J$. Then for any function $f \in E(J \times O^2, R^n)$ and for any natural number s the inequality*

$$\beta = \max_{t', t'' \in J} \left| \int_{t'}^{t''} f(t, \phi(t)) dt \right| \leq \sigma(s, \phi) \int_J L_{f,K}(t) dt + s(l+1) H_f(J, K)$$

is valid, where

$$\sigma(s, \phi) = \max\{\sigma(s, \phi_i) : 1 \leq i \leq l+1\},$$

$$\sigma(s, \phi_i) = \sup\Big\{ |\psi(t') - \psi(t'')| +$$

$$+ |\varphi_i(t') - \varphi_i(t'')| : t', t'' \in [t_{i-1}, t_i], |t' - t''| \leq \frac{t_i - t_{i-1}}{s} \Big\};$$

$$\phi_i = (\psi, \varphi_i), \tag{1.3}$$

$$\varphi_i(t) = \begin{cases} \varphi(\tau(t_{i-1}^+)), & t = t_{i-1}, \\ \varphi(\tau(t)), & t \in (t_{i-1}, t_i), \\ \varphi(\tau(t_i^-)), & t = t_i, \end{cases} \tag{1.4}$$

$i = 1, \ldots, l+1$, $t_0 = a$, $t_{l+1} = b$; $H_f(J, K) = \sup\{H_f(t', t'', x', x'') : (t', t'', x', x'') \in J^2 \times K^2\}$ (see Lemma 1.1).

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[1] Here and in the sequel by K, K_0, K_1 we denote compact subsets of the set O.

[2] Everywhere we assume that piecewise continuous functions have finite number of discontinuity points of the first kind.

Proof. There exist numbers $a_1, b_1 \in J$ such that

$$\beta = \Big| \int_{a_1}^{b_1} f(t, \phi(t))dt \Big|.$$

Let $a_1 \in [t_{p-1}, t_p)$, $b_1 \in (t_{q-1}, t_q]$, $1 \le p \le q \le l+1$. We divide the intervals $[a_1, t_p]$, $[t_{i-1}, t_i]$, $i = p+1, q-1$, $[t_{q-1}, t_q]$ into s equal parts Δ_j^p, Δ_j^i, $i = p+1, \dots, q-1$, Δ_j^q, $j = 1, \dots, s$, respectively.

It is obvious that

$$[a_1, b_1] = [a_1, t_p] \cup \big(\bigcup_{i=p+1}^{q-1} [t_{i-1}, t_i] \big) \cup [t_{q-1}, b_1] = \bigcup_{i=p}^{q} \bigcup_{j=1}^{s} \Delta_j^i.$$

Taking into account this equality and the notation (1.4), we obtain

$$\beta = \Big| \int_{a_1}^{t_p} f(t, \phi_1(t))dt + \sum_{p+1}^{q-1} \int_{t_{i-1}}^{t_i} f(t, \phi_i(t))dt + \int_{t_{q-1}}^{b_1} f(t, \phi_q(t))dt \Big| \le$$

$$\le \sum_{i=p}^{q} \sum_{j=1}^{s} \Big| \int_{\Delta_j^i} f(t, \phi_i(t))dt \Big|.$$

Let $t_j^i \in \Delta_j^i$, $i = p, \dots, q$, $j = 1, \dots, s$, be arbitrary but fixed points. Then

$$\beta \le \sum_{i=p}^{q} \sum_{j=1}^{s} \int_{\Delta_j^i} |f(t, \psi(t), \varphi_i(t)) - f(t, \psi(t_j^i), \varphi_i(t_j^i))| dt +$$

$$+ \sum_{i=p}^{q} \sum_{j=1}^{s} \Big| \int_{\Delta_j^i} f(t, \psi(t_j^i), \varphi_i(t_j^i))dt \Big| \le$$

$$\le \sum_{i=p}^{q} \sum_{j=1}^{s} \int_{\Delta_j^i} L_{f,K}(t) \sigma_i(m, \phi_i) dt + s(q-p+1) H_f(J, K) \le$$

$$\le \sigma(s, \phi) \int_J L_{f,K}(t)dt + s(l+1) H_f(J, K). \quad \square$$

Lemma 1.3. *Let* $\psi(t) \in K$, $t \in J$, *be a continuous function and* $\varphi \in \Delta(J_1, O)$, $\varphi(t) \in K$, $t \in J_1$. *Let the sequence* $\delta f_i \in E(J \times O^2, R^n)$, $i = 1, 2, \dots,$ *satisfy*

$$\int_J L_{\delta f_i, K}(t)dt \le \alpha_0 = \text{const}, \quad \lim_{i \to \infty} H_{\delta f_i}(J, K) = 0.$$

Then

$$\lim_{i\to\infty}\beta_i = 0,$$

where

$$\beta_i = \max_{t',t''\in J}\left|\int_{t'}^{t''}\delta f_i(t,\phi(t))dt\right|, \quad \phi(t) = (\psi(t),\varphi(\tau(t))).$$

Proof. Let $\varepsilon > 0$ be an arbitrary number. By virtue of Lemma 1.2

$$\beta_i \le \sigma(s,\phi)\int_J L_{\delta f_i,K}(t)dt + s(l+1)H_{\delta f_i}(J,K). \tag{1.5}$$

The functions $\phi_i(t)$, $t\in[t_{i-1},t_i]$, $i=1,\dots,l+1$, are continuous (see (1.3), (1.4)). Therefore

$$\lim_{s\to\infty}\sigma(s,\phi) = 0.$$

Consequently there exist natural numbers s_0 and i_0 such that

$$\sigma(s_0,\phi)\alpha_0 \le \varepsilon/2, \quad s_0(l+1)H_{\delta f_i}(J,K)\le\varepsilon/2, \quad i\ge i_0. \tag{1.6}$$

From (1.5) taking into consideration (1.6) we obtain

$$\beta_i\le\varepsilon, \quad i\ge i_0. \quad \square$$

By induction and integration by parts we can easily prove the following

Lemma 1.4. *Let* $m(\cdot)\in L_1(J,R_0^+)$. *Then*

$$\int_a^t m(\xi_1)d\xi_1\int_a^{\xi_1}m(\xi_2)d\xi_2\dots\int_a^{\xi_{k-1}}m(\xi_k)d\xi_k = \frac{1}{k!}\Big(\int_a^t m(\xi)d\xi\Big)^k. \tag{1.7}$$

Lemma 1.5 ([25]). *Let* $K\subset \mathrm{int}K_1$ *and there exist a compact set* $Q\subset O^2$ *with* $K^2\subset Q\subset K_1^2$ *and an infinitely differentiable function* $\chi: R^n\times R^n\to[0,1]$ *such that*

$$\chi(x_1,x_2) = \begin{cases} 1, & (x_1,x_2)\in Q,\\ 0, & (x_1,x_2)\notin K_1^2.\end{cases} \tag{1.8}$$

Lemma 1.6. *Let* $f\in E(J\times O^2,R^n)$. *Then the function*

$$g(t,x_1,x_2) = \begin{cases}\chi(x_1,x_2)f(t,x_1,x_2), & (x_1,x_2)\in K_1^2,\ t\in J,\\ 0, & (x_1,x_2)\notin K_1^2,\ t\in J,\end{cases} \tag{1.9}$$

satisfies the following conditions:

$$|g(t,x_1,x_2)| \le m_{f,K_1}(t), \quad \forall (t,x_1,x_2) \in J \times R^{2n}, \tag{1.10}$$

$$|g(t,x_1',x_2') - g(t,x_1'',x_2'')| \le L_f(t)\sum_{i=1}^{2}|x_i' - x_i''|,$$
$$\forall (t,x_1',x_2',x_1'',x_2'') \in J \times R^{4n}, \tag{1.11}$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \tag{1.12}$$
$$\alpha_1 = \sup\{|\chi_{x_1}(x_1,x_2)| + |\chi_{x_2}(x_1,x_2)| : (x_1,x_2) \in K_1^2\},$$
$$\chi_{x_i} = \frac{\partial \chi}{\partial x_i}, \quad i = 1,2.$$

Proof. By (1.9) the validity of the inequality (1.10) is obvious. Let $(x_1',x_2') \in K_1^2$, $(x_1'',x_2'') \in K_1^2$. Then (see (1.8))

$$|g(t,x_1',x_2') - g(t,x_1'',x_2'')| = \chi(x_1',x_2')|f(t,x_1',x_2') - f(t,x_1'',x_2'')| +$$
$$+|\chi(x_1',x_2') - \chi(x_1'',x_2'')||f(t,x_1'',x_2'')| \le$$
$$\le L_{f,K_1}(t)\sum_{i=1}^{2}|x_i' - x_i''| + \alpha_1 m_{f,K_1}(t)\sum_{i=1}^{2}|x_i' - x_i''| = L_f(t)\sum_{i=1}^{2}|x_i' - x_i''|.$$

Let $(x_1',x_2') \in K_1^2$, $(x_1'',x_2'') \notin K_1^2$. Then recalling that $\chi(x_1'',x_2'') = 0$, we get

$$|g(t,x_1',x_2') - g(t,x_1'',x_2'')| \le |\chi(x_1',x_2') - \chi(x_1'',x_2'')||f(t,x_1'',x_2'')| \le$$
$$\le \alpha_1 m_{f,K_1}(t)\sum_{i=1}^{2}|x_i' - x_i''| \le L_f(t)\sum_{i=1}^{2}|x_i' - x_i''|.$$

It is not difficult to see that the last inequality is valid in the case $(x_1',x_2') \notin K_1^2$ and $(x_1'',x_2'') \in K_1^2$ as well the inequality (1.11) is likewise obvious, if the points (x_1',x_2') and (x_1'',x_2'') do not belong to K_1^2. □

1.2. Theorems on Continuouty of Solution. To every element

$$\mu = (t_0,x_0,\varphi,f) \in A = J \times O \times \Delta(J_1,O) \times E(J \times O^2,R^n)$$

there corresponds the differential equation

$$\dot y(t) = f(t,y(t),h(t_0,\varphi,y)(\tau(t))) \tag{1.13}$$

with the initial condition

$$y(t_0) = x_0, \tag{1.14}$$

where the operator

$$h : J \times \Delta(J_1,O) \times C(J,R^n) \to \Delta(J_1,R^n)$$

is defined by

$$h(t_0,\varphi,y)(t)=\begin{cases}\varphi(t), & t\in[\tau(a),t_0),\\ y(t), & t\in[t_0,b];\end{cases} \tag{1.15}$$

$C(J,R^n)$ is the space of continuous functions $y:J\to R^n$ with the distance

$$\rho(y_1,y_2)=\max_{t\in J}|y_1(t)-y_2(t)|.$$

Definition 1.1. An absolutely continuous function $y(t)=y(t,\mu)\in O$, $t\in[r_1,r_2]\subset J$, is said to be a solution corresponding to the element $\mu\in A$ and defined on $[r_1,r_2]$, if $t_0\in[r_1,r_2]$, $y(t_0)=x_0$ and the function $y(t)$ satisfies the equation (1.13) almost everywhere (a.e.) on $[r_1,r_2]$.

In the space $E(J\times O^2,R^n)$ let us introduce a topology by means of the following basis of neighborhoods of zero [11]

$$B=\{V(K,\delta)\subset E(J\times O^2,R^n):K\subset O,\ \delta>0\},$$
$$V(K,\delta)=\{\delta f\in E(J\times O^2,R^n):H_{\delta f}(J,K)\le\delta\}.$$

Theorem 1.2. *Let $\tilde y(t)$ be the solution corresponding to the element $\tilde\mu=(\tilde t_0,\tilde x_0,\tilde\varphi,\tilde f)\in A$ defined on $[r_1,r_2]\subset(a,b)$; let K_1 contain some neighborhood of the set $K_0=cl\tilde\varphi(J_1)\cup\tilde y([r_1,r_2])$. Then there exist numbers $\delta_i>0$, $i=0,1$, such that to an arbitrary element*

$$\mu\in V(\tilde\mu,K_1,\delta_0,\alpha_0)=$$
$$=V(\tilde t_0,\delta_0)\times V(\tilde x_0,\delta_0)\times V(\tilde\varphi,\delta_0)\times V(\tilde f,K_1,\delta_0)\cap W(\tilde f,K_1,\alpha_0)$$

there corresponds a solution $y(t,\mu)$ defined on $[r_1-\delta_1,r_2+\delta_1]\subset J$. Moreover, for each $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)\in[0,\delta_0]$ such that for an arbitrary $\mu\in V(\tilde\mu,K_1,\delta_0,\alpha_0)$ the inequality

$$|y(t,\mu)-y(t,\tilde\mu)|\le\varepsilon,\quad t\in[r_1-\delta_1,r_2+\delta_1], \tag{1.16}$$

is fulfilled.

Here $V(\tilde t_0,\delta_0)$, $V(\tilde x_0,\delta_0)$, $V(\tilde\varphi,\delta_0)$ are closed δ-neighborhoods of the points $\tilde t_0$, $\tilde x_0$, $\tilde\varphi$ in the spaces $R^1,R^n,\Delta(J_1,R^n)$ respectively;

$$\left.\begin{aligned}&V(\tilde f,K_1,\delta_0)=\{\tilde f+\delta f:\delta f\in V(K_1,\delta)\},\\ &W(\tilde f,K_1,\alpha_0)=\\ &=\Big\{\tilde f+\delta f:\delta f\in E(J\times O^2,R^n),\ \int_J[m_{\delta f,K_1}(t)+L_{\delta f,K_1}(t)]dt\le\alpha_0\Big\}.\end{aligned}\right\} \tag{1.17}$$

Proof. Let $\varepsilon_0 > 0$ be so small that the closed ε_0-neighborhood of the set K_0 : $K(\varepsilon_0) = \{x \in R^n : \exists \hat{x} \in K_0,\ |x - \hat{x}| \le \varepsilon_0\}$ lies in intK_1.

On the basis of Lemma 1.5 there exists a compact Q, $K^2(\varepsilon_0) \subset Q \subset K_1^2$ and an infinitely differentiable function $\chi : R^n \times R^n \to [0,1]$ such that

$$\chi(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in Q, \\ 0, & (x_1, x_2) \notin K_1^2. \end{cases} \tag{1.18}$$

Now to every element $\mu \in A$ we correspond the differential equation

$$\dot{z}(t) = g(t, z(t), h(t_0, \varphi, z)(\tau(t))) \tag{1.19}$$

with the initial condition

$$z(t_0) = x_0, \tag{1.20}$$

where $g = \chi f$ and satisfies (1.10), (1.11).

It is obvious that the solution of the equation (1.19) with the initial condition (1.20) depends on the parameter

$$\mu \in G = J \times O \times \Delta(J_1, O) \times W(\tilde{f}, K_1, \alpha_0) \subset E_\mu =$$
$$= R^1 \times R^n \times \Delta(J_1, R^n) \times E(J \times O^2, R^n).$$

The topology in G is induced from E_μ.

On the complete space $C(J, R^n)$ we define a family of mappings depending on the parameter μ

$$F(\cdot, \mu) : C(J, R^n) \to C(J, R^n) \tag{1.21}$$

by the formula

$$\zeta(t) = \zeta(t, z, \mu) = x_0 + \int_{t_0}^{t} g(s, z(s), h(t_0, \varphi, z)(\tau(s)))ds,$$
$$t \in J, \quad z \in C(J, R^n).$$

It is clear that every fixed point $z(t, \mu)$, $t \in J$, of the mapping (1.21) is a solution of the equation (1.19) with the initial condition (1.20).

Let us define the k-iteration $F^k(z, \mu)$ by

$$\zeta_k(t) = \zeta_k(t, z, \mu) = x_0 + \int_{t_0}^{t} g(s, \zeta_{k-1}(s), h(t_0, \varphi, \zeta_{k-1})(\tau(s)))ds,$$

$$k = 1, 2, \dots, \quad \zeta_0(t) = z(t).$$

We will now prove that, for a sufficiently large k, $F^k(z, \mu)$ is a uniform contraction family. To this end, we estimate the difference (see (1.11))

$$|\zeta_k'(t) - \zeta_k''(t)| = |\zeta_k(t, z', \mu) - \zeta_k(t, z'', \mu)| \le$$

$$\leq \int_a^t |g(s, \zeta'_{k-1}(s), h(t_0, \varphi, \zeta'_{k-1})(\tau(s))) -$$
$$-g(s, \zeta''_{k-1}(s), h(t_0, \varphi, \zeta''_{k-1})(\tau(s)))|ds \leq$$
$$\leq \int_a^t L_f(s)\big(|\zeta'_{k-1}(s)-\zeta''_{k-1}(s)|+|h(t_0,\varphi,\zeta'_{k-1})(\tau(s))-h(t_0,\varphi,\zeta''_{k-1})(\tau(s))|\big)ds,$$
$$k = 1, 2, \dots, \tag{1.22}$$

where $L_f(s)$ has the form (1.12). We assume that $\zeta'_0(t) = z'(t)$, $\zeta''_0(t) = z''(t)$.

From the definition of the operator $h(\cdot)$ (see (1.15)) it follows

$$h(t_0, \varphi, \zeta'_{k-1})(\tau(t)) - h(t_0, \varphi, \zeta''_{k-1})(\tau(t)) = h(t_0, 0, \zeta'_{k-1} - \zeta''_{k-1})(\tau(t)).$$

Thus with $s \in [a, \gamma(t_0))$ we get (see (1.15))

$$h(t_0, 0, \zeta'_{k-1} - \zeta''_{k-1})(\tau(s)) = 0. \tag{1.23}$$

Let $\gamma(t_0) < b$. Then with $s \in [\gamma(t_0), b]$ we have

$$|h(t_0, 0, \zeta'_{k-1} - \zeta''_{k-1})(\tau(s))| = |\zeta'_{k-1}(\tau(s)) - \zeta''_{k-1}(\tau(s))| \leq$$
$$\leq \sup\big\{|\zeta'_{k-1}(\tau(\xi)) - \zeta''_{k-1}(\tau(\xi))| : \xi \in [\gamma(t_0), s]\big\} \leq$$
$$\leq \sup\big\{|\zeta'_{k-1}(\xi) - \zeta''_{k-1}(\xi)| : \xi \in [a, s]\big\}. \tag{1.24}$$

If $\gamma(t_0) > b$, then the equality (1.23) holds on the whole interval J.

From (1.22) taking into account (1.23), (1.24) it follows

$$\sup\big\{|\zeta'_{k-1}(\xi) - \zeta''_{k-1}(\xi)| : \xi \in [a, t]\big\} \leq$$
$$\leq 2\int_a^t L_f(\xi_1) \sup\big\{|\zeta'_{k-1}(\xi) - \zeta''_{k-1}(\xi)| : \xi \in [a, \xi_1]\big\} d\xi_1, \quad k = 1, 2, \dots .$$

Consequently

$$|\zeta'_k(t) - \zeta''_k(t)| \leq$$
$$\leq 2^2 \int_a^t L_f(\xi_1)d\xi_1 \int_a^{\xi_1} L_f(\xi_2) \sup\big\{|\zeta'_{k-2}(\xi) - \zeta''_{k-2}(\xi)| : \xi \in [a, \xi_2]\big\} d\xi_2.$$

Continuing this process, we obtain

$$|\zeta'_k(t) - \zeta''_k(t)| \leq 2^k \alpha_k(t) \|z' - z''\|,$$

where

$$\alpha_k(t) = \int_a^t L_f(\xi_1)d\xi_1 \int_a^{\xi_1} L_f(\xi_2)d\xi_2 \cdots \int_a^{\xi_{k-1}} L_f(\xi_k)d\xi_k = \frac{1}{k!}\Big(\int_a^t L_f(\xi)d\xi\Big)^k$$

(see (1.7)). Thus

$$\rho(F^k(z',\mu),F^k(z'',\mu)) = \|\zeta_k'(t)-\zeta_k''(t)\| \le$$
$$\le \frac{1}{k!}\Big(2\int\limits_J L_f(t)dt\Big)^k \|z'-z''\| = \tilde{\alpha}_k \rho(z',z'').$$

Now we will show the existence of a number $\alpha_2>0$ such that

$$\int\limits_J L_f(t)dt \le \alpha_2, \quad \forall f=\tilde f+\delta f \in W(\tilde f, K_1, \alpha_0). \tag{1.25}$$

Indeed, let $(x_1,x_2)\in K_1^2$ and $f\in W(\tilde f,K_1,\alpha_0)$. Then

$$|f(t,x_1,x_2)| \le m_{\tilde f,K_1}(t)+m_{\delta f,K_1}(t)=m_{f,K_1}(t), \quad t\in J. \tag{1.26}$$

Further, let $(x_1',x_2')\in K_1^2$ and $(x_1'',x_2'')\in K_1^2$. Then

$$|f(t,x_1',x_2')-f(t,x_1'',x_2'')| \le |\tilde f(t,x_1',x_2')-\tilde f(t,x_1'',x_2'')| +$$
$$+|\delta f(t,x_1',x_2')-\delta f(t,x_1'',x_2'')| \le \big(L_{\tilde f,K_1}(t)+L_{\delta f,K_1}(t)\big)\sum_{i=1}^{2}|x_i'-x_i''| =$$
$$= L_{f,K_1}(t)|x_i'-x_i''|. \tag{1.27}$$

On the basis of (1.12), taking into consideration (1.26), (1.27), (1.17) we obtain (1.25), where

$$\alpha_2=\alpha_0(1+\alpha_1)+\int\limits_J [L_{\tilde f,K_1}(t)+\alpha_1 m_{\tilde f,K_1}]dt.$$

Thus

$$\tilde\alpha_k \le \frac{(2\alpha_2)^k}{k!}.$$

Consequently, there exists a natural number k_1 for which $\tilde\alpha_{k_1}<1$. Therefore k_1-iteration of the familly (1.21) is a contraction. According to Theorem 1.1 the mapping (1.21) for every μ has a unique fixed point. Hence it follows that the equation (1.19) with the initial condition (1.20) has a unique solution $z(t,\mu)$, $t\in J$.

Now we prove that for an arbitrary $k=1,2,\dots$ the mapping

$$F^k(z(\cdot,\tilde\mu),\mu): G\to C(J,R^n)$$

is continuous at the point $\mu=\tilde\mu$.

To prove this, it suffices to show that if the sequence $\mu_i=(t_0^i,x_0^i,\varphi_i,f_i)\in G$, $i=1,2,\dots$ is convergent to $\tilde\mu=(\tilde t_0,\tilde x_0,\tilde\varphi,\tilde f)$, i.e.,

$$\lim_{i\to\infty}\big(|t_0^i-\tilde t_0|+|x_0^i-\tilde x_0|+\|\varphi_i-\tilde\varphi\|+H_{\delta f_i}(J,K_1)\big)=0, \quad \delta f_i=f_i-\tilde f,$$

then

$$\lim_{i\to\infty} F^k(z(\cdot,\tilde\mu),\mu_i)=F^k(z(\cdot,\tilde\mu),\tilde\mu)=z(\cdot,\tilde\mu). \tag{1.28}$$

The proof of the equality (1.28) will be carried out by induction.
Let $k=1$. We have

$$|\zeta_1^i(t)-\tilde z(t)|\le |x_0^i-\tilde x_0|+$$

$$+\left|\int_{t_0^i}^{t} g_i(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))ds-\int_{\tilde t_0}^{t}\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))ds\right|=$$

$$=a_1^i+a_2^i(t), \tag{1.29}$$

where

$$\zeta_1^i(t)=\zeta_1(t,\tilde z,\mu_i),\quad \tilde z(t)=z(t,\tilde\mu),\quad g_i=\chi f_i,\quad \tilde g=\chi\tilde f,$$

$$a_1^i=|x_0^i-\tilde x_0|+\left|\int_{t_0^i}^{\tilde t_0}|\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))|ds\right|,$$

$$a_2^i(t)=\left|\int_{t_0^i}^{t}[g_i(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))-\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))]ds\right|.$$

According to (1.10) we have

$$a_1^i\le |x_0^i-\tilde x_0|+\left|\int_{t_0^i}^{\tilde t_0} m_{\tilde f,K_1}(t)dt\right|.$$

Consequently,

$$\lim_{i\to\infty} a_1^i=0. \tag{1.30}$$

It is easy to see that after elementary transformations for $a_2^i(t)$ we obtain

$$a_2^i(t)=\left|\int_{t_0^i}^{t}[\tilde g(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))-\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))]ds\right|+$$

$$+\left|\int_{t_0^i}^{t}[\delta g_i(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))-\delta g_i(s,\tilde z(s),h(t_0^i,\tilde\varphi,\tilde z)(\tau(s)))]ds\right|+$$

$$+\left|\int_{t_0^i}^{t}\delta g_i(s,\tilde z(s),h(t_0^i,\tilde\varphi,\tilde z)(\tau(s)))ds\right|\le a_{21}^i+a_{22}^i+a_{23}^i(t), \tag{1.31}$$

where

$$a_{21}^i = \int_J L_{\tilde f}(t)|h(t_0^i, \varphi_i, \tilde z)(\tau(t))) - h(\tilde t_0, \tilde\varphi, \tilde z)(\tau(t)))|dt,$$

$$a_{22}^i = \int_J L_{\delta f_i}(t)|h(t_0^i, \varphi_i, \tilde z)(\tau(t))) - h(t_0^i, \tilde\varphi, \tilde z)(\tau(t))|dt,$$

$$a_{23}^i(t) = \left|\int_{t_0^i}^t \delta g_i(s, \tilde z(s), h(t_0^i, \tilde\varphi, \tilde z)(\tau(s)))ds\right|, \quad \delta g_i = g_i - \tilde g.$$

Now we will estimate $a_{21}^i, a_{22}^i, a_{23}^i(t)$. We have

$$a_{21}^i \le \int_J L_{\tilde f}(t)|h(t_0^i, \varphi_i - \tilde\varphi, 0)(\tau(t)))|dt +$$

$$+\int_J L_{\tilde f}(t)|h(t_0^i, \tilde\varphi, \tilde z)(\tau(t))) - h(\tilde t_0, \tilde\varphi, \tilde z)(\tau(t)))|dt \le$$

$$\le \|\varphi_i - \tilde\varphi\|\int_J L_{\tilde f}(t)dt + \int_{\gamma(s_1^i)}^{\gamma(s_2^i)} L_{\tilde f}(t)|\tilde\varphi(\tau(t)) - \tilde z(\tau(t))|dt,$$

$$s_1^i = \min\{t_0^i, \tilde t_0\}, \quad s_2^i = \max\{t_0^i, \tilde t_0\}$$

The function $\gamma(t)$ is continuous, therefore

$$\lim_{i\to\infty}[\gamma(s_2^i) - \gamma(s_1^i)] = 0.$$

Thus

$$\lim_{i\to\infty} a_{21}^i = 0. \tag{1.32}$$

Further,

$$a_{22}^i = \int_J L_{\delta f_i}(t)|\varphi_i(\tau(t)) - \tilde\varphi(\tau(t))|dt \le \|\varphi_i - \tilde\varphi\|\int_J L_{\delta f_i}(t)dt.$$

On the basis of (1.12) and (1.17), we have

$$\int_J L_{\delta f_i}(t)dt = \int_J [L_{\delta f_i, K_1}(t) + \alpha_1 m_{\delta f_i, K_1}(t)]dt \le \alpha_0(1+\alpha_1),$$

whence we can conclude that

$$\lim_{i\to\infty} a_{22}^i = 0. \tag{1.33}$$

Now we will estimate $a_{23}^i(t)$. Here we consider two cases.

Let $t \in [a, b_i]$, $b_i = \min\{b, \gamma(t_0^i)\}$. Then

$$a_{23}^i(t) = \left| \int_{t_0^i}^{t} \delta g_i(s, \tilde{z}(s), \tilde{\varphi}(\tau(s)))ds \right| \le \max_{t', t'' \in J} \left| \int_{t'}^{t''} \delta g_i(s, \tilde{z}(s), \tilde{\varphi}(\tau(s)))ds \right|.$$

It is not difficult to see that $b_i \ge b_0 = \min\{\gamma(a), b\}$ since $\gamma(t_0^i) > \gamma(a)$. Therefore with $t \in [b_i, b]$ we have

$$a_{23}^i(t) = \left| \int_{t_0^i}^{b} \delta g_i(s, \tilde{z}(s), \tilde{\varphi}(\tau(s)))ds + \left| \int_{b^i}^{t} \delta g_i(s, \tilde{z}(s), \tilde{z}(\tau(s)))ds \right| \le \right.$$

$$\le \max_{t', t'' \in J} \left| \int_{t'}^{t''} \delta g_i(s, \tilde{z}(s), \tilde{\varphi}(\tau(s)))ds \right| +$$

$$+ \max_{t', t'' \in [b_0, b]} \left| \int_{t'}^{t''} \delta g_i(s, \tilde{z}(s), \tilde{z}(\tau(s)))ds \right| = a_{24}^i.$$

Thus

$$a_{23}^i(t) \le a_{24}^i, \qquad t \in J.$$

It is easy to note that

$$H_{\delta g_i}([b_0, b], K_1) \le H_{\delta g_i}(J, K_1) \le H_{\delta f_i}(J, K_1).$$

By the hypothesis

$$\lim_{i \to \infty} H_{\delta f_i}(J, K_1) = 0.$$

Consequently, all the conditions of Lemma 1.3 are fulfilled (see (1.17)). Therefore

$$\lim_{i \to \infty} a_{24}^i = 0.$$

Thus

$$\lim_{i \to \infty} a_{23}^i(t) = 0 \quad \text{uniformly for } \ t \in J. \tag{1.34}$$

The conditions (1.32)-(1.34) yield (see (1.31))

$$\lim_{i \to \infty} a_2^i(t) = 0 \quad \text{for } \ t \in J. \tag{1.35}$$

Taking into consideration (1.30), (1.35), from (1.29) we obtain the equality

$$\lim_{i \to \infty} \|\zeta_1^i - \tilde{z}\| = 0.$$

The equality (1.28) for $k = 1$ is proved.

Let now the condition (1.28) hold for some $k \ge 1$. We will prove the validity of (1.28) for $k + 1$.

By elementary transformations we obtain

$$|\zeta_{k+1}^i(t) - \tilde{z}(t)| \le |x_0^i - \tilde{x}_0| +$$

$$+\left|\int_{t_0^i}^{t} g_i(s, \zeta_k^i(s), h(t_0^i, \varphi_i, \zeta_k^i)(\tau(s)))ds - \int_{\tilde{t}_0}^{t} \tilde{g}(s, \tilde{z}(s), h(\tilde{t}_0, \tilde{\varphi}, \tilde{z})(\tau(s)))ds\right| \le$$

$$\le |x_0^i - \tilde{x}_0| + \left|\int_{t_0^i}^{\tilde{t}} |\tilde{g}(s, \tilde{z}(s), h(\tilde{t}_0, \tilde{\varphi}, \tilde{z})(\tau(s)))ds|\right| +$$

$$+\left|\int_{t_0^i}^{t} [g_i(s, \tilde{z}(s), h(t_0^i, \varphi_i, \tilde{z})(\tau(s))) - \tilde{g}(s, \tilde{z}(s), h(\tilde{t}_0, \tilde{\varphi}, \tilde{z})(\tau(s)))]ds\right| +$$

$$+\left|\int_J |g_i(s, \zeta_k^i(s), h(t_0^i, \varphi_i, \zeta_k^i)(\tau(s)))ds - g_i(s, \tilde{z}(s), h(t_0^i, \varphi_i, \tilde{z})(\tau(s)))|ds\right| =$$

$$= a_1^i + a_2^i(t) + a_{3k}^i.$$

Now we estimate a_{3k}^i (see (1.25))

$$a_{3k}^i \le \int_J L_{f_i}(t)\big[|\zeta_k^i(s) - \tilde{z}(s)| + |h(t_0^i, 0, \zeta_k^i - \tilde{z})(\tau(s))|ds \le$$

$$\le \|\zeta_k^i - \tilde{z}\| \int_J L_{f_i}(s)ds + \int_{b_i}^{b} L_{f_i}(s)|\zeta_k^i(\tau(s)) - \tilde{z}(\tau(s))|ds \le$$

$$\le 2\|\zeta_k^i - \tilde{z}\| \int_J L_{f_i}(s)ds \le 2\alpha_2\|\zeta_k^i(s) - \tilde{z}\|.$$

Since

$$\lim_{i\to\infty} \|\zeta_k^i - \tilde{z}\| = 0,$$

hence we have

$$\lim_{i\to\infty} a_{3k}^i = 0. \tag{1.36}$$

Owing to (1.30), (1.35) and (1.36), we get

$$\lim_{i\to\infty} \|\zeta_{k+1}^i - \tilde{z}\| = 0.$$

The equality (1.28) for every $k = 1, 2, \dots$ is proved.

Let a number $\delta_1 > 0$ be so small that $[r_1 - \delta_1, r_2 + \delta_1] \subset J$ and

$$|z(t, \tilde{\mu}) - z(r_1, \tilde{\mu})| \le \varepsilon_0/2 \quad \text{with} \quad t \in [r_1 - \delta_1, r_1],$$
$$|z(t, \tilde{\mu}) - z(r_2, \tilde{\mu})| \le \varepsilon_0/2 \quad \text{with} \quad t \in [r_2, r_2 + \delta_1].$$

It is clear that

$$z(t,\tilde{\mu}) = \tilde{y}(t), \qquad t \in [r_1, r_2]$$

and

$$(z(t,\tilde{\mu}), h(\tilde{t}_0, \tilde{\varphi}, z(\cdot,\tilde{\mu}))(\tau(t))) \in K^2(\frac{\varepsilon_0}{2}) \subset Q, \quad t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{1.37}$$

Consequently,

$$\chi(z(t,\tilde{\mu}), h(\tilde{t}_0, \tilde{\varphi}, z(\cdot,\tilde{\mu}))(\tau(t))) = 1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

The function $z(t,\tilde{\mu})$ satisfies the equation

$$\dot{y}(t) = \tilde{f}(t, y(t), h(\tilde{t}_0, \tilde{\varphi}, y)(\tau(t))), \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$

with the initial condition

$$y(\tilde{t}_0) = \tilde{x}_0.$$

Thus

$$y(t,\tilde{\mu}) = z(t,\tilde{\mu}), \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

By Theorem 1.1 for $\varepsilon_0/2$ there exists a number $\delta_0 \in (0, \varepsilon_0)$ such that to every element $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds a solution $z(t,\mu)$ satisfying the condition

$$|z(t,\mu) - z(t,\tilde{\mu})| \le \varepsilon_0/2, \quad t \in J.$$

Thus with $t \in [r_1 - \delta_1, r_2 + \delta_1]$

$$|z(t,\mu) - z(t,\tilde{\mu})| \le \varepsilon_0/2.$$

Hence by (1.37) we conclude that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ it holds

$$(z(t,\mu), h(t_0, \varphi, z(\cdot,\mu))(\tau(t))) \in Q, \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Thus the function $z(t,\mu)$ satisfies the equation (1.13) with the initial condition (1.14), i.e.,

$$y(t,\mu) = z(t,\mu), \quad t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{1.38}$$

The first part of the theorem is proved.

By Theorem 1.1 for an arbitrary $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$

$$|z(t,\mu) - z(t,\tilde{\mu})| \le \varepsilon, \quad t \in J,$$

whence using (1.38) we obtain (1.16). □

To every element $\mu = (t_0, x_0, \varphi, f) \in A$ there corresponds the delay differential equation

$$\dot{x}(t) = f(t, x(t), x(\tau(t))) \tag{1.39}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\tau(t_0), t_0), \qquad x(t_0) = x_0. \tag{1.40}$$

Definition 1.2. The function $x(t) = x(t, \mu) \in O$, $t \in [\tau(t_0), t_1] \subset [\tau(a), b]$, is said to be a solution of the equation (1.39) with the initial condition (1.40) or a solution corresponding to the element $\mu \in A$ and defined on the interval $[\tau(t_0), t_1]$, $t_0 \in [a, t_1)$, if on $[\tau(t_0), t_0]$ it satisfies the condition (1.40), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.39) a.e.

Theorem 1.3. *Let $\tilde{x}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A$ defined on $[\tau(\tilde{t}_0), \tilde{t}_1] \subset (\tau(a), b)$; let K_1 contain some neighborhood of the set $cl\tilde{\varphi}(J_1) \cup \tilde{y}([r_1, r_2])$. Then there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to every element $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds a solution $x(t, \mu)$ defined on $[\tau(t_0), \tilde{t}_1 + \delta_1] \subset [\tau(a), b]$. Moreover, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0, \delta_0]$ such that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$*

$$|x(t, \mu) - x(t, \tilde{\mu})| \le \varepsilon, \quad t \in [s_2, \tilde{t}_1 + \delta_1], \quad s_2 = \max\{t_0, \tilde{t}_0\}.$$

Proof. Let in Theorem 1.2 $r_1 = \tilde{t}_0$, $r_2 = \tilde{t}_1$. Then $\tilde{x}(t)$ on the interval $[\tilde{t}_0, \tilde{t}_1]$ satisfies the equation

$$\dot{y}(t) = \tilde{f}(t, y(t), h(\tilde{t}_0, \tilde{\varphi}, y)(\tau(t)))$$

with the initial condition

$$y(\tilde{t}_0) = \tilde{x}_0.$$

Thus in Theorem 1.2 instead of $\tilde{y}(t)$ we can take $\tilde{x}(t)$. By this theorem there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to every element $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds a solution $y(t, \mu)$ defined on the interval $[\tilde{t}_0 - \delta_1, \tilde{t}_1 + \delta_1] \subset J$. Moreover, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$

$$|y(t, \mu) - y(t, \tilde{\mu})| \le \varepsilon, \quad t \in [\tilde{t}_0 - \delta_1, \tilde{t}_1 + \delta_1]. \tag{1.41}$$

It is easy to see that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ the function

$$x(t, \mu) = \begin{cases} \varphi(t), & t \in [\tau(t_0), t_0), \\ y(t, \mu), & t \in [t_0, \tilde{t}_0 + \delta_1] \end{cases}$$

is the solution corresponding to the element $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$, defined on the interval $[\tau(t_0), \tilde{t}_1 + \delta_1] \subset [\tau(a), b]$ (see Definition 1.2). Hence the first part of the theorem is proved.

It is obvious that

$$x(t, \mu) = y(t, \mu) \quad t \in [s_2, \tilde{t}_1 + \delta_1].$$

Therefore from (1.41) it follows the desired inequality. □

Finally we note that Theorems 1.2, 1.3 are also valid, respectively, for the equations with delays

$$\dot{y}(t) = f(t, h(t_0, \varphi, y)(\tau_1(t)), \dots, h(t_0, \varphi, y)(\tau_s(t))),$$
$$\dot{x}(t) = f(t, x(\tau_1(t)), \dots, x(\tau_s(t))),$$

where $\tau_i : R^1 \to R^1$, $i = 1, \dots, s$ are absolutly continuous functions satisfying $\tau_i(t) \le t$, $\dot{\tau}_i(t) > 0$, $i = 1, \dots, s$, while the right-hand side f belongs to $E(J \times O^s, R^n)$.

2. Lemmas on the Estimation of the Increment

Introduce the set

$$V = \big\{\delta\mu = (\delta t_0, \delta x_0, \delta\varphi, \delta f) \in A - \tilde{\mu} : |\delta t_0| \le \alpha_3 = \text{const}, \ |\delta x_0| \le \alpha_3,$$
$$\|\delta\varphi\| \le \alpha_3, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \alpha_3, \ i = 1, \dots, k\big\}, \tag{2.1}$$

where $\delta f_i \in E(J \times O^2, R^n) - \tilde{f}$, $i = 1, \dots, k$ are fixed points.

Lemma 2.1. *Let $\tilde{y}(t)$ be the solution corresponding to the element $\tilde{\mu} = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}, \tilde{f}) \in A$ defined on $[r_1, r_2] \subset (a, b)$; let K_1 contain some neighborhood of the set $K_0 = cl\tilde{\varphi}(J_1) \cup \tilde{y}([r_1, r_2])$. Then there exist numbers $\delta_2 > 0$, $\varepsilon_2 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in A$ there corresponds the solution $y(t, \tilde{\mu} + \varepsilon\delta\mu)$ defined on $[r_1 - \delta_2, r_2 + \delta_2] \subset J$. Moreover,*

$$\begin{aligned} \varphi(t) = \tilde{\varphi}(t) + \varepsilon\delta\varphi(t) \in K_1, \quad t \in J_1, \\ y(t, \tilde{\mu} + \varepsilon\delta\mu) \in K_1, \quad t \in [r_1 - \delta_2, r_2 + \delta_2], \end{aligned} \tag{2.2}$$

$$\lim_{\varepsilon \to 0} y(t, \tilde{\mu} + \varepsilon\delta\mu) = y(t, \tilde{\mu}) \ \textit{uniformly for} \ (t, \mu) \in [r_1 - \delta_2, r_2 + \delta_2] \times V. \tag{2.3}$$

Proof. Let a number $\varepsilon_0 > 0$ be so small that the ε_0-closed neighborhood $K(\varepsilon_0)$ of the set K_0 lies in $int K_1$. By Theorem 1.2 there exist numbers δ_i, $i = 0, 1$, such that to every element $\tilde{\mu} + \varepsilon\delta\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds the solution $y(t, \tilde{\mu} + \varepsilon\delta\mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset J$. It is obvious that there exist numbers $\varepsilon_1 > 0$, $\delta_2 \in (0, \delta_1]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ we have

$$\tilde{\mu} + \varepsilon\delta\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0), \quad \varphi(t) \in K_1,$$
$$t \in J_1, \ \ y(t, \tilde{\mu}) \in K\Big(\frac{\varepsilon_0}{2}\Big), \quad t \in [r_1 - \delta_2, r_2 + \delta_2].$$

From the second part of Theorem 1.2 it follows the existence of a number $\varepsilon_2 \in [0, \varepsilon_1]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$

$$y(\tilde{\mu} + \varepsilon\delta\mu) \in K_1, \quad t \in [r_1 - \delta_2, r_2 + \delta_2].$$

Moreover, the equality (2.3) is fulfilled. □

Remark 2.1. Due to the uniqueness, the solution $y(t,\mu)$ defined on the interval $[r_1-\delta_2, r_2+\delta_2]$ is a continuation of the solution $\tilde{y}(t)$. Therefore the trajectory $\tilde{y}(t)$ in the sequel is assumed to be defined on the whole interval $[r_1-\delta_2, r_2+\delta_2]$.

We set

$$\Delta y(t) = \Delta y(t,\varepsilon\delta\mu) = y(t,\tilde{\mu}+\varepsilon\delta\mu) - \tilde{y}(t),$$
$$(t,\varepsilon,\delta\mu) \in [r_1-\delta_2, r_2+\delta_2] \times [0,\varepsilon_2] \times V. \tag{2.4}$$

Lemma 2.2. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(r_2) \geq \tilde{t}_0$ and the following hypotheses hold*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \sup_{\delta\mu\in V^-} \left| \int_{t_0}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t)+\Delta y(t), \varphi(\tau(t)))dt \right| < \infty, \tag{2.5}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \sup_{\delta\mu\in V^-} \sup_{t\in[\gamma(t_0),\gamma_0]} \left| \int_{\gamma(t_0)}^{t} [\tilde{f}(s,\tilde{y}(s)+\Delta y(s), \tilde{y}(\tau(s))+\Delta y(\tau(s))) - \right.$$
$$\left. -\tilde{f}(s,\tilde{y}(s),\tilde{\varphi}(\tau(s)))]ds \right| < \infty, \tag{2.6}$$

where

$$V^- = \{\delta\mu \in V : \delta t_0 \leq 0\}, \quad t_0 = \tilde{t}_0 + \varepsilon\delta t_0, \quad \gamma_0 = \gamma(\tilde{t}_0).$$

Then there exists a number $\varepsilon_3 \in (0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_3] \times V^-$

$$\max_{t\in[\tilde{t}_0, r_2+\delta_2]} |\Delta y(t)| \leq O(\varepsilon).^3 \tag{2.7}$$

Proof. By assumption of the lemma there exists a number $\varepsilon_3 \in (0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_3] \times V^-$ the conditions

$$\gamma(t_0) > \tilde{t}_0, \tag{2.8}$$

$$\left| \int_{t_0}^{\tilde{t}_0} \tilde{f}(t,\tilde{y}(t)+\Delta y(t), \varphi(\tau(t)))dt \right| \leq O(\varepsilon), \tag{2.9}$$

$$\left| \int_{\gamma(t_0)}^{t} [\tilde{f}(s,\tilde{y}(s)+\Delta y(s), \tilde{y}(\tau(s))+\Delta y(\tau(s))) - \tilde{f}(s,\tilde{y}(s),\tilde{\varphi}(\tau(s)))]ds \right| \leq$$
$$\leq O(\varepsilon), \quad \forall t \in [\gamma(t_0), \gamma_0] \tag{2.10}$$

[3] Here and in the sequel the symbols $O(\varepsilon)$, $o(t,\varepsilon\delta\mu)$ (scalar or vector) mean that $\lim\limits_{\varepsilon\to 0}[O(\varepsilon)/\varepsilon] < \infty$, $\lim\limits_{\varepsilon\to 0}[o(t,\varepsilon\delta\mu)/\varepsilon] < \infty$ uniformly for $(t,\delta\mu)$.

are fulfilled. It is easy to see that the function $\Delta y(t)$ on the interval $[\tilde{t}_0, r_2+\delta_2]$ satisfies the equation

$$\dot{\Delta} y(t) = \frac{d}{dt}\Delta y(t) = a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu), \tag{2.11}$$

where

$$a(t,\varepsilon\delta\mu) =$$
$$= \tilde{f}(t,\tilde{y}(t)+\Delta y(t), h(t_0,\varphi,\tilde{y}+\Delta y)(\tau(t))) - \tilde{f}(t,\tilde{y}(t), h(\tilde{t}_0,\tilde{\varphi},\tilde{y})(\tau(t))),$$
$$b(t,\varepsilon\delta\mu) = \varepsilon\delta f(t,\tilde{y}(t)+\Delta y(t), h(t_0,\varphi,\tilde{y}+\Delta y)(\tau(t))).$$

Now rewrite the equation (2.11) in the integral form

$$\Delta y(t) = \Delta y(\tilde{t}_0) + \int_{\tilde{t}_0}^{t} [a(s,\varepsilon\delta\mu) + b(s,\varepsilon\delta\mu)]ds, \quad t \in [\tilde{t}_0, r_2+\delta_2].$$

Hence it follows

$$|\Delta y(t)| = |\Delta y(\tilde{t}_0)| + \left|\int_{\tilde{t}_0}^{t} a(s,\varepsilon\delta\mu)ds\right| + \int_{\tilde{t}_0}^{r_2+\delta_2} |b(s,\varepsilon\delta\mu)|ds =$$
$$= |\Delta y(\tilde{t}_0)| + a_1(t,\varepsilon\delta\mu) + b_1(t,\varepsilon\delta\mu). \tag{2.12}$$

We will estimate $\Delta y(\tilde{t}_0)$. Taking into consideration (2.8), we get

$$|\Delta y(\tilde{t}_0)| = |y(\tilde{t}_0, \tilde{\mu}+\varepsilon\delta\mu) - y(\tilde{t}_0)| =$$
$$= \Big|\tilde{x}_0 + \varepsilon\delta x_0 + \int_{t_0}^{\tilde{t}_0} [\tilde{f}(t,\tilde{y}(t)+\Delta y(t), h(t_0,\varphi,\tilde{y}+\Delta y)(\tau(t))) + b(t,\varepsilon\delta\mu)]dt -$$
$$-\tilde{x}_0\Big| \le \varepsilon|\delta x_0| + \left|\int_{t_0}^{\tilde{t}_0} \tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))dt\right| + \int_{t_0}^{\tilde{t}_0} |b(t,\varepsilon\delta\mu)|dt. \tag{2.13}$$

It is obvious that (see (2.1), (2.2))

$$|b(t,\varepsilon\delta\mu)| \le \varepsilon\alpha_3 m_{\delta f}(t), \quad t \in [\tilde{t}_0, r_2+\delta_2], \quad m_{\delta f}(t) = \sum_{i=1}^{k} m_{\delta f_i, K_1}(t). \tag{2.14}$$

Therefore

$$\int_{t_0}^{\tilde{t}_0} |b(t,\varepsilon\delta\mu)|dt \le o(\varepsilon\delta\mu). \tag{2.15}$$

From (2.13), taking into account (2.1), (2.9) and (2.15), we obtain

$$|\Delta y(\tilde{t}_0)| \le O(\varepsilon). \tag{2.16}$$

To estimate $a_1(t,\varepsilon\delta\mu)$, $t\in[\tilde t_0, r_2+\delta_2]$, we consider three cases.
Let $t\in[\tilde t_0,\gamma(t_0)]$. Then

$$a_1(t,\varepsilon\delta\mu)=\left|\int_{\tilde t_0}^{t}[\tilde f(s,\tilde y(s)+\Delta y(s),\varphi(\tau(s)))-\tilde f(s,\tilde y(s),\tilde\varphi(\tau(s)))]ds\right|\le$$

$$\le\int_{\tilde t_0}^{t}L_{\tilde f,K_1}(s)(|\Delta y(s)|+\varepsilon\alpha_3)ds=\int_{\tilde t_0}^{t}L_{\tilde f,K_1}(s)|\Delta y(s)|ds+o(\varepsilon).\quad(2.17)$$

If $t\in[\gamma(t_0),\gamma_0]$, then on the basis of (2.17) and (2.10) we get

$$a_1(t,\varepsilon\delta\mu)=a_1(\gamma(t_0),\varepsilon\delta\mu)+\Bigg|\int_{\gamma(t_0)}^{t}[\tilde f(s,\tilde y(s)+\Delta y(s),\tilde y(\tau(s))+\Delta y(\tau(s)))-$$

$$-\tilde f(s,\tilde y(s),\tilde\varphi(\tau(s)))]ds\Bigg|\le\int_{\tilde t_0}^{t}L_{\tilde f,K_1}(s)|\Delta y(s)|ds+O(\varepsilon).$$

Let $t\in[\gamma_0,r_2+\delta_2]$. After elementary transformations we obtain

$$a_1(t,\varepsilon\delta\mu)=a_1(\gamma_0,\varepsilon\delta\mu)+$$

$$+\int_{\gamma_0}^{t}|\tilde f(s,\tilde y(s)+\Delta y(s),\tilde y(\tau(s))+\Delta y(\tau(s)))-\tilde f(s,\tilde y(s),\tilde y(\tau(s)))|ds\le$$

$$\le\int_{\tilde t_0}^{\gamma_0}L_{\tilde f,K_1}(s)|\Delta y(s)|ds+O(\varepsilon)+\int_{\gamma_0}^{t}L_{\tilde f,K_1}(s)(|\Delta y(s)|+|\Delta y(\tau(s))|)ds=$$

$$=\int_{\tilde t_0}^{t}L_{\tilde f,K_1}(s)|\Delta y(s)|ds+\int_{\tilde t_0}^{\tau(t)}L_{\tilde f,K_1}(\gamma(s))|\Delta y(s)|\dot\gamma(s)ds+O(\varepsilon).\quad(2.18)$$

It is clear that $[\tilde t_0,\tau(t)]\subset[\tilde t_0,t]$ with $t\in[\gamma_0,r_2+\delta_2]$. Therefore

$$a_1(t,\varepsilon\delta\mu)\le\int_{\tilde t_0}^{t}L(s)|\Delta y(s)|ds+O(\varepsilon),\quad\forall(t,\varepsilon,\delta\mu)\in[\gamma_0,r_2+\delta_2]\times[0,\varepsilon_3]\times V^-,$$

where

$$L(s)=L_{\tilde f,K_1}(s)+\chi(s)\dot\gamma(s)L_{\tilde f,K_1}(\gamma(s))\qquad(2.19)$$

and $\chi(s)$ is the characteristic function of the interval $[\tau(a),\tau(b)]$.

Now, on the basis the obtained estimates, for $a_1(t,\varepsilon\delta\mu)$ write out the final estimate

$$a_1(t,\varepsilon\delta\mu) \le \int_{\tilde t_0}^{t} L(s)|\Delta y(s)|ds + O(\varepsilon), \tag{2.20}$$
$$\forall (t,\varepsilon,\delta\mu)\in[\tilde t_0, r_2+\delta_2]\times[0,\varepsilon_3]\times V^-.$$

By virtue of (2.14) we have

$$b_1(\varepsilon\delta\mu)\le O(\varepsilon). \tag{2.21}$$

According to (2.16), (2.20) and (2.21), from the inequality (2.12) it follows

$$|\Delta y(t)| \le O(\varepsilon) + \int_{\tilde t_0}^{t} L(s)|\Delta y(s)|ds, \quad t\in[\tilde t_0, r_2+\delta_2].$$

By virtue of Gronwall's inequality we have

$$|\Delta y(t)| \le O(\varepsilon)\exp\Bigg(\int_{\tilde t_0}^{r_2+\delta_2} L(s)ds\Bigg), \quad t\in[\tilde t_0, r_2+\delta_2].$$

Hence it follows the desired inequality (2.7). □

Lemma 2.3. *Let $\tau(\tilde t_0)<\tilde t_0$, $\tau(r_2)\ge \tilde t_0$ and*

$$\lim_{\omega\to\omega_0^-}\tilde f(\omega)=f_0^-, \quad \omega=(t,x_1,x_2)\in R^-_{\tilde t_0}\times O^2,$$
$$R^-_{\tilde t_0}=(-\infty,\tilde t_0],\ \ \omega_0^-=(\tilde t_0,\tilde x_0,\tilde\varphi(\tau(\tilde t_0^-))). \tag{2.22}$$

Next, let there exist neighborhoods $V^-(\tilde t_0)$, $V^-(\omega_1^0)$, $V^-(\omega_2^-)$,[4] *$\omega_1^0=(\gamma_0,\tilde y(\gamma_0),\tilde x_0)$, $\omega_2^-=(\gamma_0,\tilde y(\gamma_0),\tilde\varphi(\tilde t_0))$ such that the functions $\dot\gamma(t)$, $t\in V^-(\tilde t_0)$ $\tilde f(\omega_1)-\tilde f(\omega_2)$, $(\omega_1,\omega_2)\in V^-(\omega_1^0),\times V^-(\omega_2^-)$ are bounded. Then there exists a number $\varepsilon_3\in(0,\varepsilon_2]$ such that the for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V^-$ the inequality (2.7) is fulfilled. Moreover,*

$$\Delta y(\tilde t_0)=\varepsilon[\delta x_0 - f_0^-\delta t_0]+o(\varepsilon\delta\mu). \tag{2.23}$$

Proof. From (2.22) it follows the existence of a nighborhood $V^-(\omega_0^-) = V^-(\tilde t_0)\times V(\tilde x_0)\times V(\tilde\varphi(\tau(\tilde t_0^-)))$ such that the function $\tilde f(\omega)$, $\omega\in V^-(\omega_0^-)$, is bounded.

[4] $V^-(\tilde t_0)=\{t\in V(\tilde t_0): t\le \tilde t_0\}$, $V(\tilde t_0)$ is some neighborhood of the point $\tilde t_0$,
$V^-(\omega_1^0)=V^-(\gamma_0)\times V(\tilde y(\gamma_0))\times V(\tilde x_0)$, $V^-(\omega_2^-)=V^-(\gamma_0)\times V(\tilde y(\gamma_0))\times V(\tilde\varphi(\tilde t_0))$.

Let $\bar{\varepsilon} \in (0, \varepsilon_2]$ be so small that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \bar{\varepsilon}] \times V^-$ the conditions

$$[t_0, \tilde{t}_0] \in V^-(\tilde{t}_0); \;\; (t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) \in V^-(\omega_0^-), \; t \in [t_0, \tilde{t}_0];$$
$$(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t))) \in V^-(\omega_1^0),$$
$$(t, \tilde{y}(t), \tilde{\varphi}(\tau(t))) \in V^-(\omega_2^-), \; t \in [\gamma(t_0), \gamma_0],$$

are fulfilled.

Consequently, the functions $\dot{\gamma}(t)$, $\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \, \varphi(\tau(t)))$, $t \in [t_0, \tilde{t}_0]$, $\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t)))$, $\tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))$, $t \in [\gamma(t_0), \gamma_0]$ are bounded.

It is obvious that

$$\gamma_0 - \gamma(t_0) = \int_{t_0}^{\tilde{t}_0} \dot{\gamma}(t) dt \leq O(\varepsilon).$$

Thus the conditions of Lemma 2.2 are fulfilled. Therefore there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ the inequality (2.7) is valid.

Now we prove the second part of the lemma. We have (see (2.13))

$$\Delta y(\tilde{t}_0) = \varepsilon[\delta x_0 - f_0^- \delta t_0] +$$
$$+ \int_{t_0}^{\tilde{t}_0} [\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) - f_0^-] dt + \int_{t_0}^{\tilde{t}_0} b(t, \varepsilon\delta\mu) dt. \qquad (2.24)$$

It is obvious that

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0, \tilde{t}_0]} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) - f_0^-| = 0 \quad \text{uniformly for} \quad \delta\mu \in V^-.$$

Consequently the second addend of the right-hand side of (2.24) has the order $o(\varepsilon\delta\mu)$. Taking into account this and the relation (2.15), from (2.24) we obtain the formula (2.23). □

Lemma 2.4. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(r_2) \geq \tilde{t}_0$ and the following hypotheses hold*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{\delta\mu \in V^+} \left| \int_{\tilde{t}_0}^{t_0} \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t))) dt \right| < \infty, \qquad (2.25)$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{\delta\mu \in V^+} \sup_{t \in [\gamma_0, \gamma(t_0)]} \left| \int_{\gamma_0}^{t} [\tilde{f}(s, \tilde{y}(s) + \Delta y(s), \varphi(\tau(s))) - \right.$$
$$\left. - \tilde{f}(s, \tilde{y}(s), \tilde{y}(\tau(s)))] ds \right| < \infty, \qquad (2.26)$$

where $V^+ = \{\delta\mu \in V : \delta t_0 \geq 0\}$. *Then there exists a number* $\varepsilon_3 \in (0, \varepsilon_2]$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$

$$\max_{t \in [t_0, r_2+\delta_2]} |\Delta y(t)| \leq O(\varepsilon). \tag{2.27}$$

Proof. By assumption of the lemma there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$ the conditions

$$t_0 < \gamma_0, \quad \gamma(t_0) < r_2 + \delta_2, \tag{2.28}$$

$$\left| \int_{\tilde{t}_0}^{t_0} \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))dt \right| \leq O(\varepsilon), \tag{2.29}$$

$$\left| \int_{\gamma_0}^{t} [\tilde{f}(s, \tilde{y}(s) + \Delta y(s), \varphi(\tau(s))) - \tilde{f}(s, \tilde{y}(s), \tilde{y}(\tau(s)))]ds \right| \leq$$

$$\leq O(\varepsilon) \quad \forall t \in [\gamma_0, \gamma(t_0)] \tag{2.30}$$

are fulfilled.

The function $\Delta y(t)$ on the interval $[t_0, r_2+\delta_2]$ satisfies the equation (2.11), which we rewrite in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int_{t_0}^{t} [a(s, \varepsilon\delta\mu) + b(s, \varepsilon\delta\mu)]ds, \quad t \in [t_0, r_2 + \delta_2].$$

Hence it follows

$$|\Delta y(t)| \leq |\Delta y(t_0)| + \int_{t_0}^{t} |a(s, \varepsilon\delta\mu)|ds + \int_{t_0}^{r_2+\delta_2} |b(s, \varepsilon\delta\mu)|ds =$$

$$= |\Delta y(t_0)| + a_2(t, \varepsilon\delta\mu) + b_2(\varepsilon\delta\mu). \tag{2.31}$$

We will estimate $\Delta y(t_0)$. Taking into consideration of (2.28) and (2.29), we get

$$|\Delta y(t_0)| = |y(t_0, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{y}(t_0)| =$$

$$= \left|\tilde{x}_0 + \varepsilon\delta x_0 - \left[\tilde{x}_0 + \int_{\tilde{t}_0}^{t_0} \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))dt\right]\right| \leq$$

$$\leq \varepsilon|\delta x_0| + \left| \int_{\tilde{t}_0}^{t_0} \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))dt \right| \leq O(\varepsilon). \tag{2.32}$$

To estimate $a_2(t, \varepsilon\delta\mu)$, $t \in [t_0, r_2 + \delta_2]$, we consider three cases.

Let $t \in [t_0, \gamma_0]$. Then analogously to (2.17) we obtain

$$a_2(t, \varepsilon\delta\mu) \le \int\limits_{t_0}^{t} L_{\tilde{f},K_1}(s)|\Delta y(s)|ds + O(\varepsilon). \tag{2.33}$$

Let $t \in [\gamma_0, \gamma(t_0)]$. Then on the basis (2.33) and (2.30) we get

$$a_2(t, \varepsilon\delta\mu) \le a_2(\gamma_0, \varepsilon\delta\mu) + \Bigg| \int\limits_{\gamma_0}^{\gamma(t_0)} [\tilde{f}(s, \tilde{y}(s) + \Delta y(s), \varphi(\tau(s)) -$$

$$-\tilde{f}(s, \tilde{y}(s), \tilde{y}(\tau(s)))]ds\Bigg| \le \int\limits_{t_0}^{t} L_{\tilde{f},K_1}(s)|\Delta y(s)|ds + O(\varepsilon).$$

Let $t \in [\gamma(t_0), r_2 + \delta_2]$. Then analogously to (2.18) and (2.19) it can be proved that

$$a_2(t, \varepsilon\delta\mu) \le \int\limits_{t_0}^{t} L(s)|\Delta y(s)|ds + O(\varepsilon).$$

Now for $a_2(t, \varepsilon\delta\mu)$ write out the final estimate

$$a_2(t, \varepsilon\delta\mu) \le \int\limits_{t_0}^{t} L(s)|\Delta y(s)|ds + O(\varepsilon), \tag{2.34}$$

$$\forall (t, \varepsilon, \delta\mu) \in [t_0, r_2 + \delta_2] \times [0, \varepsilon_3] \times V^+.$$

By virtue of (2.14), for $b_2(\varepsilon\delta\mu)$, we obtain

$$b_2(\varepsilon\delta\mu) \le O(\varepsilon). \tag{2.35}$$

From (2.31), taking into account (2.32), (2.34) and (2.35), we get

$$|\Delta y(t)| \le O(\varepsilon) + \int\limits_{t_0}^{t} L(s)|\Delta y(s)|ds, \quad t \in [t_0, r_2 + \delta_2].$$

By virtue of Gronwall's inequality, we have

$$|\Delta y(t)| \le O(\varepsilon) \exp\Bigg(\int\limits_{t_0}^{r_2+\delta_2} L(s)ds \Bigg), \quad t \in [t_0, r_2 + \delta_2].$$

Thus the inequality (2.27) is proved. □

Lemma 2.5. *Let* $\tau(\tilde{t}_0) < \tilde{t}_0$ $\tau(r_2) \geq \tilde{t}_0$ *and the following conditions are fulfilled*

$$\lim_{\omega \to \omega_0^+} \tilde{f}(\omega) = f_0^+, \quad \omega \in R_{\tilde{t}_0}^+ \times O^2,$$
$$R_{\tilde{t}_0}^+ = [\tilde{t}_0, \infty), \ \ \omega_0^+ = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0^+))). \tag{2.36}$$

Let, moreover, there exist neighborhoods $V^+(\tilde{t}_0)$, $V^+(\omega_1^0)$, $V^+(\omega_2^-)$, $\omega_2^+ = (\gamma_0, \tilde{y}(\gamma_0), \tilde{\varphi}(\tilde{t}_0^+))$ *such that the functions* $\dot{\gamma}(t)$, $t \in V^+(\tilde{t}_0)$ $\tilde{f}(\omega_1) - \tilde{f}(\omega_2)$, $(\omega_1, \omega_2) \in V^+(\omega_1^0), \times\ V^+(\omega_2^+)$ *are bounded. Then there exists a number* $\varepsilon_3 \in (0, \varepsilon_2]$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$ *the inequality* (2.27) *is fulfilled. Moreover,*

$$\Delta y(t_0) = \varepsilon[\delta x_0 - f_0^+ \delta t_0] + o(\varepsilon\delta\mu). \tag{2.37}$$

This lemma can be proved as Lemma 2.4 with insignificant changes (see the proof of Lemma 2.3).

Lemma 2.6. *Let* $\tau(r_2) < \tilde{t}_0$ *and the condition* (2.5) *be fulfilled. Then there exist numbers* $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ *it holds*

$$\max_{t \in [\tilde{t}_0, r_2 + \delta_3]} |\Delta y(t)| \leq O(\varepsilon). \tag{2.38}$$

Proof. By assumption of the lemma there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ the relation (2.9) is fulfilled and

$$\gamma(t_0) > r_2 + \delta_3. \tag{2.39}$$

Analogously to the proof of Lemma 2.2 we obtain (see (2.12))

$$|\Delta y(t)| \leq |\Delta y(\tilde{t}_0)| + \int_{\tilde{t}_0}^{t} |a(s, \varepsilon\delta\mu)| ds + \int_{\tilde{t}_0}^{r_2 + \delta_3} |b(s, \varepsilon\delta\mu)| ds =$$

$$= |\Delta y(\tilde{t}_0)| + a_1(t, \varepsilon\delta\mu) + b_3(t, \varepsilon\delta\mu), \quad t \in [\tilde{t}_0, r_2 + \delta_3].$$

Since (2.39) holds, for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_0, r_2 + \delta_3] \times [0, \varepsilon_3] \times V^-$ we have

$$a_1(t, \varepsilon\delta\mu) \leq \int_{\tilde{t}_0}^{t} L_{\tilde{f}, K_1}(s) |\Delta y(s)| ds + O(\varepsilon).$$

Besides (see (2.14))

$$b_3(\varepsilon\delta\mu) \leq O(\varepsilon).$$

Hence, taking into account (2.16), we get

$$|\Delta y(t)| \le O(\varepsilon) + \int\limits_{\tilde{t}_0}^{t} L_{\tilde{f},K_1}(s)|\Delta y(s)|ds, \quad t \in [\tilde{t}_0, r_2 + \delta_3].$$

Therefore by Gronwall's inequality we obtain (2.38). □

Lemma 2.7. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (2.22) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ the conditions (2.23) and (2.38) are fulfilled.*

This lemma, using Lemma 2.6, is proved analogously to Lemma 2.3.

Lemma 2.8. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (2.25) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$*

$$\max_{t \in [t_0, r_2 + \delta_3]} |\Delta y(t)| \le O(\varepsilon). \tag{2.40}$$

Proof. By assumption of the lemma there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$ the condition (2.29) is fulfilled and

$$\gamma_0 > r_2 + \delta_3. \tag{2.41}$$

Analogously to the proof of Lemma 2.4 we obtain (see (2.31))

$$|\Delta y(t)| \le |\Delta y(t_0)| + \int\limits_{t_0}^{t} |a(s, \varepsilon\delta\mu)|ds + \int\limits_{t_0}^{r_2+\delta_3} |b(s, \varepsilon\delta\mu)|ds =$$
$$= |\Delta y(t_0)| + a_2(t, \varepsilon\delta\mu) + b_4(t, \varepsilon\delta\mu), \quad t \in [t_0, r_2 + \delta_3].$$

Since (2.41) is fulfilled, for an arbitrary $(t, \varepsilon, \delta\mu) \in [t_0, r_2+\delta_3] \times [0, \varepsilon_3] \times V^+$ we have

$$a_2(t, \varepsilon\delta\mu) \le \int\limits_{t_0}^{t} L_{\tilde{f},K_1}(s)|\Delta y(s)|ds + O(\varepsilon).$$

Besides (see (2.14))

$$b_4(\varepsilon\delta\mu) \le O(\varepsilon).$$

After this, taking into account (2.32), we get

$$|\Delta y(t)| \le O(\varepsilon) + \int\limits_{t_0}^{t} L_{\tilde{f},K_1}(s)|\Delta y(s)|ds, \quad t \in [t_0, r_2 + \delta_3].$$

Hence, by Gronwall's inequality we obtain (2.40). □

Lemma 2.9. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (2.36) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$ (2.23) and (2.38) are fulfilled.*

This lemma, using Lemma 2.8, is proved analogously to Lemma 2.3.

Lemma 2.10. *Let $\tau(t_0) = \tilde{t}_0$ and the condition*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{\delta\mu \in V^-} \Bigg\{ \Bigg| \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt +$$

$$+ \int_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t)))dt \Bigg| \Bigg\} < \infty, \qquad (2.42)$$

be fulfilled. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ the inequality (2.7) is valid.

Proof. Let $\varepsilon_3 \in (0, \varepsilon_2]$ be so small that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$

$$\Bigg| \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt +$$

$$+ \int_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t)))dt \Bigg| \leq O(\varepsilon).$$

Since $\gamma(t_0) \in [t_0, \tilde{t}_0]$, the expression for $\Delta y(\tilde{t}_0)$ has the form

$$\Delta y(\tilde{t}_0) = \varepsilon \delta x_0 + \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt +$$

$$+ \int_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t)))dt + \int_{t_0}^{\tilde{t}_0} b(t, \varepsilon\delta\mu)dt. \qquad (2.43)$$

Hence, on the basis of the previous inequality (see (2.15)), we obtain

$$|\Delta y(\tilde{t}_0)| \leq O(\varepsilon).$$

It is easy to see that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_0, r_2 + \delta_2] \times [0, \varepsilon_3] \times V^-$ the inequality (see (2.19))

$$a_1(t, \varepsilon\delta\mu) \leq \int_{\tilde{t}_0}^{t} |\tilde{f}(s, \tilde{y}(s) + \Delta y(s), \tilde{y}(\tau(s)) + \Delta y(\tau(s))) -$$

$$-\tilde{f}(s,\tilde{y}(s),\tilde{y}(\tau(s)))|ds \le \int\limits_{\tilde{t}_0}^{t} L(s)|\Delta y(s)|ds + O(\varepsilon)$$

is valid.

After this the inequality (2.7) is proved in the standard way (see the proof of Lemma 2.2). □

Lemma 2.11. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$ *and there exist the finite limits*

$$\lim_{\omega\to\omega_3} \tilde{f}(\omega) = f_2^-, \quad \lim_{\omega\to\omega_4^-} \tilde{f}(\omega) = f_3^-, \quad \omega \in R_{\tilde{t}_0}^- \times O^2,$$

$$\omega_3 = (\tilde{t}_0, \tilde{x}_0, \tilde{x}_0), \quad \omega_4^- = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tilde{t}_0^-)); \qquad \lim_{t\to\tilde{t}_0} \dot{\gamma}(t) = \dot{\gamma}^-, \quad t \in R_{\tilde{t}_0}^-.$$

Then there exists a number $\varepsilon_3 \in (0,\varepsilon_2]$ *such that for an arbitrary* $(\varepsilon,\delta\mu) \in [0,\varepsilon_3]\times V^-$ *the inequality* (2.7) *is valid. Moreover,*

$$\Delta y(\tilde{t}_0) = \varepsilon\{\delta x_0 - [f_3^- + (f_2^- - f_3^-)\dot{\gamma}^-]\delta t_0\} + o(\varepsilon\delta\mu). \qquad (2.44)$$

Proof. First of all we prove the equality (2.44). It is easy to see that

$$\tilde{t}_0 - \gamma(t_0) = \gamma(\tilde{t}_{0c}) - \gamma(t_0) = \int\limits_{t_0}^{\tilde{t}_0} \dot{\gamma}(t)dt = -\varepsilon\dot{\gamma}^-\delta t_0 + o(\varepsilon\delta\mu).$$

Consequently

$$\gamma(t_0) = \tilde{t}_0 + \varepsilon\dot{\gamma}^-\delta t_0 + o(\varepsilon\delta\mu). \qquad (2.45)$$

Further, with $\varepsilon \in [0,\varepsilon_2]$

$$\int\limits_{t_0}^{\gamma(t_0)} \tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))dt = \varepsilon(\dot{\gamma}^- - 1)f_3^-\delta t_0 +$$

$$+\int\limits_{t_0}^{\gamma(t_0)} [\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t))) - f_3^-]dt = \varepsilon(\dot{\gamma}^- - 1)f_3^-\delta t_0 + \alpha(\varepsilon\delta\mu), \qquad (2.46)$$

$$\int\limits_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t,\tilde{y}(t)+\Delta y(t),\tilde{y}(\tau(t))+\Delta y(\tau(t)))dt = -\varepsilon\dot{\gamma}^- f_2^-\delta t_0 +$$

$$+\int\limits_{\gamma(t_0)}^{\tilde{t}_0} [\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\tilde{y}(\tau(t))+\Delta y(\tau(t))) - f_2^-]dt =$$

$$= -\varepsilon\dot{\gamma}^- f_2^-\delta t_0 + \beta(\varepsilon\delta\mu). \qquad (2.47)$$

It is obvious that

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,\gamma(t_0)]}|\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))-f_3^-|=0,$$

$$\lim_{\varepsilon\to 0}\sup_{t\in[\gamma(t_0),\tilde{t}_0]}|\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\tilde{y}(\tau(t))+\Delta y(\tau(t)))-f_2^-|=0$$

uniformly for $\delta\mu\in V^-$. Therefore

$$\alpha(\varepsilon\delta\mu)=o(\varepsilon\delta\mu),\quad \beta(\varepsilon\delta\mu)=o(\varepsilon\delta\mu). \tag{2.48}$$

If in (2.43) we use the relations obtained above (see also (2.15)), then we obtain (2.44).

It is clear that the conditions (2.46)–(2.48) guarantee that (2.42) is valid. Consequently, by Lemma 2.10 the first part of the lemma is also valid. □

Lemma 2.12. *Let* $\tau(\tilde{t}_0)=\tilde{t}_0$ *and the conditions*

$$\lim_{\omega\to\omega_3}\tilde{f}(\omega)=f_2^-,\quad \omega\in R_{\tilde{t}_0}^-\times O^2;\quad \lim_{t\to\tilde{t}_0}\dot{\gamma}(t)=1,\ \ t\in R_{\tilde{t}_0}^-,$$

be fulfilled. Let, moreover, there exist a neighborhood $V^-(\omega_4^-)$ *such that the function* $\tilde{f}(\omega)$, $\omega\in V^-(\omega_4^-)$ *is bounded. Then there exists a number* $\varepsilon_3\in(0,\varepsilon_2]$ *such that for an arbitrary* $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V^-$ *the relation* (2.7) *is valid. Moreover,*

$$\Delta y(\tilde{t}_0)=\varepsilon[\delta x_0-f_2^-\delta t_0]+o(\varepsilon\delta\mu). \tag{2.49}$$

Proof. It is clear (see (2.45)) that

$$\gamma(t_0)=\tilde{t}_0+o(\varepsilon\delta\mu).$$

Next, there exists a number $\bar{\varepsilon}\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\bar{\varepsilon}]\times V^-$ the condition

$$(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))\in V^-(\omega_4^-),\ \ t\in[t_0,\gamma(t_0)].$$

is fulfilled. Consequently the function $\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))$ with $(t,\varepsilon,\delta\mu)\in[t_0,\,\gamma(t_0)]\times[0,\bar{\varepsilon}]\times V^-$ is bounded. Thus

$$\left|\int_{t_0}^{\gamma(t_0)}[\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))dt\right|=o(\varepsilon\delta\mu).$$

It is obvious that

$$\int_{\gamma(t_0)}^{\tilde{t}_0}\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\tilde{y}(\tau(t))+\Delta y(\tau(t)))dt=-\varepsilon f_2^-\delta t_0+o(\varepsilon\delta\mu).$$

Thus the condition (2.42) is fulfilled. Consequently the first part of the lemma is proved (see Lemma 2.10.). Finally, from (2.43) on the basis of the last relations (see also (2.15)) we obtain (2.49). □

Lemma 2.13. *Let $\tau(t_0) = \tilde{t}_0$ and the conditions*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V^+}\left|\int_{\tilde{t}_0}^{t_0}\tilde{f}(t,\tilde{y}(t),\tilde{y}(\tau(t)))dt\right| < \infty,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V^+}\sup_{t\in[t_0,\gamma(t_0)]}\left|\int_{t_0}^{t}[\tilde{f}(s,\tilde{y}(s)+\Delta y(s),\varphi(\tau(s))) - \right.$$

$$\left. -\tilde{f}(s,\tilde{y}(s),\tilde{y}(\tau(s)))]ds\right| < \infty$$

be fulfilled. Then there exists a number $\varepsilon_3 \in (0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_3]\times V^+$ the inequality (2.7) is valid.

Proof. By assumption of the lemma there exists a number $\varepsilon_3 \in (0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V^+$ the conditions

$$\gamma(t_0) < r_2 + \delta_2,$$

$$\left|\int_{\tilde{t}_0}^{t_0}\tilde{f}(t,\tilde{y}(t),\tilde{y}(\tau(t)))dt\right| \le O(\varepsilon),$$

$$\left|\int_{t_0}^{t}[\tilde{f}(s,\tilde{y}(s)+\Delta y(s),\varphi(\tau(s))) - \tilde{f}(s,\tilde{y}(s),\tilde{y}(\tau(s)))]ds\right| \le O(\varepsilon)$$

$$\forall t\in[t_0,\gamma(t_0)]$$

are fulfilled.

It is obvious that the inequality (2.32) is valid. In order to estimate $a_2(t,\varepsilon\delta\mu)$, $t\in[t_0,r_2+\delta_2]$ (see (2.31)), we consider two cases.

Let $t\in[t_0,\gamma(t_0)]$. Then we have (see (2.31))

$$a_2(t,\varepsilon\delta\mu) = \left|\int_{t_0}^{t}[\tilde{f}(s,\tilde{y}(s)+\Delta y(s),\varphi(\tau(s))) - \tilde{f}(s,\tilde{y}(s),\tilde{y}(\tau(s)))]ds\right| \le O(\varepsilon).$$

Let $t\in[\gamma(t_0),r_2+\delta_2]$. Then (see (2.18), (2.19))

$$a_2(t,\varepsilon\delta\mu) \le a_2(\gamma(t_0),\varepsilon\delta\mu) + \int_{\gamma(t_0)}^{t}|a(s,\varepsilon\delta\mu)|ds \le$$

$$\le O(\varepsilon) + \int_{\gamma(t_0)}^{t}L_{\tilde{f},K_1}(s)(|\Delta y(s)| + |\Delta y(\tau(s))|)ds \le$$

$$\leq O(\varepsilon) + \int_{t_0}^{t} L(s)|\Delta y(s)|ds.$$

After this in the standard way we can estimate $|\Delta y(t)|$ (see (2.31)) and prove the inequality (2.27). □

Lemma 2.14. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and the folowing conditions*

$$\lim_{\omega \to \omega_3} \tilde{f}(\omega) = f_2^+, \quad \omega \in R_{\tilde{t}_0}^+ \times O^2$$

be fulfilled. Let, moreover, there exist neighborhoods $V^+(\tilde{t}_0)$, $V^+(\omega_4^+)$ such that the functions $\dot{\gamma}(t)$, $t \in V^+(\tilde{t}_0)$, $\tilde{f}(\omega)$, $\omega \in V^-(\omega_4^+)$ are bounded. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^+$ the inequality (2.7) *is fulfilled. Moreover,*

$$\Delta y(t_0) = \varepsilon[\delta x_0 - f_2^+ \delta t_0] + o(\varepsilon\delta\mu). \tag{2.50}$$

This lemma, by Lemma 2.13, is proved analogously to Lemma 2.3.

3. Differentiability of Solution

3.1. Preliminary Notes. We denote by $E_1(J \times O^2, R^n)$ the space of n-dimensional functions $f : J \times O^2 \to R^n$ satisfying the conditions:

1) for any fixed $t \in J$ the function f is continuously differentiable with respect to $(x_1, x_2) \in O^2$;

2) for any fixed $(x_1, x_2) \in O^2$ the function f and the matrix functions

$$f_{x_i} = \big(f_{x_j^i}^p\big)_{p,j=1}^{n,n}, \quad i = 1, 2,$$

are measurable with respect to t;

For an arbitrary $K \subset O$ and $f \in E_1(J \times O^2, R^n)$ there exists a function $m_{f,K}(\cdot) \in\in L_1(J, R_0^+)$ such that

$$|f(t, x_1, x_2)| + \sum_{i=1}^{2} |f_{x_i}(\cdot)| \leq m_{f,K}(t), \quad \forall (t, x_1, x_2) \in J \times K^2.$$

By the modulus $|f_{x_i}|$ of the matrix f_{x_i} we mean the Euclidean modulus, i.e.,

$$|f_{x_i}|^2 = \sum_{p,j=1}^{n} |f_{x_j^i}^p|^2. \tag{3.1}$$

Lemma 3.1. *The inclusion*

$$E_1(J \times O^2, R^n) \subset E(J \times O^2, R^n) \tag{3.2}$$

is valid.

Proof. Let $f \in E_1(J \times O^2, R^n)$, $K \subset O$ be an arbitrary compact. In order to prove the inclusion (3.2), it suffices to show the existence of such a function $L_{f,K}(\cdot) \in L_1(J, R_0^+)$ for which

$$|f(t, x_1', x_2') - f(t, x_1'', x_2'')| \le$$
$$\le L_{f,K}(t)\sum_{i=1}^{2}|x_i' - x_i''|, \quad \forall (t, x_1', x_2', x_1'', x_2'') \in J \times K^4.$$

Introduce the function $g(t, x_1, x_2)$ (see (1.9)). It is obvious with $(x_1, x_2) \notin K_1^2$

$$g_{x_i}(t, x_1, x_2) = 0, \quad i = 1, 2.$$

Thus there exists a function $m_{g,K_1}(\cdot) \in L_1(J, R_0^+)$ such that

$$\sum_{i=1}^{n}|g_{x_i}(t, x_1, x_2)| \le m_{g,K_1}(t), \quad \forall (t, x_1, x_2) \in J \times R^{2n}.$$

Let (x_1', x_2') and (x_1'', x_2'') be arbitrary points from K^2. Then (see (1.8)) we get

$$|f(t, x_1', x_2') - f(t, x_1'', x_2'')| = |g(t, x_1', x_2') - g(t, x_1'', x_2'')| =$$
$$= \left|\int_0^1 \frac{d}{ds} g(t, x_1'' + s(x_1' - x_1''), x_2'' + s(x_2' - x_2''))ds\right| \le$$
$$\le \int_0^1 \left[\sum_{i=1}^{2}|g_{x_i}(t, x_1'' + s(x_1' - x_1''), x_2'' + s(x_2' - x_2''))||x_i' - x_i''|\right] ds \le$$
$$\le m_{g,K_1}(t)\sum_{i=1}^{2}|x_i' - x_i''|.$$

Thus as $L_{f,K}(t)$ we can take $m_{g,K_1}(t)$. □

Now we consider the linear differential equation with delayed argument

$$\dot{x}(t) = A(t)x(t) + B(t)x(\tau(t)) + f(t), \quad t \in [t_0, b], \tag{3.3}$$
$$x(t) = \varphi(t), \; t \in [\tau(t_0), t_0), \quad x(t_0) = x_0, \tag{3.4}$$

where $A(t)$, $B(t)$ are summable $n \times n$ matrix functions, $f : J \to R^n$ is a summable function, $\varphi \in \Delta(J_1, R^n)$, $t_0 \in [a, b)$, $x_0 \in R^n$.

Lemma 3.2 (Cauchy's formula). *The solution $x(t)$, $t \in [t_0, b]$ of the equation (3.3) with the initial condition (3.4) can be represented in the form*

$$x(t) = X(t_0, t)x_0 + \int_{\tau(t_0)}^{t_0} Y(\gamma(s), t)B(\gamma(s))\varphi(s)\dot{\gamma}(s)ds + \int_{t_0}^{t} Y(s, t)f(s)ds, \tag{3.5}$$

where $Y(s,t)$ is the matrix function satisfying the equation

$$\frac{\partial Y(s,t)}{\partial s} = -Y(s,t)A(s) - B(\gamma(s))Y(\gamma(s),t)\dot{\gamma}(s), \quad s \in [a,t], \quad (3.6)$$

and the condition

$$Y(s,t) = \begin{cases} E, & s = t, \\ \Theta, & s > t. \end{cases} \quad (3.7)$$

Here E is the identity matrix, Θ is the zero matrix.

This lemma is proved in a standard way.[5]

Lemma 3.3. *Let $\tilde{t}_1 \in (a,b]$, and $Y(s,t)$ be the solution of the equation (3.6) with the condition (3.7). Then for each $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for an arbitrary $t_1 \in J : |t_1 - \tilde{t}_1| \le \delta$ the inequality*

$$|Y(s,t_1) - Y(s,\tilde{t}_1)| \le \varepsilon, \quad \forall s \in [a,s_1], \;\; s_1 = \min\{t_1,\tilde{t}_1\}$$

is fulfilled.

This lemma is a simple corollary of a theorem analogous to Theorem 1.3, which is valid for equations with advanced argument.

Lemma 3.4. *The solution $Y(s,t)$ is continuous on the set*

$$\Pi = \{(s,t) : a \le s \le t, \; t \in J\}.$$

Proof. Let $(s,t) \in \Pi$ and $s < t$. Then there exists a number $\delta_1 > 0$ such that $s + \Delta s < \min\{t + \Delta t, t\}$ with $|\Delta s| \le \delta_1$, $|\Delta t| \le \delta_1$, i.e. $(s+\Delta s, t+\Delta t) \in \Pi$. On the basis of Lemma 3.3 for each $\varepsilon > 0$ there exists $\delta_2 \in (0,\delta_1)$ such that for an arbitrary Δs, Δt satisfying the conditions $|\Delta s| \le \delta_2$, $|\Delta t| \le \delta_2$,the inequality

$$|Y(s+\Delta s, t+\Delta t) - Y(s+\Delta s, t)| \le \varepsilon/2$$

is fulfilled.

On the other hand the function $Y(s,t)$ is continuous on $[a,t]$, i.e., there exists a number $\delta_3 \in (0,\delta_1)$ such that

$$|Y(s+\Delta s, t) - Y(s,t)| \le \varepsilon/2, \quad |\Delta s| \le \delta_3.$$

Consequently with $|\Delta s| \le \delta$, $|\Delta t| \le \delta$, $\delta = \min\{\delta_2,\delta_3\}$ we have

$$a(s,t,\Delta s,\Delta t) = |Y(s+\Delta s, t+\Delta t) - Y(s,t)| \le$$
$$\le |Y(s+\Delta s, t+\Delta t) - Y(s+\Delta s, t)| + |Y(s+\Delta s, t) - Y(s,t)| \le \varepsilon. \quad (3.8)$$

Let $s = t$ and the increments Δs, Δt are such that $(t+\Delta s, t+\Delta t) \in \Pi$, i.e., $\Delta s \le \Delta t$.

If $\Delta s \le 0$, then $t + \Delta s \le \min\{t, t+\Delta t\}$. Therefore the smallness of $a(t,t,\Delta s,\Delta t)$ for small Δs, Δt is proved analogouosly (see (3.8)).

[5] For various classes of linear differential equations with deviating argument representation formulas of solutions are given in [1], [4-6], [9], [10], [13], [14], [18], [21], [22].

If $\Delta s \geq 0$, then we will use the inequalites

$$a(t,t,\Delta s,\Delta t) \leq |Y(t+\Delta s,t+\Delta t) - Y(t,t+\Delta t)| + |Y(t,t+\Delta t) - Y(t,t)| =$$

$$= a(t,\Delta t,\Delta s) + a(t,\Delta t),$$

$$|Y(s,t)| \leq M = \text{const}, \quad (s,t) \in \Pi. \tag{3.9}$$

The inequality (3.9) will be proved later.

Now we estimate $a(t,\Delta t,\Delta s)$. We have (see (3.6)):

$$a(t,\Delta t,\Delta s) \leq \int\limits_t^{t+\Delta s} \left| \frac{\partial Y(s,t+\Delta t)}{\partial s} \right| ds \leq$$

$$\leq M \int\limits_t^{t+\Delta s} \big(|A(s)| + \chi(\gamma(s))\big) |B(\gamma(s))| \dot{\gamma}(s) ds,$$

where $\chi(s)$ is the characteristic function of the interval $[\tau(a),\tau(b)]$.

Hence it follows

$$\lim_{\substack{\Delta t \to 0 \\ \Delta s \to 0}} a(t,\Delta t,\Delta s) = 0.$$

The smallness of $a(t,\Delta t)$ for small Δt follows from Lemma 3.3. Thus the continuity of the function $Y(s,t)$ on Π is proved.

Prove now the inequality (3.9). From the equation (3.6) taking into account (3.7) we get:

$$|Y(s,t)| \leq |E| +$$

$$+ \int\limits_s^t \big(|A(\xi)||Y(\xi,t)| + |B(\gamma(\xi))||Y(\gamma(\xi,t))|\dot{\gamma}(\xi)\big) d\xi, \, s \in [a,t]. \tag{3.10}$$

We set

$$g(s,t) = \max_{\xi \in [s,t]} |Y(\xi,t)|, \quad g(s,t) = 0, \quad s > t.$$

The following inequalites are obvious:

$$|Y(s,t)| \leq g(s,t), \quad |Y(\gamma(s),t)| \leq g(\gamma(s),t) \leq g(s,t), \quad s \in [a,t].$$

From the inequality (3.10) we obtain (see (3.1))

$$g(s,t) \leq \sqrt{n} + \int\limits_a^t \big(|A(\xi)| + \chi(\gamma(\xi))|B(\gamma(\xi))|\dot{\gamma}(\xi)\big) g(\xi,t) d\xi.$$

For any fixed t the function $g(s,t)$ is continuous with respect to $s \in [a,t]$. Therefore by Gronwall's lemma we obtain

$$g(s,t) \leq \sqrt{n}\exp\left(\int_J (|A(\xi)| + \chi(\gamma(\xi))|B(\gamma(\xi))|\dot{\gamma}(\xi))\,d\xi\right) = M. \quad \Box$$

3.2. Theorems on differentiability of the solution.

Lemma 3.5. *Let $\tilde{x}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A$, defined on $[\tau(\tilde{t}_0), \tilde{t}_1] \subset (\tau(a), b)$. Let K_1 contain some neighborhood of the set $cl\tilde{\varphi}(J_1) \cup \tilde{x}([\tilde{t}_0, \tilde{t}_1])$. Then there exist numbers $\delta_2 > 0$, $\varepsilon_2 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in A$ there corresponds the solution $x(t, \tilde{\mu} + \varepsilon\delta\mu)$, defined on $[\tau(t_0), \tilde{t}_1 + \delta_2] \subset (\tau(a), b)$. Moreover*

$$x(t, \tilde{\mu} + \varepsilon\delta\mu) \in K_1, \quad t \in [\tau(t_0), \tilde{t}_1 + \delta_2]. \tag{3.11}$$

Proof. In Lemma 2.1 we assume that

$$r_1 = \tilde{t}_0, \quad r_2 = \tilde{t}_1, \quad \tilde{y}(t) = \tilde{x}(t). \tag{3.12}$$

Then there exist numbers $\delta_2 > 0$, $\varepsilon_2 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in A$ there corresponds the solution $y(t, \tilde{\mu} + \varepsilon\delta\mu)$, defined on $[\tilde{t}_1 - \delta_2, \tilde{t}_1 + \delta_2] \subset (\tau(a), b)$. Moreover,

$$\varphi(t) \in K_1, \;\; t \in J_1, \quad y(t, \tilde{\mu} + \varepsilon\delta\mu) \in K_1, \;\; t \in [\tilde{t}_1 - \delta_2, \tilde{t}_1 + \delta_2].$$

It is easy to see that

$$x(t, \tilde{\mu} + \varepsilon\delta\mu) = h(t_0, \varphi, y(\cdot, \tilde{\mu} + \varepsilon\delta\mu))(t) \in K_1, \;\; t \in [\tau(t_0), \tilde{t}_1 + \delta_2]. \quad \Box \tag{3.13}$$

Remark 3.1. Due to uniqueness, the solution $x(t, \tilde{\mu})$ on the interval $[\tau(\tilde{t}_0), \tilde{t}_1 + \delta_2]$ is a continuation of the solution $\tilde{x}(t)$. Therefore the trajectory $\tilde{x}(t)$ in the sequel is assumed to be defined on the whole interval $[\tau(\tilde{t}_0), \tilde{t}_1 + \delta_2]$.

By virtue of Lemma 3.5 and Remark 3.1, it can be defined

$$\Delta x(t) = \Delta x(t, \varepsilon\delta\mu) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau(a), s_1), \\ x(t, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{x}(t), & t \in [s_1, \tilde{t}_1 + \delta_2], \\ & s_1 = \min\{t_0, \tilde{t}_0\}. \end{cases} \tag{3.14}$$

It is obvious (see (2.4), (3.12), (3.13), (2.3)) that

$$\Delta x(t) = \Delta y(t), \quad t \in [s_2, \tilde{t}_1 + \delta_2], \quad s_2 = \max\{t_0, \tilde{t}_0\}, \tag{3.15}$$

$$\lim_{\varepsilon \to 0} \Delta y(t) = 0, \;\; \text{uniformly for } \; (t, \delta\mu) \in [\tilde{t}_0 - \delta_2, \tilde{t}_1 + \delta_2] \times V. \tag{3.16}$$

Theorem 3.1. *Let* $\tilde{f} \in E_1(J \times O^2, R^n)$[6], $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(\tilde{t}_1) > \tilde{t}_0$ *and there exist the finite limits*

$$\lim_{\omega \to \omega_0^-} \tilde{f}(\omega) = f_0^-, \quad \omega \in R_{\tilde{t}_0}^- \times O^2, \quad \omega_0^- = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0^-))), \tag{3.17}$$

$$\begin{gathered} \lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^-)} [\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f_1^-, \quad \omega_i \in R_{\gamma_0}^- \times O^2, \quad i = 1,2, \\ \omega_1^0 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{x}_0), \ \omega_2^- = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(\tilde{t}_0^-));^7 \ \lim_{t\to t_0^-} \dot{\gamma}(t) = \dot{\gamma}^-, t \in R_{\tilde{t}_0}^-. \end{gathered} \tag{3.18}$$

Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V^-$

$$\Delta x(t, \varepsilon\delta\mu) = \varepsilon\delta x(t, \delta\mu) + o(t, \varepsilon\delta\mu), \tag{3.19}$$

where

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\delta x_0 - \left\{Y(\tilde{t}_0, t)f_0^- + Y(\gamma_0, t)f_1^-\dot{\gamma}^-\right\}\delta t_0 + \alpha(t, \delta\mu), \tag{3.20}$$

$$\alpha(t, \delta\mu) = \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\delta\varphi(s)\dot{\gamma}(s)ds + \int_{\tilde{t}_0}^{t} Y(s, t)\delta f[s]ds, \tag{3.21}$$

$\tilde{f}_{x_2}[t] = \tilde{f}_{x_2}(t, \tilde{x}(t), \tilde{x}(\tau(t)))$, $\delta f[t] = \delta f(t, \tilde{x}(t), \tilde{x}(\tau(t)))$, $Y(s,t)$ *is a matrix function satisfying the equation*

$$\frac{\partial Y(s,t)}{\partial s} = -Y(s,t)\tilde{f}_{x_1}[s] - Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\dot{\gamma}(s), \quad s \in [\tilde{t}_0, t],$$

and the condition (3.7).

Proof. It is easy to see that $\tilde{x}(\gamma_0) = \tilde{y}(\gamma_0)$ (see (3.12)) and the assumptions of Lemma 2.3 are fulfilled. Therefore there exists a number $\bar{\varepsilon} \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \bar{\varepsilon}] \times V^-$ we have (see (2.7), (2.23), (3.15))

$$\max_{t \in [\tilde{t}_0, \tilde{t}_1 + \delta_2]} |\Delta x(t)| \le O(\varepsilon), \tag{3.22}$$

$$\Delta x(\tilde{t}_0) = \varepsilon[\delta x_0 - f_0^- \delta t_0] + o(\varepsilon\delta\mu). \tag{3.23}$$

Let numbers $\delta_3 \in (0, \delta_2]$ and $\varepsilon_3 \in (0, \bar{\varepsilon}]$ be so small that for each $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ is the relation

$$\tilde{t}_0 < \gamma(t_0) < \gamma_0 < \tilde{t}_1 - \delta_3 \tag{3.24}$$

is valid.

---

[6] In all theorems of this section, in the sequelit is assumed that $\tilde{f} \in E_1(J \times O^2, R^n)$.

[7] Since $\tilde{x}(\gamma_0) = \tilde{y}(\gamma_0)$ (see (3.12)), here and in the sequel we preserve notation used in §2.

The function $\Delta x(t)$, $t \in [\tau(\tilde{t}_0), \tilde{t}_1 + \delta_3]$ on the interval $[\tilde{t}_0, \tilde{t}_1 + \delta_2]$ satisfies the equation

$$\dot{\Delta} x(t) = \tilde{f}_{x_1}[t]\Delta x(t) + \tilde{f}_{x_2}[t]\Delta x(\tau(t)) + \varepsilon\delta f[t] + \sum_{i=1}^{2} R_i(t, \varepsilon\delta\mu), \tag{3.25}$$

where

$$R_1(t, \varepsilon\delta\mu) = \tilde{f}(t, \tilde{x}(t) + \Delta x(t), \tilde{x}(\tau(t)) + \Delta x(\tau(t))) - \tilde{f}[t] - \\ -\tilde{f}_{x_1}[t]\Delta x(t) - \tilde{f}_{x_2}[t]\Delta x(\tau(t)), \tag{3.26}$$

$$R_2(t, \varepsilon\delta\mu) = \varepsilon(\delta f(t, \tilde{x}(t) + \Delta x(t), \tilde{x}(\tau(t)) + \Delta x(\tau(t))) - \delta f[t]). \tag{3.27}$$

By means of the Cauchy formula (see Lemma 3.2) the solution of the equation (3.25) can be represented in the form

$$\Delta x(t) = Y(\tilde{t}_0, t)\Delta x(\tilde{t}_0) + \varepsilon \int_{\tilde{t}_0}^{t} Y(s, t)\delta f[s]ds + \sum_{i=0}^{2} h_i(t, \tilde{t}_0, \varepsilon\delta\mu), \tag{3.28}$$

$$t \in [\tilde{t}_0, \tilde{t}_1 + \delta_2],$$

where

$$h_0(t, \tilde{t}_0, \varepsilon\delta\mu) = \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta x(s)\dot{\gamma}(s)ds, \tag{3.29}$$

$$h_i(t, \tilde{t}_0, \varepsilon\delta\mu) = \int_{\tilde{t}_0}^{t} Y(s, t)R_i(s, \varepsilon\delta\mu)ds, \quad i = 1, 2. \tag{3.30}$$

It is obvious (see (3.23) and Lemma 3.4),

$$Y(\tilde{t}_0, t)\Delta x(\tilde{t}_0) = \varepsilon Y(\tilde{t}_0, t)[\delta x_0 - f_0^- \delta t_0] + o(t, \varepsilon\delta\mu). \tag{3.31}$$

Now we transform $h_0(t, \tilde{t}_0, \varepsilon\delta\mu)$. We have:

$$h_0(t, \tilde{t}_0, \varepsilon\delta\mu) = \varepsilon \int_{\tau(\tilde{t}_0)}^{t_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds +$$

$$+\int_{t_0}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta x(s)\dot{\gamma}(s)ds = \varepsilon \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds +$$

$$+ \int_{\gamma(t_0)}^{\gamma_0} Y(s, t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds + o(t, \varepsilon\delta\mu). \tag{3.32}$$

Owing to the relations (3.24), the expression $h_1(t, \tilde{t}_0, \varepsilon\delta\mu)$ with $[\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3]$ can be represented as

$$h_1(t, \tilde{t}_0, \varepsilon\delta\mu) = \int_{\tilde{t}_0}^{\gamma(t_0)} Y(s,t)R_1(s,\varepsilon\delta\mu)ds + \int_{\gamma(t_0)}^{\gamma_0} Y(s,t)R_1(s,\varepsilon\delta\mu)ds +$$

$$+\int_{\gamma_0}^{t} Y(s,t)R_1(s,\varepsilon\delta\mu)ds = \sum_{i=3}^{5} a_i(t,\varepsilon\delta\mu). \tag{3.33}$$

Now we estimate the first term of the expression (3.33). We have (see (3.26))

$$|a_3(t,\varepsilon\delta\mu)| \leq \|Y\| \int_{\tilde{t}_0}^{\gamma(t_0)} \left[|\tilde{f}(t, \tilde{x}(t) + \Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\delta\varphi(\tau(t))) - \right.$$

$$\left. -\tilde{f}(t, \tilde{x}(t), \tilde{\varphi}(\tau(t))) - \tilde{f}_{x_1}[t]\Delta x(t) - \varepsilon \tilde{f}_{x_2}[t]\delta\varphi(\tau(t))|\right] dt \leq$$

$$\leq \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1+\delta_3} \left| \int_0^1 \frac{d}{d\xi}\tilde{f}(t, \tilde{x}(t) + \xi\Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\xi\delta\varphi(\tau(t))) - \right.$$

$$\left. -\tilde{f}(t, \tilde{x}(t), \tilde{\varphi}(\tau(t))) - \tilde{f}_{x_1}[t]\Delta x(t) - \varepsilon \tilde{f}_{x_2}[t]\delta\varphi(\tau(t))\right| dt \leq$$

$$\leq \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1+\delta_3} \left[\int_0^1 (|\tilde{f}_{x_1}(t, \tilde{x}(t) + \xi\Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\xi\delta\varphi(\tau(t))) - \right.$$

$$-\tilde{f}_{x_1}[t]|\,|\Delta x(t)| + \varepsilon|\tilde{f}_{x_2}(t, \tilde{x}(t) + \xi\Delta x(t), \tilde{\varphi}(\tau(t)) +$$

$$\left. +\varepsilon\xi\delta\varphi(\tau(t))) - \tilde{f}_{x_2}[t]|\,|\delta\varphi(\tau(t))|)d\xi \right] dt \leq$$

$$\leq \|Y\|[O(\varepsilon)\sigma_1(\varepsilon\delta\mu) + \varepsilon\alpha_3\sigma_2(\varepsilon\delta\mu)], \tag{3.34}$$

where

$$\sigma_i(\varepsilon\delta\mu) = \int_{\tilde{t}_0}^{\tilde{t}_1+\delta_3} \left[\int_0^1 |\tilde{f}_{x_i}(t, \tilde{x}(t) + \xi\Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\xi\delta\varphi(\tau(t))) - \tilde{f}_{x_i}[t]|d\xi \right] dt,$$

$$i = 1, 2.$$

Since $\Delta x(t) \to 0$ as $\varepsilon \to 0$, $t \in [\tilde{t}_0, \tilde{t}_1 + \delta_2]$, by Lebesgue's theorem

$$\lim_{\varepsilon \to 0} \sigma_i(\varepsilon\delta\mu) = 0, \quad i = 1, 2, \quad \text{uniformly for} \quad \delta\mu \in V^-.$$

Thus $a_3(t, \varepsilon\delta\mu)$ has the order $o(t, \varepsilon\delta\mu)$.

Rewrite the second term of the expression (3.33) as

$$a_4(t,\varepsilon\delta\mu)=\sum_{i=3}^{4}\sigma_i(t,\varepsilon\delta\mu)-\int\limits_{\gamma(t_0)}^{\gamma_0}Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds,$$

where

$$\sigma_3(t,\varepsilon\delta\mu)=-\int\limits_{\gamma(t_0)}^{\gamma_0}Y(s,t)\tilde f_{x_1}[s]\Delta x(s)ds,$$

$$\sigma_4(t,\varepsilon\delta\mu)=\int\limits_{\gamma(t_0)}^{\gamma_0}Y(s,t)\big(\tilde f(s,\tilde x(s)+\Delta x(s),\tilde x(\tau(s))+\Delta x(\tau(s)))-\tilde f[s]\big)ds.$$

It is clear (see (3.22)) that $\sigma_3(t,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.
Next, write $\sigma_4(t,\varepsilon\delta\mu)$ as

$$\sigma_4(t,\varepsilon\delta\mu)=\int\limits_{\gamma(t_0)}^{\gamma_0}Y(s,t)\big(\tilde f(s,\tilde x(s)+\Delta x(s),\tilde x(\tau(s))+$$

$$+\Delta x(\tau(s)))-\tilde f[s]-f_1^-\big)ds+\int\limits_{\gamma(t_0)}^{\gamma_0}Y(s,t)f_1^-ds=\sum_{i=5}^{6}\sigma_i(t,\varepsilon\delta\mu).$$

It is obvious that if $s\in[\gamma(t_0),\gamma_0]$, then $\tau(s)\in[t_0,\tilde t_0]$. Therefore (see (3.14), (3.15)) with $s\in[\gamma(t_0),\gamma_0]$

$$\begin{aligned}&\tilde x(\tau(s))+\Delta x(\tau(s))=x(\tau(s),\tilde\mu+\varepsilon\delta\mu)=\\&=y(\tau(s),\tilde\mu+\varepsilon\delta\mu)=\tilde y(\tau(s))+\Delta y(\tau(s)).\end{aligned}$$

From this equality, taking into consideration (3.16), (3.22) and $\tilde y(\tilde t_0)=\tilde x_0$, we obtain

$$\lim_{\substack{\varepsilon\to0\\ s\in[\gamma(t_0),\gamma_0]}}(s,\tilde x(s)+\Delta x(s),\tilde x(\tau(s))+\Delta x(\tau(s)))=\lim_{s\to\gamma_0}(s,\tilde x(s),\tilde y(\tau(s)))=\omega_1^0.$$

It is easy to note that when $s\in[\gamma(t_0),\gamma_0]$, then $\tilde f[s]=\tilde f(s,\tilde x(s),\tilde\varphi(\tau(s)))$ and

$$\lim_{\substack{\varepsilon\to0\\ s\in[\gamma(t_0),\gamma_0]}}(s,\tilde x(s),\tilde\varphi(\tau(s)))=\omega_2^-.$$

Thus

$$\lim_{\varepsilon\to0}\sup_{s\in[\gamma(t_0),\gamma_0]}|\tilde f(s,\tilde x(s)+\Delta x(s),\tilde x(\tau(s))+\Delta x(\tau(s)))-\tilde f[s]-f_1^-|=0$$

uniformly for $\delta\mu\in V^-$.

The function $Y(s,t)$ is continuous on $[\gamma(t_0),\gamma_0]\times[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\subset\Pi$ (see (3.24), Lemma 3.4). Besides

$$\gamma_0-\gamma(t_0)=-\varepsilon\dot\gamma^-\,\delta t_0+o(t,\varepsilon\delta\mu).$$

Consequently $\sigma_5(t,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.

By the equality

$$\lim_{s\to\gamma_0^-}Y(s,t)=Y(\gamma_0,t)\quad\text{uniformly for}\quad t\in[\tilde t_1-\delta_2,\tilde t_1+\delta_2],$$

for $\sigma_6(t,\varepsilon\delta\mu)$ we get

$$\sigma_6(t,\varepsilon\delta\mu)=-\varepsilon Y(\gamma_0,t)f_1^-\dot\gamma^-\,\delta t_0+o(t,\varepsilon\delta\mu).$$

For the last term of the expression (3.33) analogously (see (3.34), (3.22)) we obtain

$$|a_5(t,\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon)\int_{\gamma_0}^{\tilde t_1+\delta_3}\Bigg[\int_0^1\{|\tilde f_{x_1}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta x(\tau(s)))-$$
$$-\tilde f_{x_1}[s]|+|\tilde f_{x_2}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta x(\tau(s)))-\tilde f_{x_2}[s]|\}d\xi\Bigg]ds\le$$
$$\le\|Y\|O(\varepsilon)\sigma_7(\varepsilon\delta\mu).$$

It is obvious that when $t\ge\gamma_0$, then $\tau(t)\ge\tilde t_0$; therefore by (3.22) we establish that

$$\lim_{\varepsilon\to0}\sigma_7(\varepsilon\delta\mu)=0\quad\text{uniformly for}\quad\delta\mu\in V^-.$$

Thus

$$a_5(t,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu).$$

From (3.33) on the basis of the relations obtained above we get

$$h_1(t,\tilde t_0,\varepsilon\delta\mu)=-\varepsilon Y(\gamma_0,t)f_1^-\dot\gamma^-\,\delta t_0-\int_{t_0}^{\tilde t_0}Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\Delta x(s)\dot\gamma(s)ds+$$
$$+o(t,\varepsilon\delta\mu).\tag{3.35}$$

Finally we estimate $h_2(t,\tilde t_0,\varepsilon\delta\mu)$. We have (see (3.27))

$$|h_2(t,\tilde t_0,\varepsilon\delta\mu)|\le\varepsilon\|Y\|\int_{\tilde t_0}^{\tilde t_1+\delta_3}\big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+\Delta x(\tau(t)))-\delta f[t]\big|dt=$$
$$=\varepsilon\|Y\|a_6(\varepsilon\delta\mu).$$

We represent $a_6(\varepsilon\delta\mu)$ as the sum of three addends $a_7(\varepsilon\delta\mu)$, $a_8(\varepsilon\delta\mu)$, $a_9(\varepsilon\delta\mu)$:

$$a_6(\varepsilon\delta\mu)=$$

$$= \int\limits_{\tilde t_0}^{\gamma(t_0)} \big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde\varphi(\tau(t))+\varepsilon\delta\varphi(\tau(t)))-\delta f(t,\tilde x(t),\tilde\varphi(\tau(t)))\big|dt+$$

$$+\int\limits_{\gamma(t_0)}^{\gamma_0} \big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+\Delta x(\tau(t)))-\delta f(t,\tilde x(t),\tilde\varphi(\tau(t)))\big|dt+$$

$$+\int\limits_{\gamma_0}^{\tilde t_1+\delta_3} \big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+\Delta x(\tau(t)))-\delta f[t]\big|dt.$$

According to (2.1), (3.11), (3.22), we get

$$a_7(\varepsilon\delta\mu)\le\sum_{i=1}^k\int\limits_{\tilde t_0}^{\tilde t_1+\delta_3} L_{\delta f_i,K_1}(t)\big(|\Delta x(t)|+\varepsilon|\delta\varphi(\tau(t))|\big)dt\le O(\varepsilon). \quad (3.36)$$

Next (see (3.11)),

$$a_8(\varepsilon\delta\mu)\le\sum_{i=1}^k\lambda_i\int\limits_{\gamma(t_0)}^{\gamma_0}\big|\delta f_i(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+$$

$$+\Delta x(\tau(t)))-\delta f_i(t,\tilde x(t),\tilde\varphi(\tau(t)))\big|dt\le 2\sum_{i=1}^k|\lambda_i|\int\limits_{\gamma(t_0)}^{\gamma_0} m_{\delta f_i,K_1}(t)dt.$$

Since $\gamma(t_0)\to\gamma_0$ as $\varepsilon\to 0$, we have

$$\lim_{\varepsilon\to 0}a_8(\varepsilon\delta\mu)=0 \quad \text{uniformly for} \quad \delta\mu\in V^-.$$

It is clear that

$$a_9(\varepsilon\delta\mu)\le\sum_{i=1}^k|\lambda_i|\int\limits_{\gamma(t_0)}^{\tilde t_1+\delta_3}\big|L_{\delta f_i,K_1}(t)\big(|\Delta x(t)|+|\Delta x(\tau(t))|\big)dt\le O(\varepsilon). \quad (3.37)$$

Using the estimates obtained above, we have

$$h_2(t,\tilde t_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu). \quad (3.38)$$

From (3.28) according to (3.31), (3.32), (3.35), (3.38) we obtain the desired formula (3.19), where $\delta x(t,\delta\mu)$ has the form (3.20). □

Theorem 3.2. *Let $\tau(\tilde t_0)<\tilde t_0$, $\tau(\tilde t_1)>\tilde t_0$ and there exist the finite limits*

$$\lim_{\omega\to\omega_0^+}\tilde f(\omega)=f_0^+,\ \ \omega\in R^+_{\tilde t_0}\times O^2,\ \ \omega_0^+=(\tilde t_0,\tilde x_0,\tilde\varphi(\tau(\tilde t_0^+))); \quad (3.39)$$

$$\lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^+)}[\tilde f(\omega_1)-\tilde f(\omega_2)]=f_1^+,\ \ \omega_i\in R^+_{\gamma_0}\times O^2,\ \ i=1,2,$$

$$\omega_2^+ = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(\tilde{t}_0^+)); \lim_{t\to\tilde{t}_0} \dot{\gamma}(t) = \dot{\gamma}^+, \quad t\in R_{\tilde{t}_0}^+.$$

Then there exist numbers $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in [\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\times[0,\varepsilon_3]\times V^+$ *is valid* (3.19), *where*

$$\delta x(t,\delta\mu)=Y(\tilde{t}_0,t)\delta x_0-\{Y(\tilde{t}_0,t)f_0^+ + Y(\gamma_0,t)f_1^+\dot{\gamma}^+\}\delta t_0+\alpha(t,\delta\mu) \quad (3.40)$$

(*see* (3.21)).

Proof. By assumption of the theorem the conditions of Lemma 2.5 are fulfilled. Therefore (see (2.27), (2.37), (3.15)) there exists a number $\bar{\varepsilon}\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\bar{\varepsilon}]\times V^+$ it holds

$$\max_{t\in[t_0,\tilde{t}_1+\delta_2]}|\Delta x(t)|\le O(\varepsilon), \quad (3.41)$$

$$\Delta x(t_0)=\varepsilon[\delta x_0 - f_0^+\delta t_0]+o(\varepsilon\delta\mu). \quad (3.42)$$

Let the numbers $\delta_3\in(0,\delta_2]$, $\varepsilon_3\in(0,\bar{\varepsilon}]$ be so small that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V^+$

$$\tau(t_0)<\tilde{t}_0<\gamma_0<\gamma(t_0)<\tilde{t}_1-\delta_3. \quad (3.43)$$

The function $\Delta x(t)$, $t\in[\tau(t_0),\tilde{t}_1+\delta_3]$, on the interval $[t_0,\tilde{t}_1+\delta_3]$ satisfies the equation (3.25). Therefore by means of the Cauchy formula the expression $\Delta x(t)$ can be represented as

$$\Delta x(t)=Y(t_0,t)\Delta x(t_0)+\varepsilon\int_{t_0}^{t}Y(s,t)\delta f[s]ds+\sum_{i=0}^{2}h_i(t,t_0,\varepsilon\delta\mu), \quad (3.44)$$

$$t\in[t_0,\tilde{t}_1+\delta_3],$$

where $h_i(t,t_0,\varepsilon\delta\mu)$, $i=0,1,2$, have the form (3.29), (3.30) respectively.

Since $t_0\in[\tilde{t}_0,\tau(\tilde{t}_1-\delta_3)]$ (see (3.43)) and $Y(s,t)$ is continuous on $[\tilde{t}_0,\tau(\tilde{t}_1-\delta_3)]\times[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\subset\Pi$, we have

$$Y(t_0,t)\Delta x(t_0)=\varepsilon Y(\tilde{t}_0,t)[\delta x_0-f_0^+\delta t_0]+o(t,\varepsilon\delta\mu). \quad (3.45)$$

Now we transform $h_0(t,t_0,\varepsilon\delta\mu)$. We have (see (3.43)):

$$h_0(t,t_0,\varepsilon\delta\mu)=\varepsilon\int_{\tau(t_0)}^{\tilde{t}_0}Y(\gamma(s),t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds+$$

$$+\int_{\tilde{t}_0}^{t_0}Y(\gamma(s),t)\tilde{f}_{x_2}[\gamma(s)]\Delta x(s)\dot{\gamma}(s)ds=\varepsilon\int_{\tau(\tilde{t}_0)}^{\tilde{t}_0}Y(\gamma(s),t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds+$$

$$+\int_{\gamma_0}^{\gamma(t_0)}Y(s,t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds+o(t,\varepsilon\delta\mu). \quad (3.46)$$

By the relation (3.43) the exprssion $h_1(t,t_0,\varepsilon\delta\mu)$ with $t\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]$ may be represented as the sum of three addends $a_2(t,t_0,\varepsilon\delta\mu)$, $a_3(t,t_0,\varepsilon\delta\mu)$, $a_4(t,t_0,\varepsilon\delta\mu)$:

$$h_1(t,t_0,\varepsilon\delta\mu)=\int_{t_0}^{\gamma_0}Y(s,t)R_1(s,\varepsilon\delta\mu)ds+$$

$$+\int_{\gamma_0}^{\gamma(t_0)}Y(s,t)R_1(s,\varepsilon\delta\mu)ds+\int_{\gamma(t_0)}^{t}Y(s,t)R_1(s,\varepsilon\delta\mu)ds. \tag{3.47}$$

The first addend of the expression (3.47) is estimated analogously (see (3.34)).

Consequently,

$$|a_2(t,t_0,\varepsilon\delta\mu)|\le\|Y\|\big[O(\varepsilon)\sigma_1(t_0,\varepsilon\delta\mu)+\varepsilon\alpha_3\sigma_2(t_0,\varepsilon\delta\mu)\big], \tag{3.48}$$

where

$$\sigma_i(t_0,\varepsilon\delta\mu)=\int_{t_0}^{\tilde{t}_1+\delta_3}\Bigg[\int_0^1|\tilde{f}_{x_i}(t,\tilde{x}(t)+\xi\Delta x(t),\tilde{\varphi}(\tau(t))+\varepsilon\xi\delta\varphi(\tau(t)))-$$

$$-\tilde{f}_{x_i}[t]|d\xi\Bigg]dt,\quad i=1,2. \tag{3.49}$$

It is obvious (see (3.15)) that

$$\Delta x(t)=\Delta y(t),\quad t\in[t_0,\tilde{t}_1+\delta_3].$$

In (3.49) we change under the integral the function $\Delta x(t)$ by $\Delta y(t)$, which allow us to write the following inequality

$$\sigma_i(t_0,\varepsilon\delta\mu)\le\int_{\tilde{t}_0}^{\tilde{t}_1+\delta_3}\Bigg[\int_0^1|\tilde{f}_{x_i}(t,\tilde{x}(t)+\xi\Delta y(t),\tilde{\varphi}(\tau(t))+\varepsilon\xi\delta\varphi(\tau(t)))-\tilde{f}_{x_i}[t]|d\xi\Bigg]dt,$$

$$i=1,2.$$

Hence, using (3.16), it follows

$$\lim_{\varepsilon\to0}\sigma_i(t_0,\varepsilon\delta\mu)=0,\quad i=1,2,\quad\text{for}\quad\delta\mu\in V^+.$$

Thus $a_2(t,t_0,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.

The second addend of the expression (3.47) may be represented as

$$a_3(t,t_0,\varepsilon\delta\mu)=\sum_{i=3}^{4}\sigma_i(t,t_0,\varepsilon\delta\mu)-\int_{t_0}^{\gamma(t_0)}Y(s,t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds,$$

where

$$\sigma_3(t,t_0,\varepsilon\delta\mu) = -\int\limits_{\gamma_0}^{\gamma(t_0)} Y(s,t)\tilde{f}_{x_1}[s]\Delta x(s)ds,$$

$$\sigma_4(t,t_0,\varepsilon\delta\mu) = \int\limits_{\gamma_0}^{\gamma(t_0)} Y(s,t)\big(\tilde{f}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) - \tilde{f}[s]\big)ds.$$

It is clear (see (3.41)) that $\sigma_3(t,t_0,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.
Further, rewrite $\sigma_4(t,t_0,\varepsilon\delta\mu)$ as

$$\sigma_4(t,t_0,\varepsilon\delta\mu) = \int\limits_{\gamma_0}^{\gamma(t_0)} Y(s,t)\big(\tilde{f}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) - \tilde{f}[s] + f_1^+\big)ds -$$
$$- \int\limits_{\gamma_0}^{\gamma(t_0)} Y(s,t)f_1^+ ds = \sum_{i=5}^{6}\sigma_i(t,t_0,\varepsilon\delta\mu).$$

It is obvious that if $s\in[\gamma_0,\gamma(t_0)]$, then $\tau(s)\in[\tilde{t}_0,t_0]$. Therefore

$$\lim_{\substack{\varepsilon\to 0\\ s\in[\gamma_0,\gamma(t_0)]}} (s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) = \lim_{s\to\gamma_0^+}(s,\tilde{x}(s),\tilde{\varphi}(\tau(s))) = \omega_2^+,$$

$$\lim_{\substack{\varepsilon\to 0\\ s\in[\gamma_0,\gamma(t_0)]}} (s,\tilde{x}(s),\tilde{x}(\tau(s))) = \omega_1^0.$$

Thus

$$\lim_{\varepsilon\to 0}\sup_{s\in[\gamma_0,\gamma(t_0)]} |\tilde{f}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) - \tilde{f}[s] + f_1^+| = 0$$

uniformly for $\delta\mu\in V^+$.

The function $Y(s,t)$ is continuous on $[\gamma_0,\gamma(t_0)]\times[\tilde{t}_1-\tilde{t}_3,\tilde{t}_1+\delta_3]\subset\Pi$ (see (3.43), Lemma 3.4). Besides

$$\gamma(t_0)-\gamma_0 = \varepsilon\dot{\gamma}^+\delta t_0 + o(\varepsilon\delta\mu).$$

Consequently $\sigma_5(t,t_0,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.
The equality

$$\lim_{s\to\gamma_0^+} Y(s,t) = Y(\gamma_0,t) \quad \text{uniformly for} \quad t\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]$$

allows us to write for $\sigma_6(t,t_0,\varepsilon\delta\mu)$ the relaiton

$$\sigma_6(t,t_0,\varepsilon\delta\mu) = -\varepsilon Y(\gamma_0,t)f_1^+\dot{\gamma}^+\delta t_0 + o(t,\varepsilon\delta\mu).$$

For the last addend of the expression (3.47) in a simalar way (see (3.34), (3.41)) we obtain

$$|a_4(t,t_0,\varepsilon\delta\mu)|\le$$

$$\le\|Y\|O(\varepsilon)\int\limits_{\gamma(t_0)}^{\tilde t_1+\delta_3}\Big[\int\limits_0^1\{|\tilde f_{x_1}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta x(\tau(s)))-$$

$$-\tilde f_{x_1}[s]|+|\tilde f_{x_2}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta x(\tau(s)))-\tilde f_{x_2}[s]|\}d\xi\Big]ds\le$$

$$\le\|Y\|O(\varepsilon)\sigma_7(t_0,\varepsilon\delta\mu).$$

It is obvious that

$$\Delta x(\tau(t))=\Delta y(\tau(t)),\quad t\in[\gamma(t_0),\tilde t_1+\delta_3].$$

Consequently the inequality

$$\sigma_7(t_0,\varepsilon\delta\mu)\le\int\limits_{\gamma_0}^{\tilde t_1+\delta_3}\Big[\int\limits_0^1\{|\tilde f_{x_1}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta y(\tau(s)))-$$

$$-\tilde f_{x_1}[s]|+|\tilde f_{x_2}(s,\tilde x(s)+\xi\Delta x(s),\tilde x(\tau(s))+\xi\Delta y(\tau(s)))-\tilde f_{x_2}[s]|\}d\xi\Big]ds$$

is valid. From this inequality, taking into account (3.16), we establish that

$$\lim_{\varepsilon\to 0}\sigma_7(\varepsilon\delta\mu)=0\quad\text{uniformly for}\quad\delta\mu\in V^+.$$

Thus $a_4(t,t_0,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$.

Owing to obtained relations for $h_1(t,t_0,\varepsilon\delta\mu)$, we get

$$h_1(t,t_0,\varepsilon\delta\mu)=-\varepsilon Y(\gamma_0,t)f_1^+\dot\gamma^+\delta t_0-$$

$$-\int\limits_{\tilde t_0}^{t_0}Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\Delta x(s)\dot\gamma(s)ds+o(t,\varepsilon\delta\mu).\qquad(3.50)$$

Estimate now $h_2(t,t_0,\varepsilon\delta\mu)$. We have:

$$|h_2(t,t_0,\varepsilon\delta\mu)|\le\varepsilon\|Y\|\int\limits_{t_0}^{\tilde t_1+\delta_3}\big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+\Delta x(\tau(t)))-$$

$$-\delta f[t]\big|dt=\varepsilon\|Y\|a_5(t_0,\varepsilon\delta\mu).\qquad(3.51)$$

We represent $a_5(t_0,\varepsilon\delta\mu)$ in the form of three relations

$$a_5(t_0,\varepsilon\delta\mu)=$$

$$= \int_{t_0}^{\gamma_0} \big|\delta f(t, \tilde{x}(t) + \Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\delta\varphi(\tau(t))) - \delta f(t, \tilde{x}(t), \tilde{\varphi}(\tau(t)))\big| dt +$$

$$+ \int_{\gamma_0}^{\gamma(t_0)} \big|\delta f(t, \tilde{x}(t) + \Delta x(t), \varphi(\tau(t))) - \delta f[t]\big| dt +$$

$$+ \int_{\gamma(t_0)}^{\tilde{t}_1+\delta_3} \big|\delta f(t, \tilde{x}(t) + \Delta x(t), \tilde{x}(\tau(t)) + \Delta x(\tau(t))) - \delta f[t]\big| dt = \sum_{i=6}^{8} a_i(t_0, \varepsilon\delta\mu).$$

The estimate of these relations according to (2.1), (3.11), (3.41) yields

$$a_6(t_0, \varepsilon\delta\mu) \le \sum_{i=1}^{k} \int_{t_0}^{\tilde{t}_1+\delta_3} L_{\delta f_i, K_1}(t)\big(|\Delta x(t)| + \varepsilon|\delta\varphi(\tau(t))|\big) dt \le O(\varepsilon).$$

Further (see (3.11)),

$$a_7(t_0, \varepsilon\delta\mu) \le \sum_{i=1}^{k} |\lambda_i| \int_{\gamma_0}^{\gamma(t_0)} \big|\delta f_i(t, \tilde{x}(t) + \Delta x(t), \varphi(\tau(t))) - \delta f_i[t]\big| dt \le$$

$$\le 2\sum_{i=1}^{k} |\lambda_i| \int_{\gamma_0}^{\gamma(t_0)} m_{\delta f_i, K_1}(t) dt. \tag{3.52}$$

Since $\gamma(t_0) \to \gamma_0$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon \to 0} a_7(t_0, \varepsilon\delta\mu) = 0 \quad \text{uniformly for} \quad \delta\mu \in V^+. \tag{3.53}$$

It is obvious that

$$a_8(t_0, \varepsilon\delta\mu) \le \sum_{i=1}^{k} \lambda_i \int_{\gamma(t_0)}^{\tilde{t}_1+\delta_3} \big|L_{\delta f_i, K_1}(t)\big(|\Delta x(t)| + |\Delta x(\tau(t))|\big) dt \le O(\varepsilon). \tag{3.54}$$

The estimates obtained above allow us to conclude that

$$h_2(t, t_0, \varepsilon\delta\mu) = o(t, \varepsilon\delta\mu). \tag{3.55}$$

Finally we note that with $t \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3]$

$$\varepsilon \int_{t_0}^{t} Y(s, t)\delta f[s] ds = \varepsilon \int_{\tilde{t}_0}^{t} Y(s, t)\delta f[s] ds + o(t, \varepsilon\delta\mu). \tag{3.56}$$

From (3.44) taking into account (3.45), (3.46), (3.50), (3.55) and (3.56), we obtain (3.19), where $\delta x(t, \delta\mu)$ has the form (3.40). $\square$

Theorem 3.3. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(\tilde{t}_1) > \tilde{t}_0$; the function $(\tilde{\varphi}(t), \tilde{\varphi}(\tau(t)), \dot{\tau}(t))$ is continuous at the point $\tilde{t}_0$, while the function $\tilde{f}(\omega)$ is continuous at the points $\omega_0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0)))$, ω_1^0, $\omega_2^0 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(\tilde{t}_0))$. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V$ the relation* (3.19) *is fulfilled , where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\delta x_0 -$$
$$-\big\{Y(\tilde{t}_0, t)\tilde{f}(\omega_0) + Y(\gamma_0, t)[\tilde{f}(\omega_1^0) - \tilde{f}(\omega_2^0)]\dot{\gamma}(tdt_0)\big\}\delta t_0 + \alpha(t, \delta\mu).$$

It is not difficult to note that Theorem 3.3 is a simple corollary of Theorems 3.1, 3.2.

Theorem 3.4. *Let $\tau(\tilde{t}_1) < \tilde{t}_0$ and the condition* (3.17) *is fulfilled. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V^-$ the relation* (3.19) *is fulfilled, where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\big[\delta x_0 - f_0^- \delta t_0\big] + \alpha(t, \delta\mu). \tag{3.57}$$

Proof. From Lemma 2.7 it follows the existence of the numbers $\bar{\varepsilon}_3 \in (0, \varepsilon_2]$, $\bar{\delta}_3 \in (0, \delta_2]$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \bar{\delta}_3, \tilde{t}_1 + \bar{\delta}_3] \times [0, \bar{\varepsilon}_3] \times V^-$ (3.23) and

$$\max_{t \in [\tilde{t}_0, \tilde{t}_1 + \bar{\delta}_3]} |\Delta x(t)| \le O(\varepsilon) \tag{3.58}$$

are fulfilled.

Let $\delta_3 \in (0, \bar{\delta}_3]$, $\varepsilon_3 \in (0, \bar{\varepsilon}_3]$ be so small that

$$\gamma(t_0) > \tilde{t}_1 + \delta_3, \quad \tilde{t}_0 < \tilde{t}_1 - \delta_3. \tag{3.59}$$

It is clear that if $(s, t) \in [t_0, \tilde{t}_0] \times [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3]$, then $\gamma(s) > t$; therefore

$$Y(\gamma(s), t) = 0.$$

Consequently (see (3.31)) we get

$$h_0(t, \tilde{t}_0, \varepsilon\delta\mu) = \varepsilon \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds + o(t, \varepsilon\delta\mu).$$

In the case under consideration it is easy to note (see (3.30)) that

$$|h_1(t, \tilde{t}_0, \varepsilon\delta\mu)| \le \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1 + \delta_3} \Big[\big|\tilde{f}(t, \tilde{x}(t) + \Delta x(t), \tilde{\varphi}(\tau(t)) + \varepsilon\delta\varphi(\tau(t))) -$$
$$-\tilde{f}(t, \tilde{x}(t), \tilde{\varphi}(\tau(t))) - \tilde{f}_{x_1}[t]\Delta x(t) - \varepsilon\tilde{f}_{x_2}[t]\delta\varphi(\tau(t))\big|\Big] dt,$$
$$|h_2(t, \tilde{t}_0, \varepsilon\delta\mu)| \le$$

$$\le \varepsilon \|Y\| \int\limits_{\tilde{t}_0}^{\tilde{t}_1+\delta_3} \big|\delta f(t,\tilde{x}(t)+\Delta x(t),\tilde{\varphi}(\tau(t))+\varepsilon\delta\varphi(\tau(t)))-\delta f(t,\tilde{x}(t),\tilde{\varphi}(\tau(t)))\big|dt.$$

In a similar way (see (3.34), (3.36)) using (3.58) we establish that

$$h_i(t,\tilde{t}_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu), \quad i=1,2.$$

By (3.23) and (3.59), the relation (3.31) is fulfilled .

From (3.28), taking into account the relations obtained above, we have (3.19), where $\delta x(t,\delta\mu)$ has the form (3.57). □

Theorem 3.5. *Let $\tau(\tilde{t}_1)<\tilde{t}_0$ and the condition* (3.39) *be fulfilled. Then there exist numbers $\delta_3>0$, $\varepsilon_3>0$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\times[0,\varepsilon_3]\times V^+$ the relation* (3.19) *is valid, where*

$$\delta x(t,\delta\mu)=Y(\tilde{t}_0,t)\big[\delta x_0-f_0^+\delta t_0\big]+\alpha(t,\delta\mu). \tag{3.60}$$

Proof. By Lemma 2.9 there exist numbers $\bar{\varepsilon}_3\in(0,\varepsilon_2]$, $\bar{\delta}_3\in(0,\delta_2]$ such that (3.42) are fulfilled and

$$\max_{t\in[t_0,\tilde{t}_1+\bar{\delta}_3]}|\Delta x(t)|\le O(\varepsilon). \tag{3.61}$$

Let the numbers $\delta_3\in(0,\bar{\delta}_3]$, $\varepsilon_3\in(0,\bar{\varepsilon}_3]$ be so small that

$$\gamma_0>\tilde{t}_1+\delta_3, \quad t_0<\tilde{t}_1-\delta_3. \tag{3.62}$$

It is clear that if $(s,t)\in[\tilde{t}_0,t_0]\times[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]$, then $\gamma(s)>t$; therefore

$$Y(\gamma(s),t)=0.$$

Consequently (see (3.46)) we get

$$h_0(t,t_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s),t)\tilde{f}_{x_2}[\gamma(s)]\Delta\varphi(s)\dot{\gamma}(s)ds+o(t,\varepsilon\delta\mu).$$

In the case under consideration it is easy to see (see (3.30)) that

$$|h_1(t,t_0,\varepsilon\delta\mu)|\le\|Y\|\int\limits_{t_0}^{\tilde{t}_1+\delta_3}\Big[\big|\tilde{f}(t,\tilde{x}(t)+\Delta x(t),\tilde{\varphi}(\tau(t))+\varepsilon\delta\varphi(\tau(t)))-$$

$$-\tilde{f}(t,\tilde{x}(t),\tilde{\varphi}(\tau(t)))-\tilde{f}_{x_1}[t]\Delta x(t)-\varepsilon\tilde{f}_{x_2}[t]\delta\varphi(\tau(t))\big|\Big]dt,$$

$$|h_2(t,t_0,\varepsilon\delta\mu)|\le$$

$$\le \varepsilon \|Y\| \int\limits_{t_0}^{\tilde{t}_1+\delta_3} \big|\delta f(t,\tilde{x}(t)+\Delta x(t),\tilde{\varphi}(\tau(t))+\varepsilon\delta\varphi(\tau(t)))-\delta f(t,\tilde{x}(t),\tilde{\varphi}(\tau(t)))\big|dt.$$

Analogously, using (3.61) and (3.16) (see the proof of Theorem 3.2) it can be proved that

$$h_i(t, t_0, \varepsilon\delta\mu) = o(t, \varepsilon\delta\mu), \quad i = 1, 2.$$

It is obvious that by virtue of (3.42) and (3.62) the relation (3.45) is fulfilled.

From (3.44), taking into account the relations obtained above, we have (3.19), where $\delta x(t, \delta\mu)$ has the form (3.60). □

Theorem 3.6. *Let $\tau(\tilde{t}_1) < \tilde{t}_0$, the function $\tilde{\varphi}(\tau(t))$ be continuous at the point $\tilde{t}_0$, and the function $\tilde{f}(\omega)$ be continuous at the point ω_0. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V$ the relation (3.19) is valid, where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\{\delta x_0 - \tilde{f}(\omega_0)\delta t_0\} + \alpha(t, \delta\mu).$$

This theorem is a simple corollary of Theorems of 3.4, 3.5.

Theorem 3.7. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and the assumptions of Lemma 2.11 be fulfilled. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V^-$ the relation (3.19) is fulfilled, where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\{\delta x_0 - [f_3^- + (f_2^- - f_3^-)\dot{\gamma}^-]\delta t_0\} + \alpha_1(t, \delta\mu), \quad (3.63)$$

$$\alpha_1(t, \delta\mu) = \int_{\tilde{t}_0}^{t} Y(s, t)\delta f[s]ds.$$

Proof. By Lemma 2.11 there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V^-$ the relation (3.22) is fulfilled and

$$\Delta x(\tilde{t}_0) = \varepsilon\{\delta x_0 - [f_3^- + (f_2^- - f_3^-)\dot{\gamma}^-]\delta t_0\} + o(t, \varepsilon\delta\mu).$$

Let the number $\delta_3 \in (0, \delta_2]$ be so small that

$$\tilde{t}_0 < \tilde{t}_1 - \delta_3.$$

The function $Y(s, t)$, $(s, t) \in [t_0, \tilde{t}_0] \times [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3]$ is continuous. Therefore

$$Y(\tilde{t}_0, t)\Delta x(\tilde{t}_0) = \varepsilon Y(\tilde{t}_0, t)\{\delta x_0 - [f_3^- + (f_2^- - f_3^-)\dot{\gamma}^-]\delta t_0\} + o(t, \varepsilon\delta\mu).$$

It is clear that

$$h_0(t, \tilde{t}_0, \varepsilon\delta\mu) = 0.$$

In the case under consideration we have

$$|h_1(t, \tilde{t}_0, \varepsilon\delta\mu)| \le \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1+\delta_3} \big[\big|\tilde{f}(t, \tilde{x}(t) + \Delta x(t), \tilde{x}(\tau(t)) + \Delta x(\tau(t))) -$$
$$- \tilde{f}[t] - \tilde{f}_{x_1}[t]\Delta x(t) - \tilde{f}_{x_2}[t]\Delta x(\tau(t))\big|\big]\, dt,$$

$$|h_2(t,\tilde t_0,\varepsilon\delta\mu)|\le$$

$$\le\varepsilon\|Y\|\int\limits_{\tilde t_0}^{\tilde t_1+\delta_3}\big|\delta f(t,\tilde x(t)+\Delta x(t),\tilde x(\tau(t))+\Delta x(\tau(t)))-\delta f[t]\big|dt.$$

From this inequalites, using (3.22) analogously (see (3.34),(3.37)) can be obtained that

$$h_i(t,\tilde t_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu),\quad i=1,2.$$

On the basis of the obtained relations, from (3.28) it immediately follows (3.19), where $\delta x(t,\delta\mu)$ has the form (3.63). □

Theorem 3.8. *Let $\tau(\tilde t_0)=\tilde t_0$ and there exist the finite limits $\lim\limits_{\omega\to\omega_3}\tilde f(\omega)=f_2^+$, $\lim\limits_{\omega\to\omega_4^+}\tilde f(\omega)=f_3^+$, $\omega\in R_{\tilde t_0}^+\times O^2$, $\omega_4^+=(\tilde t_0,\tilde x_0,\tilde\varphi(\tilde t_0^+))$; $\lim\limits_{t\to t_0}\dot\gamma(t)=\dot\gamma^+$, $t\in R^+\tilde t_0$. Then there exist numbers $\delta_3>0$, $\varepsilon_3>0$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V^+$ the relation (3.19) is fulfilled, where*

$$\delta x(t,\delta\mu)=Y(\tilde t_0,t)\big\{\delta x_0-[f_3^-+(f_2^--f_3^-)\dot\gamma^-]\delta t_0\big\}+\alpha_1(t,\delta\mu).\quad(3.64)$$

Proof. By Lemma 2.14 there exists a number $\bar\varepsilon\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\bar\varepsilon]\times V^+$ the relation (3.41) is fulfilled and

$$\Delta x(t_0)=\varepsilon[\delta x_0-f_2^+\delta t_0]+o(t,\varepsilon\delta\mu).\quad(3.65)$$

Let the numbers $\varepsilon_3\in(0,\bar\varepsilon]$, $\delta_3\in(0,\delta_2]$ be so small that

$$\gamma(t_0)<\tilde t_1-\delta_3.$$

Obviously (see (3.29))

$$h_0(t,t_0,\varepsilon\delta\mu)=\int\limits_{\tau(t_0)}^{t_0}Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\Delta x(s)\dot\gamma(s)ds=$$

$$=\int\limits_{t_0}^{\gamma(t_0)}Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds.$$

Further, with $t\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]$

$$h_1(t,t_0,\varepsilon\delta\mu)=\int\limits_{t_0}^{\gamma(t_0)}Y(s,t)R_1(s,\varepsilon\delta\mu)ds+\int\limits_{\gamma(t_0)}^{t}Y(s,t)R_1(s,\varepsilon\delta\mu)ds=$$

$$=\sum_{i=5}^{6}a_i(t,t_0,\varepsilon\delta\mu).\quad(3.66)$$

Represent the first addend of the expression (3.66) as

$$a_5(t,t_0,\varepsilon\delta\mu) = -\int_{t_0}^{\gamma(t_0)} Y(s,t)\tilde{f}_{x_1}[s]\Delta x(s)ds +$$

$$+\int_{t_0}^{\gamma(t_0)} Y(s,t)\big(\tilde{f}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) - \tilde{f}[s]\big)ds -$$

$$-\int_{t_0}^{\gamma(t_0)} Y(s,t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds =$$

$$= \sum_{i=7}^{8}\sigma_i(t,t_0,\varepsilon\delta\mu) - \int_{t_0}^{\gamma(t_0)} Y(s,t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds.$$

It is obvious (see (3.41)) that $\sigma_7(t,t_0,\varepsilon\delta\mu)$ has the order $o(t,\varepsilon\delta\mu)$. Next, write $\sigma_8(t,t_0,\varepsilon\delta\mu)$ as

$$\sigma_8(t,t_0,\varepsilon\delta\mu) =$$

$$\int_{t_0}^{\gamma(t_0)} Y(s,t)\big(\tilde{f}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) - \tilde{f}[s] + f_2^+ - f_3^+\big)ds +$$

$$+\int_{t_0}^{\gamma(t_0)} Y(s,t)[f_3^+ - f_2^+]ds = \sum_{i=9}^{10}\sigma_i(t,t_0,\varepsilon\delta\mu).$$

It is clear that if $s\in[\gamma_0,\gamma(t_0)]$, then $\tau(s)\in[\tilde{t}_0,t_0]$. Therefore

$$\lim_{\substack{\varepsilon\to 0\\ s\in[t_0,\gamma(t_0)]}}(s,\tilde{x}(s)+\Delta x(s),\varphi(\tau(s))) = \lim_{s\to\tilde{t}_0^+}(s,\tilde{x}(s),\tilde{\varphi}(\tau(s))) = \omega_4^+,$$

$$\lim_{\substack{\varepsilon\to 0\\ s\in[t_0,\gamma(t_0)]}}(s,\tilde{x}(s),\tilde{x}(\tau(s))) = \omega_3.$$

On the basis of these equalities, using the relation

$$\gamma(t_0) - t_0 = \varepsilon(\dot{\gamma}^+ - 1)\delta t_0 + o(\varepsilon\delta\mu),$$

in a standard way we obtain

$$\sigma_9(t,t_0,\varepsilon\delta\mu) = (t,\varepsilon\delta\mu),$$

$$\sigma_{10}(t,t_0,\varepsilon\delta\mu) = -\varepsilon Y(\tilde{t}_0,t)[f_3^+ - f_2^+](\dot{\gamma}^+ - 1)\delta t_0 + o(\varepsilon\delta\mu).$$

The second addend $a_6(t,t_0,\varepsilon\delta\mu)$ of the expression (3.66) is estimated analogously to $a_4c(t,t_0,\varepsilon\delta\mu)$ (see the proof of Theorem 3.2), i.e.,

$$a_6(t,t_0,\varepsilon\delta\mu) = o(t,\varepsilon\delta\mu).$$

According to the obtained relations, in the case under consideration for $h_1(t,t_0,\varepsilon\delta\mu)$ we get

$$h_1(t,t_0,\varepsilon\delta\mu) = -\varepsilon Y(\tilde{t}_0,t)[f_3^+ - f_2^+](\dot{\gamma}^+ - 1)\delta t_0 - \\ - \int\limits_{t_0}^{\gamma(t_0)} Y(s,t)\tilde{f}_{x_2}[s]\Delta x(\tau(s))ds + o(\varepsilon\delta\mu).$$

It remains to estimate $h_2(t,t_0,\varepsilon\delta\mu)$. We have (see (3.51), (3.54)):

$$|h_2(t,t_0,\varepsilon\delta\mu)| \le \varepsilon\|Y\|\big(a_7(t_0,\varepsilon\delta\mu) + a_8(t_0,\varepsilon\delta\mu)\big) \le O(\varepsilon).$$

Finally note that the equalities (3.45), (3.56) (see (3.64)) are valid. From (3.44), taking into account the obtained relations, we have (3.19), where $\delta x(t,\delta\mu)$ has the form (3.64). □

Theorem 3.9. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and the conditions of Theorems 3.7, 3.8 be fulfilled. Moreover,*

$$f_3^- + (f_2^- - f_3^-)\dot{\gamma}^- = f_3^+ + (f_2^+ - f_3^+)\dot{\gamma}^+ = \hat{f}.$$

Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t,\varepsilon,\delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0,\varepsilon_3] \times V$ the relation (3.19) is valid, where

$$\delta x(t,\delta\mu) = Y(\tilde{t}_0,t)\big[\delta x_0 - \hat{f}\delta t_0\big] + \alpha_1(t,\delta\mu).$$

This theorem is a corollary of Theorems 3.7, 3.8.

Theorem 3.10. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and*

$$\lim_{\omega\to\omega_3} \tilde{f}(\omega) = f_2^-,\ \ \omega \in R_{\tilde{t}_0}^- \times O^2,\ \ \lim_{t\to t_0^-} \dot{\gamma}(t) = 1,\ \ t \in R_{\tilde{t}_0}^-.$$

Let, moreover, there exists a neighborhood $V^-(\omega_4^-)$, such that the function $\tilde{f}(\omega)$, $\omega \in V^-(\omega_4^-)$ is bounded. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t,\varepsilon,\delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0,\varepsilon_3] \times V^-$ the relation (3.19) is fulfilled , where

$$\delta x(t,\delta\mu) = Y(\tilde{t}_0,t)\big[\delta x_0 - f_2^-\delta t_0\big] + \alpha_1(t,\delta\mu).$$

This theorem, by Lemma 2.12, is proved analogously to Theorem 3.7.

Theorem 3.11. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and*

$$\lim_{\omega\to\omega_3} \tilde{f}(\omega) = f_2^+,\ \ \omega \in R_{\tilde{t}_0}^+ \times O^2,\ \ \lim_{t\to t_0^+} \dot{\gamma}(t) = 1,\ \ t \in R_{\tilde{t}_0}^+.$$

Let, moreover, there exist the neighborhood $V^+(\omega_4^+)$, such that the function $\tilde{f}(\omega)$, $\omega \in V^+(\omega_4^+)$ is bounded. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t,\varepsilon,\delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0,\varepsilon_3] \times V^+$ the relation (3.19) is valid, where

$$\delta x(t,\delta\mu) = Y(\tilde{t}_0,t)\big[\delta x_0 - f_2^+\delta t_0\big] + \alpha_1(t,\delta\mu).$$

This theorem, by Lemma 2.14, is proved analogously to Theorem 3.8. The difference consists in estimation of the expression $\sigma_8(t, t_0, \varepsilon\delta\mu)$. Namely, the integrand in $\sigma_8(t, t_0, \varepsilon\delta\mu)$ is bounded, while

$$\gamma(t_0) - t_0 = \gamma(\tilde{t}_0) + \varepsilon\dot{\gamma}^+\delta t_0 + o(\varepsilon\delta\mu) = o(\varepsilon\delta\mu) \quad (\dot{\gamma}^+ = 1).$$

This allows us to conclude that

$$\sigma_8(t, t_0, \varepsilon\delta\mu) = o(t, \varepsilon\delta\mu).$$

Theorem 3.12. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and the function $(\tilde{\varphi}(t), \dot{\tau}(t))$ be continuous at the point $\tilde{t}_0$. Let the function $\tilde{f}(\omega)$ be continuous at the point ω_3 and be bounded in the neighborhood of the point $\omega_4^0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tilde{t}_0))$. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V$ (3.19) is fulfilled, where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\big[\delta x_0 - \tilde{f}(\omega_3)\delta t_0\big] + \alpha_1(t, \delta\mu).$$

It is easy to note that $\dot{c\gamma}(\tilde{t}_0) = 1$. Consequently this theorem is a corollary of Theorems 3.10, 3.11.

Theorem 3.13. *Let $\tau(\tilde{t}_1) = \tilde{t}_0$ and the conditions (3.17), (3.18) are fulfilled. Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1, \tilde{t}_1] \times [0, \varepsilon_3] \times V^-$ the relation (3.19) is valid, where*

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\delta x_0 - \big[Y(\tilde{t}_0, t)f_0^- + Y(\tilde{t}_1, t)f_1^-\dot{\gamma}^-\big]\delta t_0 + \alpha(t, \delta\mu).$$

This theorem is proved analogously to Theorem 3.1.

Theorem 3.14. *Let $\tau(\tilde{t}_1) = \tilde{t}_0$ and*

$$\lim_{\omega \to \omega_0^+} \tilde{f}(\omega) = f_0^+.$$

Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ such that for an arbitrary $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V^+$ the relation (3.19) is fulfilled, where

$$\delta x(t, \delta\mu) = Y(\tilde{t}_0, t)\big[\delta x_0 - f_0^+\delta t_0\big] + \alpha(t, \delta\mu).$$

This theorm, taking into account $Y(s, t) = 0$, $(s, t) \in [\tilde{t}_1, \gamma(t_0)] \times [\tilde{t}_1 - \delta_3, \tilde{t}_1]$, is proved analogously to Theorem 3.2.

CHAPTER II
CONTINUOUS DEPENDENCE AND DIFFERENTIABILITY OF SOLUTION OF NEUTRAL DIFFERENTIAL EQUATIONS

4. Continuous Dependence of Solution

4.1. Preliminary Notes. Let $\eta : R^1 \to R^1$ be a continuously differentiable function satisfying $\eta(t) < t$, $\dot{\eta}(t) > 0$; let $\Delta_1(J_2, R^n)$ be the space of continuously differentiable functions $\varphi : J_2 \to R^n$, $J_2 = [\rho(a), b]$, $\rho(a) = \min\{\tau(a), \eta(a)\}$ with the norm $\|\varphi\|_1 = \sup\{|\varphi(a)| + |\dot{\varphi}(t)| : t \in J_2\}$; $C(J)$ be the space of measurable and bounded $n \times n$ matrix functions $C(t)$, $t \in J$, with the norm: $\|C\| = \sup\limits_{t \in J} |C(t)|$.

Consider the linear neutral differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(\tau(t)) + C(t)\dot{x}(\eta(t)) + f(t), \quad t \in [t_0, b], \tag{4.1}$$

$$x(t) = \varphi(t), \quad t \in [\rho(t_0), t_0), \qquad x(t_0) = x_0, \tag{4.2}$$

where $A(t)$, $B(t)$ are summable on J $n \times n$ matrix functions $C \in C(J)$, $f : J \to R^n$ is a summable function, $\varphi \in \Delta_1(J_2, R^n)$, $t_0 \in [a, b)$, $x_0 \in R^n$.

Lemma 4.1 (Cauchy's formula). *The solution $x(t)$, $t \in [t_0, b]$ of the equation (4.1) with the initial condition (4.2) may be represented as*

$$x(t) = \Phi(t_0, t)x_0 + \int_{\tau(t_0)}^{t_0} Y(\gamma(s), t)B(\gamma(s))\dot{\gamma}(s)\varphi(s)ds +$$

$$+ \int_{\eta(t_0)}^{t_0} Y(\sigma(s), t)C(\sigma(s))\dot{\sigma}(s)\dot{\varphi}(s)ds + \int_{t_0}^{t} Y(s, t)f(s)ds, \tag{4.3}$$

where $\Phi(s, t)$, $Y(s, t)$ are matrix functions satisfying the set of equations with advanced argument

$$\frac{\partial \Phi(s, t)}{\partial s} = -Y(s, t)A(s) - Y(\gamma(s), t)B(\gamma(s))\dot{\gamma}(s), \quad s \in [a, t], \tag{4.4}$$

$$Y(s, t) = \Phi(s, t) + Y(\sigma(s))C(\sigma(s), t)\dot{\sigma}(s), \quad s \in [a, t]. \tag{4.5}$$

Moreover, $Y(s, t)$ satisfies the condition

$$Y(s, t) = \begin{cases} E, & s = t, \\ \Theta, & s > t. \end{cases}$$

Here $\sigma(t)$ is the inverse function of $\eta(t)$.

This lemma is proved by a standard way (see the footnote 5).

It is easy to note that the equation (4.5) allows us to express $Y(s,t)$ by the function $\Phi(s,t)$. To this end on the set J^2 define the function $m(s,t)$ taking values from the set of non-negative integer numbers. Namely for $s > t$ put $m(s,t) = 0$, while for $s \le t$ let $m(s,t)$ be the natural number such that

$$s \in (\eta^{m(s,t)+1}(t), \eta^{m(s,t)}(t)].$$

Here as always

$$\eta^i(t) = \eta(\eta^{i-1}(t)), \quad i = 1, 2, \dots, \qquad \eta^0(t) = t.$$

It is clear that for any fixed $t \in J$ the function $m(s,t)$, $s \in [a,t]$ is piecewise continuous and for an arbitrary $(s,t) \in J^2$, we have $m(s,t) \in [0, m(a,b)]$.

Solving the equation (4.5) by the method of steps from right to left, we obtain

$$Y(s,t) = \sum_{i=0}^{m(s,t)} \Phi(\sigma^i(s), t) C_i[s], \qquad s \in [a,t], \tag{4.6}$$

where

$$C_i[s] = \prod\nolimits_{j=1}^{i} C(\sigma^j(s)) \dot{\sigma}(\sigma^{j-1}(s)); \tag{4.7}$$

it is assumed that $C_0[s] = E$.

Substituting (4.6) into (4.4), we obtain the equation with advanced arguments

$$\frac{\partial \Phi(s,t)}{\partial s} = -\sum_{i=0}^{m(s,t)} \Phi(\sigma^i(s), t) C_i[s] A(s) -$$

$$- \sum_{i=0}^{m(\gamma(s),t)} \Phi(\sigma^i(\gamma(s)), t) C_i[\gamma(s)] B(\gamma(s)) \dot{\gamma}(s), \quad s \in [a,t]. \tag{4.8}$$

It is obvious that (see (4.5))

$$\Phi(s,t) = \begin{cases} E, & s = t, \\ \Theta, & s > t. \end{cases} \tag{4.9}$$

Below some properties of the matrix functions $\Phi(s,t)$, $Y(s,t)$ are established which are used in proving theorems on differentiability of solution (see § 5).

Lemma 4.2. *Let $\tilde{t}_1 \in (a,b]$, while $\Phi(s,t)$ be a solution of the equation (4.8) with the condition (4.9). Then for an arbitrary $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for an arbitrary $t_1 \in J : |t_1 - \tilde{t}_1| \le \delta$ the inequality*

$$|\Phi(s,t_1) - \Phi(s,\tilde{t}_1)| \le \varepsilon, \quad \forall s \in [a, s_1], \ \ s_1 = \min\{t_1, \tilde{t}_1\},$$

is fulfilled.

This lemma is a simple corollary of a theorem analogous to Theorem 1.3, which is valid for equations with advanced arguments.

Lemma 4.3. *The function* $\Phi(s,t)$ *is continuous on the set*

$$\Pi = \{(s,t) : a \le s \le t,\ t \in J\}.$$

This lemma, using Lemma 4.2, with minor modifications can be proved analogously to Lemma 3.4.

Lemma 4.4. *Let* $\xi_0,\ \xi_1 \in (a,b),\quad \xi_0 < \eta^{m_0}(\xi_1),\quad m_0 = m(\xi_0,\xi_1)$ *and there exist the right and left finite limits of the function* $C(s)$ *at the points* $\sigma^i(\xi_0),\ i = 1,\dots,m_0$. *Then there exists a number* $\delta > 0$ *such that*

$$\lim_{s\to\xi_0^-} Y(s,t) = Y_{\xi_0}^-(t), \quad \lim_{s\to\xi_0^+} Y(s,t) = Y_{\xi_0}^+(t)$$

uniformly for $t \in [\xi_1 - \delta, \xi_1 + \delta]$,

where the functions $Y_{\xi_0}^-(t),\ Y_{\xi_0}^+(t)$ *are continuous on* $[\xi_1 - \delta, \xi_1 + \delta]$.

Proof. By properties of the function $\eta(t)$, there exist numbers $\delta,\ \bar\delta > 0$ such that for every $t \in [\xi_1 - \delta, \xi_1 + \delta] \subset J$ we have $m(\xi_0, t) = m(\xi_0,\xi_1) = m_0$ and

$$[\xi_0 - \bar\delta, \xi_0 + \bar\delta] \subset (\eta^{m_0+1}(t), \eta^{m_0}(t))$$

Moreover,

$$\sigma^{m_0}(\xi_0 + \bar\delta) < \xi_1 - \delta.$$

Thus with $(s,t) \in [\xi_0 - \bar\delta, \xi_0 + \bar\delta] \times [\xi_1 - \delta, \xi_1 + \delta] = \Pi_0$ we get

$$m(s,t) = m_0, \quad (\sigma^{m_0}(s), t) \in \Pi.$$

Consequently

$$Y(s,t) = \sum_{i=0}^{m_0} \Phi(\sigma^i(s), t) C_i[s] \quad \text{for} \quad (s,t) \in \Pi_0$$

and the functions (see Lemma 4.3)

$$\Phi(\sigma^i(s), t), \quad i = 1,\dots,m_0,$$

are continuous on Π_0.

By assumption the functions $C_i[s],\ i = 1,\dots,m_0$, have one-sided limits at the point ξ_0 (see (4.7)).

Thus

$$\lim_{s\to\xi_0^-} Y(s,t) = \sum_{i=0}^{m_0} \Phi(\sigma^i(\xi_0), t) C_i^- = Y_{\xi_0}^-(t),$$

$$\lim_{s\to\xi_0^+} Y(s,t) = \sum_{i=0}^{m_0} \Phi(\sigma^i(\xi_0), t) C_i^+ = Y_{\xi_0}^+(t),$$

where C_i^- and C_i^+ are the right and left limits of the function $C_i[s]$ at the point ξ_0.

From these equalites it follows the continuity of the functions $Y_{\xi_0}^-(t)$ and $Y_{\xi_0}^+(t)$, respectively, for $t \in [\xi_1 - \delta, \xi_1 + \delta]$. □

Lemma 4.5. *The solution $x(t)$, $t \in [t_0, b]$, of the equation*

$$\dot{x}(t) = C(t)\dot{x}(\eta(t)) + f(t), \quad t \in [t_0, b], \tag{4.10}$$

with the initial condition (4.2) *may be represented as*

$$x(t) = x_0 + \int_{\eta(t_0)}^{t_0} Y(\sigma(s), t)C(\sigma(s))\dot{\sigma}(s)\dot{\varphi}(s)ds + \int_{t_0}^{t} Y(s,t)f(s)ds, \tag{4.11}$$

where $Y(s,t)$ is a matrix function satisfying the equation

$$Y(s,t) = E + Y(\sigma(s))C(\sigma(s))\dot{\sigma}(s) \tag{4.12}$$

and having the following form

$$Y(s,t) = \sum_{i=0}^{m(s,t)} \chi(\sigma^i(s), t)C_i[s], \quad s \in [a, t], \tag{4.13}$$

$$\chi(s,t) = \begin{cases} 1, & s \le t, \\ 0, & s > t. \end{cases}$$

Proof. In the case under consideration the equation (4.4) has the form

$$\frac{\partial \Phi(s,t)}{\partial s} = 0, \quad s \in [a, t].$$

Consequently, taking into account (4.9), we obtain $\Phi(s,t) = E$, $\Phi(\sigma^i(s), t) = \Phi(\sigma^i(s), t)E$. On the basis of these equalities from (4.3), (4.5), (4.6) it follows (4.11), (4.12), (4.13), respectively. □

Lemma 4.6. *If the function $x(t)$, $t \in [\rho(t_0), b]$, has the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [\rho(t_0), t_0), \\ x_0 + \displaystyle\int_{\eta(t_0)}^{t_0} Y(\sigma(s), t)C(\sigma(s))\dot{\sigma}(s)\dot{\varphi}(s)ds + \int_{t_0}^{t} Y(s,t)f(s)ds, & t \in [t_0, b], \end{cases}$$

where $Y(s,t)$ is a solution of the equation (4.12) *(see* (4.13)*), then on the interval $[t_0, b]$ it satisfies the equation* (4.10).

Proof. Divide the interval $[t_0, b]$ into the subintervals $[\xi_i, \xi_{i+1}]$, $i = 0, \dots, l$, where $\xi_0 = t_0$, $\xi_i = \sigma^i(t_0)$, $i = 1, \dots, l$, $\xi_{l+1} = b$.

Let $t \in (\xi_0, \xi_1)$. Then $\eta(t) \in [\eta(t_0), t_0]$ and $m(s,t) = 1$ for $s \in [\eta(t_0), \eta(t)]$, $m(s,t) = 0$ for $s \in (\eta(t), t]$.

Thus (see (4.13))

$$Y(s,t) = \begin{cases} E + C_1[s], & s \in [\eta(t_0), \eta(t)], \\ E, & s \in (\eta(t), t]. \end{cases}$$

It is easy to see that (see (4.12), (4.7))

$$\int_{\eta(t_0)}^{t_0} Y(\sigma(s), t) C(\sigma(s)) \dot{\sigma}(s) \dot{\varphi}(s) ds = \int_{\eta(t_0)}^{t_0} \big(Y(s,t) - E\big) \dot{\varphi}(s) ds =$$

$$= \int_{\eta(t_0)}^{\eta(t)} C(\sigma(s)) \dot{\sigma}(s) \dot{\varphi}(s) ds, \int_{t_0}^{t} Y(s,t) f(s) ds = \int_{t_0}^{t} f(s) ds.$$

Consequently, for $t \in (\xi_0, \xi_1)$

$$x(t) = x_0 + \int_{\eta(t_0)}^{\eta(t)} C(\sigma(s)) \dot{\sigma}(s) \dot{\varphi}(s) ds + \int_{t_0}^{t} f(s) ds. \tag{4.14}$$

Thus, the function $x(t)$ is absolutely continuous on the interval $[\xi_0, \xi_1]$. From (4.14), we obtain

$$\dot{x}(t) = C(t) \dot{\varphi}(\eta(t)) + f(t) = C(t) \dot{x}(\eta(t)) + f(t).$$

Let $t \in (\xi_1, \xi_2)$. Then

$$Y(s,t) = \begin{cases} E + C_1[s] + C_2[s], & s \in [\eta(t_0), \eta^2(t)], \\ E + C_1[s], & s \in (\eta^2(t), \eta(t)], \\ E, & s \in (\eta(t), t]. \end{cases}$$

We have

$$x(t) = x_0 + \int_{\eta(t_0)}^{t_0} C_1[s] \dot{\varphi}(s) ds + \int_{\eta(t_0)}^{\eta^2(t)} C_2[s] \dot{\varphi}(s) ds +$$

$$+ \int_{t_0}^{t} f(s) ds + \int_{t_0}^{\eta(t)} C_1[s] f(s) ds.$$

It is obvious that $x(t)$, $t \in [\xi_1, \xi_2]$, is absolutely continuous and is a continuation of (4.14).

Find the derivative of the function $x(t)$:

$$\dot{x}(t) = C(t)C(\eta(t))\dot{\varphi}(\eta^2(t)) + C(t)f(\eta(t)) + f(t) =$$
$$= C(t)\big[C(\eta(t))\dot{\varphi}(\eta^2(t)) + f(\eta(t))\big] + f(t) = C(t)\dot{x}(\eta(t)) + f(t).$$

Continuing this process with respect to $i = 1, \dots, l-1$, we establish that $x(t)$ on $[t_0, b]$ is a solution of the equation (4.10). □

Lemma 4.7. *Let $\tilde{C}$, $C_k \in C(J)$, $k = 1, 2, \dots$, and $\tilde{Y}$, Y_k be the corresponding solutions (see (4.12), (4.13)), respectively. Then from the equality*

$$\lim_{k\to\infty} \|C_k - \tilde{C}\| = 0$$

it follows

$$\lim_{k\to\infty} \|Y_k - \tilde{Y}\| = 0,$$

where

$$\|Y_k - \tilde{Y}\| = \sup\{|Y_k(s,t) - \tilde{Y}(s,t)| : (s,t) \in J^2\}.$$

Lemma 4.8. *Let $\tilde{C}$, $C \in C(J)$, $C = \tilde{C} + \varepsilon\delta C$, $\varepsilon > 0$, $\|\delta C\| \le const$, $\tilde{Y}$, Y be the corresponding solutions (see (4.12), (4.13)). Then there exists a number α_1, not depending on δC, such that*

$$\|\tilde{Y} - Y\| \le \varepsilon\alpha_1. \tag{4.15}$$

On the basis of (4.12) the above lemmas are easily proved.

Lemma 4.9. *Let $C(s)$, $s \in J$, be a continuous matrix function, $K \subset O$ and a sequence $g_i \in E(J \times O, R^n)$, $i = 1, 2, \dots$, satisfy the conditions*

$$\int_J |g_i(t,x)|dt \le \alpha_2, \ \ \forall x \in K, \quad \lim_{i\to\infty} H_{g_i}(J, K) = 0. \tag{4.16}$$

Then

$$\lim_{i\to\infty} H_{Cg_i}(J, K) = 0. \tag{4.17}$$

Proof. There exists a sequence of continuously differentiable matrix functions $P_m(s)$, $s \in J$, such that

$$\lim_{m\to\infty} \|C - P_m\| = 0.$$

Further, we have

$$\left|\int_{t'}^{t''} C(s)g_i(s,x)ds\right| \le \int_J |P_m(s) - C(s)||g_i(s,x)|ds + \left|\int_{t'}^{t''} P_m(s)g_i(s,x)ds\right| \le$$
$$\le \alpha_2\|P_m - C\| + \alpha_{mi}(t', t'', x).$$

Integrating by parts yields

$$\alpha_{mi}(t',t'',x) = \left| \int\limits_{t'}^{t''} P_m(s)\Big(\frac{d}{ds}\int\limits_{t'}^{s} g_i(\xi,x)d\xi\Big)ds \right| \le$$

$$\le \left| P_m(t'')\int\limits_{t'}^{t''} g_i(s,x)ds - \int\limits_{t'}^{t''} \dot{P}_m(s)\Big(\int\limits_{t'}^{s} g_i(\xi,x)d\xi\Big)ds \right| \le$$

$$\le \Big(|P_m(t'')| + \int\limits_{J} |\dot{P}_m(s)|ds\Big) H_{g_i}(J,K).$$

It is obvious that with a fixed m

$$\lim_{i\to\infty} \max_{\substack{t',t''\in J\\ x\in K}} \alpha_{mi}(t',t'',x) = 0.$$

Consequently

$$\lim_{i\to\infty} H_{Cg_i}(J,K) \le \alpha_2\|P_m - C\|.$$

Hence we obtain (4.17). □

Lemma 4.10. *Let $C(s)$, $s\in J$, be a piecewise continuous matrix function and a sequence $g_i\in E(J\times O,R^n)$, $i=1,2,\dots$, satisfy the conditions* (4.16). *Then*

$$\lim_{i\to\infty} H_{Cg_i}(J,K) = 0. \tag{4.18}$$

Proof. It is not defficult to see that

$$H_{Cg_i}(J,K) \le \sum_{p=1}^{l} H_{Cg_i}(e_p,K), \tag{4.19}$$

where $e_p\subset J$ are the subintervals of continuity of $C(s)$.

It is clear that

$$\lim_{i\to\infty} H_{g_i}(e_p,K) \le \lim_{i\to\infty} H_{g_i}(J,K) = 0.$$

On the basis of the previous lemma we have

$$\lim_{i\to\infty} H_{Cg_i}(e_p,K) = 0, \quad p=1,\dots,l.$$

Hence, taking into account (4.19), we obtain (4.18). □

Lemma 4.11. *Let* $C(s)$, $s \in J$ *be a piecewise continuous matrix function,* $Y(s,t)$, $s \in [a,t]$, $t \in J$, *be a corresponding solution (see* (4.12), (4.13)) *and a sequence* $g_i \in E(J \times O, R^n)$, $i = 1, 2, \dots,$ *satisfy the conditions* (4.16). *Then*

$$\lim_{i\to\infty} \max_{t',t''\in J, x\in K} \left| \int_{t'}^{t''} Y(s,t) g_i(s,x) ds \right| = 0 \quad \textit{uniformly for} \quad t \in J. \tag{4.20}$$

Proof. Using the expression (4.13) and the condition $Y(s,t) = \Theta$, $s > t$, we get:

$$\sup \left\{ \int_{t'}^{t''} Y(s,t) g_i(s,x) ds : t', t'' \in J,\ x \in K \right\} \le$$

$$\le \sup \left\{ \int_{t'}^{t''} Y(s,t) g_i(s,x) ds : t', t'' \in [a,t],\ x \in K \right\} \le$$

$$\le \sum_{k=0}^{m(a,t)} \sum_{j=0}^{k} \sup \left\{ \left| \int_{t'}^{t''} C_j[s] g_i(s,x) ds \right| : t', t'' \in (\eta^{k+1}(t), \eta^k(t)] \cap J,\ x \in K \right\} \le$$

$$\le \sum_{k=0}^{m(a,t)} \sum_{j=0}^{k} H_{C_j g_i}([a, \eta^k(b)], K) \quad \text{for every} \quad k = 0, \dots, m(a,b). \tag{4.21}$$

The matrix functions $C_j[s]$, $j = 0, \dots, k$, are piecewise continuous on $[a, \eta^k(b)]$, therefore, by Lemma 4.10 we obtain

$$\lim_{i\to\infty} H_{C_j g_i}([a, \eta^k(b)], K) = 0, \quad k = 0, \dots, m(a,b), \quad j = 0, \dots, k.$$

Consequently, by (4.21) the relation (4.20) is valid. □

Lemma 4.12. *Let* $C \in C(J)$ *and a sequence* $g_i \in E(J \times O, R^n)$, $i = 1, 2, \dots,$ *satisfy the conditions:*

$$|g_i(s,x)| \le \alpha_3, \quad \forall (t,x) \in J \times K, \quad \lim_{i\to\infty} H_{g_i}(J,K) = 0. \tag{4.22}$$

Then

$$\lim_{i\to\infty} H_{C g_i}(J,K) = 0.$$

Proof. There exists a sequence of continuous matrix functions $P_m(t)$, $t \in J$, $m = 1, 2, \dots,$ such that

$$\lim_{i\to\infty} \int_J |C(s) - P_m(s)| ds = 0.$$

It is obvious that

$$\left|\int\limits_{t'}^{t''} C(s)g_i(s,x)ds\right| \le \alpha_3 \int\limits_J |C(s)-P_m(s)|ds + \left|\int\limits_{t'}^{t''} P_m(s)g_i(s,x)ds\right|.$$

Thus with a fixed m we get

$$\lim_{i\to\infty} H_{Cg_i}(J,K) \le \alpha_3 \int\limits_J |C(s)-P_m(s)|ds.$$

Hence it follows the desired equality. □

Lemma 4.13. *Let* $C \in C(J)$*,* $Y(s,t)$ *be the corresponding solution (see* (4.12)*,* (4.13)*) and a sequence* $g_i \in E(J\times O, R^n)$*,* $i=1,2,\dots,$ *satisfy the condition* (4.22)*. Then*

$$\lim_{i\to\infty} \max_{(t',t'',x)\in J\times K} \left|\int\limits_{t'}^{t''} Y(s,t)g_i(s,x)ds\right| = 0 \quad \textit{uniformly for} \quad t\in J.$$

This lemma, by Lemma 4.12, follows from the inequality (4.21).

4.2. Theorems on Continuous Dependence of Solution. To every element

$$\mu = (t_0, x_0, \varphi, C, f) \in A_1 = J\times O\times \Delta_1(J_2, O)\times C(J)\times E(J\times O^2, R^n)$$

there corresponds the differential equation

$$\dot y(t) = C(t)h(t_0,\dot\varphi,\dot y)(\eta(t)) + f(t, y(t), h(t_0,\varphi,y)(\tau(t))) \qquad (4.23)$$

with the initial condition

$$y(t_0) = x_0, \qquad (4.24)$$

where the operator $h(\cdot)$ is defined by

$$h(t_0,\varphi,y)(t) = \begin{cases} \varphi(t), & t\in[\rho(a), t_0), \\ y(t), & t\in[t_0, b]. \end{cases} \qquad (4.25)$$

The solution of the equation (4.23) is defined according to Definition 1.1, with a natural modification.

Theorem 4.1. *Let* $\tilde y(t)$ *be a solution, defined on* $[r_1, r_2] \subset (a,b)$ *corresponding to the element* $\tilde\mu = (\tilde t_0, \tilde x_0, \tilde\varphi, \tilde C, \tilde f) \in A_1$*, Let* $\tilde C(t)$*,* $t\in J$*, be a pieceswise-continuous matrix function,* K_1 *contain some neighborhood of the set* $\tilde\varphi(J_2)\cup\tilde y([r_1,r_2])$*. Then there exist numbers* $\delta_i > 0$*,* $i=0,1,$ *such that to an arbitrary element*

$$\mu \in V(\tilde\mu, K_1, \delta_0, \alpha_0) =$$

$$= V(\tilde t_0,\delta_0)\times V(\tilde x_0,\delta_0)\times V(\tilde\varphi,\delta_0)\times V(\tilde C,\delta_0)\times V(\tilde f, K_1,\delta_0)\cap W(\tilde f, K_1, \alpha_0)$$

there corresponds the solution $y(t,\mu)$ defined on $[r_1-\delta_1, r_2+\delta_1] \subset J$. Moreover, for each $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ the inequality

$$|y(t,\mu) - y(t,\tilde{\mu})| \le \varepsilon, \quad t \in [r_1 - \delta_1, r_2 + \delta_1], \tag{4.26}$$

is fulfilled.

Proof. In a way similar to the proof of Theorem 1.2, to every $\mu \in A_1$ there corresponds the equation

$$\dot{z}(t) = C(t)h(t_0, \dot{\varphi}, \dot{z})(\eta(t)) + g(t, z(t), h(t_0, \varphi, z)(\tau(t))) \tag{4.27}$$

with the initial condition

$$z(t_0) = x_0. \tag{4.28}$$

The function g has the form (1.9) and satisfies the conditions (1.10), (1.11).

It is easy to see that the equation (4.27) for $t \in [r_1, t_0]$ may be considered (see (4.25)) as the ordinary differential equation

$$\dot{z}_1(t) = C(t)\dot{\varphi}(\eta(t)) + g(t, z_1(t), \varphi(\tau(t))), \tag{4.29}$$

$$z_1(t_0) = x_0, \tag{4.30}$$

while for $t \in [t_0, r_2]$ as the neutral differential equation

$$\dot{z}_2(t) = C(t)\dot{z}_2(\eta(t)) + g(t, z_2(t), z_2(\tau(t))), \tag{4.31}$$

$$z_2(t) = \varphi(t), \quad t \in [\rho(t_0), t_0), \qquad z_2(t_0) = x_0. \tag{4.32}$$

It is clear that if $z_1(t)$, $t \in [r_1, t_0]$, is a solution of the equation (4.29) with the initial condition (4.30), and $z_2(t)$, $t \in [t_0, r_2]$, is a solution of the equation (4.31) with the initial condition (4.32), then the function

$$z(t) = \begin{cases} z_1(t), & t \in [r_1, t_0), \\ z_2(t), & t \in [t_0, r_2] \end{cases}$$

will be a solution of the equation (4.27) with the initial condition (4.28) defined on the interval $[r_1, r_2]$.

Write the equation (4.29) with the condition (4.30) in the integral form

$$z_1(t) = x_0 + \int_{t_0}^{t} [C(s)\dot{\varphi}(\eta(s)) + g(s, z_1(s), \varphi(\tau(s)))]ds, \quad t \in [a, t_0], \tag{4.33}$$

and the equation (4.31) with the condition (4.32) in the equivalent form

$$z_2(t) = x_0 + \int_{t_0}^{\sigma(t_0)} Y(s,t)C(s)\dot{\varphi}(\eta(s))ds +$$

$$+\int_{t_0}^{t} Y(,t)g(s,z_2(s),z_2(\tau(s)))ds, \quad t\in[t_0,b], \tag{4.34}$$

where $Y(s,t)$ is the matrix function corresponding to $C(s)$ (see (4.13)).

We introduce the notation

$$Y_1(s,t,t_0)=\begin{cases}E, & a\le t<t_0,\\ Y(s,t), & t_0\le t<\sigma=\min\{\sigma(t_0),b\},\\ \Theta, & \sigma\le t\le b,\end{cases}$$
$$Y_2(s,t,t_0)=\begin{cases}E, & a\le t<t_0,\\ Y(s,t), & t_0\le t\le b.\end{cases} \tag{4.35}$$

On the basis of this notation the equations (4.33) and (4.34), and consequently the equation (4.27), may be written in the form of the equivalent integral equation

$$z(t)=x_0+$$
$$+\int_{t_0}^{t}[Y_1(s,t,t_0)C(s)\dot\varphi(\eta(s))+Y_2(s,t,t_0)g(s,z(s),h(t_0,\varphi,z)(\tau(s)))]ds. \tag{4.36}$$

It is obvious that the solution of the equation (4.36) is dependent on the parameter

$$\mu\in G_1=J\times O\times\Delta_1(J_2,O)\times C(J)\times W(\tilde f,K_1,\alpha_0)\subset E_\mu=$$
$$=R^1\times R^n\times\Delta_1(J_2,R^n)\times C(J)\times E(J\times O^2,R^n).$$

The topology in G_1 is induced from E_μ.

On the space $C(J,R^n)$ define a family of mappings depending on $\mu\in G_1$

$$F(\cdot,\mu):C(J,R^n)\to C(J,R^n) \tag{4.37}$$

by the formula

$$\zeta(t)=\zeta(t,z,\mu)=x_0+$$
$$+\int_{t_0}^{t}[Y_1(s,t,t_0)C(s)\dot\varphi(\eta(s))+Y_2(s,t,t_0)g(s,z(s),h(t_0,\varphi,z)(\tau(s)))]ds,$$
$$t\in J, \quad z\in C(J,R^n).$$

We define the iterations $F^k(z,\mu)$:

$$\zeta_k(t)=\zeta_k(t,z,\mu)=x_0+$$

$$+\int_{t_0}^{t}[Y_1(s,t,t_0)C(s)\dot\varphi(\eta(s))+Y_2(s,t,t_0)g(s,\zeta_{k-1}(s),h(t_0,\varphi,\zeta_{k-1})(\tau(s)))]ds,$$

$$k=1,2,\dots,\quad \zeta_0(t)=z(t).$$

Using the inequality (1.11), we get

$$|\zeta'_k(t)-\zeta''_k(t)|\le$$
$$\le \|Y_2\|\int_a^t L_f(s)\big(|\zeta'_{k-1}(s)-\zeta''_{k-1}(s)|+|h(t_0,\varphi,\zeta'_{k-1})(\tau(s))-$$
$$-h(t_0,\varphi,\zeta''_{k-1})(\tau(s))|\big)ds,\quad k=1,2,\dots,$$

where $\|Y_2\|=\sup\{|Y_2(s,t,t_0)|:s,t,t_0\in J\}$; it is assumed that $\zeta'_0(t)=z'(t)$, $\zeta''_0(t)=z''(t)$.

On the basis of this inequality in a way similar to the proof of Theorem 1.2 it can be proved that some iteration of the mapping (4.37) is a uniform contraction. Thus for every $\mu\in G_1$ the equation (4.27) with the initial condition (4.28) has a unique solution $z(t,\mu)$, $t\in J$.

Now for an arbitrary $k=1,2,\dots$ we prove that the mapping

$$F_k(z(\cdot,\tilde\mu),\mu):G_1\to C(J,R^n)$$

is continuous at the point $\mu=\tilde\mu$.

To this end is suffices to show that if the sequence $\mu_i=(t_0^i,x_0^i,\varphi_i,C_i,f_i)\in G_1$, $i=1,2,\dots$ tends to $\tilde\mu=(\tilde t_0,\tilde x_0,\tilde\varphi,\tilde C,\tilde f)$, i.e., if

$$\lim_{i\to\infty}\big(|t_0^i-\tilde t_0|+|x_0^i-\tilde x_0|+\|\varphi_i-\tilde\varphi\|_1+\|C_i-\tilde C\|+H_{\delta f_i}(J,K_1)\big)=0,\quad \delta f_i=f_i-\tilde f,$$

then

$$\lim_{i\to\infty}F_k(z(\cdot,\tilde\mu),\mu_i)=F_k(z(\cdot,\tilde\mu),\tilde\mu)=z(\cdot,\tilde\mu). \tag{4.38}$$

The proof will be carried out by induction. Let $k=1$. Then

$$|\zeta_1^i(t)-\tilde z(t)|\le|x_0^i-\tilde x_0|+$$
$$+\Big|\int_{t_0^i}^t Y_1^i(s,t,t_0^i)C_i(s)\dot\varphi_i(\eta(s))ds-\int_{\tilde t_0}^t\tilde Y_1(s,t,\tilde t_0)\tilde C(s)\dot{\tilde\varphi}(\eta(s))ds\Big|+$$
$$+\Big|\int_{t_0^i}^t Y_2^i(s,t,t_0^i)g_i(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))ds-$$
$$-\int_{\tilde t_0}^t\tilde Y_2(s,t,\tilde t_0)\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))ds\Big|=|x_0^i-\tilde x_0|+a_i(t)+b_i(t).$$

Here $\zeta_1^i(t) = \zeta_1(t, \tilde{z}, \mu_i)$, $\tilde{z}(t) = z(t, \tilde{\mu})$; $\tilde{g} = \chi \tilde{f}$, $g_i = \chi f_i$ (see (1.9)); Y_1^i, Y_2^i are matrices corresponding to $C_i(s)$, $\tilde{Y}_1$, $\tilde{Y}_2$ are matrices corresponding to $\tilde{C}(s)$ (see (4.35)).

First of all we estimate $a_i(t)$

$$a_i(t) \leq \left| \int_{t_0^i}^{\tilde{t}_0} |\tilde{Y}_1(s,t,\tilde{t}_0)\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))| ds + \right.$$

$$\left. + \int_J |Y_1^i(s,t,t_0^i)C_i(s)\dot{\varphi}_i(\eta(s)) - \tilde{Y}_1(s,t,\tilde{t}_0)\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s)) ds \right| =$$

$$= a_i^1(t) + a_i^2(t). \tag{4.39}$$

It is obvious that

$$a_i^1(t) \leq \|\tilde{Y}_1\| \int_{t_0^i}^{\tilde{t}_0} |\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))| ds. \tag{4.40}$$

It is not difficult to see that

$$a_i^2(t) \leq \int_J |Y_1^i(s,t,t_0^i) - \tilde{Y}_1(s,t,\tilde{t}_0)|\,|C_i(s)|\,|\dot{\varphi}_i(\eta(s))| ds +$$

$$+ \int_J |\tilde{Y}_1(s,t,\tilde{t}_0)| C_i(s)\dot{\varphi}_i(\eta(s)) - \tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))| ds \leq$$

$$\leq \|C_i\|\,\|\varphi_i\|_1 \int_J |\tilde{Y}_1(s,t,\tilde{t}_0) - Y_1^i(s,t,t_0^i)| ds +$$

$$+ \|\tilde{Y}_1\| \sup_{s \in J} |C_i(s)\dot{\varphi}_i(\eta(s)) - \tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))|(b-a). \tag{4.41}$$

Next,

$$\sup_{s \in J} |C_i(s)\dot{\varphi}_i(\eta(s)) - \tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))| \leq$$

$$\leq \|\varphi_i\|_1 \|\tilde{C} - C_i\| + \|\varphi_i - \tilde{\varphi}\|_1 \|\tilde{C}\|. \tag{4.42}$$

Let $\tilde{t}_0 > t_0^i$ and assume that i_0 is so large that $\sigma(t_0^i) > \tilde{t}_0$ for $i \geq i_0$. Then, taking into consideration (4.35), we get

$$\int_J |\tilde{Y}_1(s,t,\tilde{t}_0) - Y_1^i(s,t,t_0^i)| ds =$$

$$= \int_{t_0^i}^{\tilde{t}_0} |E - Y_i(s,t)| ds + \int_{\tilde{t}_0}^{\sigma(t_0^i)} |\tilde{Y}(s,t) - Y_i(s,t)| ds + \int_{\sigma(t_0^i)}^{\sigma(\tilde{t}_0)} |\tilde{Y}(s,t)| ds \leq$$

$$\leq \|Y_i - E\|(\tilde{t}_0 - t_0^i) + \|\tilde{Y} - Y_i\|(b-a) + \|\tilde{Y}\|(\sigma(\tilde{t}_0) - \sigma(t_0^i)),$$

where Y_i, $\tilde{Y}$ are solutions of the equation (4.13) corresponding to $C_i(t)$ and $\tilde{C}(t)$, respectively.

From the latter inequality by Lemma 4.7 we conclude that

$$\lim_{i\to\infty} \int_J |\tilde{Y}_1(s,t,\tilde{t}_0) - Y_1^i(s,t,t_0^i)|ds = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.43}$$

Let now $\tilde{t}_0 < t_0^i$. Choose a number i_0 so large that $\sigma(\tilde{t}_0) > t_0^i$ for $i \geq i_1$. Then

$$\int_J |\tilde{Y}_1(s,t,\tilde{t}_0) - Y_1^i(s,t,t_0^i)|ds =$$

$$= \int_{\tilde{t}_0}^{t_0^i} |\tilde{Y}(s,t) - E|ds + \int_{t_0^i}^{\sigma(\tilde{t}_0)} |\tilde{Y}(s,t) - Y_i(s,t)|ds + \int_{\sigma(\tilde{t}_0)}^{\sigma(t_0^i)} |Y_i(s,t)|ds \leq$$

$$\leq \|\tilde{Y} - E\|(t_0^i - \tilde{t}_0) + \|\tilde{Y} - Y_i\|(b-a) + \|Y_i\|(\sigma(t_0^i) - \sigma(\tilde{t}_0)).$$

Hence it follows (4.43).

The inequalities (4.40) and (4.41), in view of (4.42) and (4.43), yield

$$\lim_{i\to\infty} a_i(t) = 0, \quad \text{uniformly for} \quad t \in J. \tag{4.44}$$

Now we estimate the addend $b_i(t)$. We have

$$b_i(t) \leq \left| \int_{t_0^i}^{\tilde{t}_0} \tilde{Y}_2(s,t,\tilde{t}_0)\tilde{g}(s,\tilde{z}(s),h(\tilde{t}_0,\tilde{\varphi},\tilde{z})(\tau(s)))ds \right| +$$

$$+\left| \int_{t_0^i}^{t} [Y_2^i(s,t,t_0^i)g_i(s,\tilde{z}(s),h(t_0^i,\varphi_i,\tilde{z})(\tau(s))) - \right.$$

$$\left. -\tilde{Y}_2(s,t,\tilde{t}_0)\tilde{g}(s,\tilde{z}(s),h(\tilde{t}_0,\tilde{\varphi},\tilde{z})(\tau(s)))]ds \right| = b_i^1(t) + b_i^2(t).$$

It is obvious that

$$b_i^1(t) \leq \|\tilde{Y}_2\| \left| \int_{t_0^i}^{\tilde{t}_0} m_{\tilde{f},K_1}(t)dt \right|.$$

Thus

$$\lim_{i\to\infty} b_i^1(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.45}$$

Further,

$$b_i^2(t) \le \max_{t',t''\in J}\left|\int\limits_{t'}^{t''} Y_2^i(s,t,t_0^i)\delta g_i(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s)))ds\right| +$$

$$+\int\limits_J |Y_2^i(s,t,t_0^i)-\tilde Y_2(s,t,\tilde t_0)|\,|\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))|ds +$$

$$+\int\limits_J |Y_2^i(s,t,t_0^i)|\,|\tilde g(s,\tilde z(s),h(t_0^i,\varphi_i,\tilde z)(\tau(s))) -$$

$$-\tilde g(s,\tilde z(s),h(\tilde t_0,\tilde\varphi,\tilde z)(\tau(s)))|ds =$$

$$= b_i^3(t)+b_i^4(t)+b_i^5(t), \qquad \delta g_i = g_i-\tilde g.$$

First of all we estimate $b_i^3(t)$:

$$b_i^3(t) \le \max_{t',t''\in J}\left|\int\limits_{t'}^{t''} Y_2^i(s,t,t_0^i)\delta g_i(s,\tilde z(s),h(t_0^i,\tilde\varphi,\tilde z)(\tau(s)))ds\right| +$$

$$+\|Y_2^i\|\int\limits_J L_{\delta g_i,K_1}(s)|h(t_0^i,\varphi_i,\tilde z)(\tau(s))-h(t_0^i,\tilde\varphi,\tilde z)(\tau(s))|ds = \beta_i^1(t)+\beta_i^2(t).$$

It is easy to see that

$$\beta_i^1(t) \le \max_{t',t''\in[a,t_0^i]}\left|\int\limits_{t'}^{t''}\delta g_i(s,\tilde z(s),\tilde\varphi(\tau(s)))ds\right| +$$

$$+\max_{t',t''\in[t_0^i,s_i]}\left|\int\limits_{t'}^{t''}Y_i(s,t)\delta g_i(s,\tilde z(s),\tilde\varphi(\tau(s)))ds\right| +$$

$$+\max_{t',t''\in[s_i,b]}\left|\int\limits_{t'}^{t''}Y_i(s,t)\delta g_i(s,\tilde z(s),\tilde z(\tau(s)))ds\right| \le$$

$$\le \max_{t',t''\in J}\left|\int\limits_{t'}^{t''}\delta g_i(s,\tilde z(s),\tilde\varphi(\tau(s)))ds\right| +$$

$$+\max_{t',t''\in J}\left|\int\limits_{t'}^{t''}\tilde Y(s,t)\delta g_i(s,\tilde z(s),\tilde\varphi(\tau(s)))ds\right| +$$

$$+\max_{t',t''\in J}\left|\int\limits_{t'}^{t''}\tilde Y(s,t)\delta g_i(s,\tilde z(s),\tilde z(\tau(s)))ds\right| + 2\|\tilde Y-Y_i\|\int\limits_J m_{\delta f_i,K_1}(t)dt, \quad (4.46)$$

where $s_i = \min\{\gamma(t_0^i), b\}$.

By the assumption

$$\int_J m_{\delta f_i, K_1}(t)dt \le \alpha_0. \tag{4.47}$$

Further

$$H_{\delta g_i}(J, K_1) \le H_{\delta f_i}(J, K_1),$$

consequently $H_{\delta g_i}(J, K_1) \to 0$ as $i \to \infty$. After this, using (4.47) and Lemmas 1.3, 4.7, 4.11, we get

$$\lim_{i\to\infty} \beta_i^1(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.48}$$

For $\beta_i^2(t)$, taking into consideration (4.25), we easily ascertain the validity of the inequality (see the proof of Theorem 1.2)

$$\beta_i^2(t) \le \alpha_0(1 + \alpha_1)\|Y_2^i\| \, \|\varphi_i - \tilde{\varphi}\|_1. \tag{4.49}$$

From (4.48), (4.49) we obtain

$$\lim_{i\to\infty} b_i^3(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.50}$$

Now we estimate $b_i^4(t)$. For $\tilde{t}_0 > t_0^i$ we have

$$b_i^4(t) \le \int_{t_0^i}^{\tilde{t}_0} |Y_i(s,t) - E| m_{\tilde{g},K_1}(s)ds + \int_{\tilde{t}_0}^{b} |Y_i(s,t) - \tilde{Y}(s,t)| m_{\tilde{g},K_1}(s)ds.$$

If $\tilde{t}_0 < t_0^i$, then

$$b_i^4(t) \le \int_{\tilde{t}_0}^{t_0^i} |E - \tilde{Y}(s,t)| m_{\tilde{g},K_1}(s)ds + \int_{t_0^i}^{b} |Y_i(s,t) - \tilde{Y}(s,t)| m_{\tilde{g},K_1}(s)ds.$$

Consequently,

$$\lim_{i\to\infty} b_i^4(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.51}$$

Finally we estimate $b_i^5(t)$. Let $\tilde{t}_0 < t_0^i$. Then

$$b_i^5(t) \le \|Y_2^i\| \int_J L_{\tilde{g},K_1}(s) |h(t_0^i, \varphi_i, \tilde{z})(\tau(s)) - h(\tilde{t}_0, \tilde{\varphi}, \tilde{z})(\tau(s))| ds \le$$

$$\le \|Y_2^i\| \left\{ \int_a^{\tilde{\sigma}} L_{\tilde{g},K_1}(s) |\varphi_i(\tau(s)) - \tilde{\varphi}(\tau(s))| ds + \right.$$

$$\left. + \int_{\tilde{\sigma}}^{\sigma_i} L_{\tilde{g},K_1}(s) |\varphi_i(\tau(s)) - \tilde{z}(\tau(s))| ds \right\},$$

where $\tilde{\sigma} = \min\{\gamma(\tilde{t}_0), b\}$, $\sigma_i = \min\{\gamma(t_0^i), b\}$. If $\tilde{t}_0 > t_0^i$, then

$$b_i^5(t) \leq \|Y_2^i\| \Big\{ \int\limits_a^{\sigma_i} L_{\tilde{g},K_1}(s)|\varphi_i(\tau(s)) - \tilde{\varphi}(\tau(s))|ds +$$

$$+ \int\limits_{\sigma_i}^{\tilde{\sigma}} L_{\tilde{g},K_1}(s)|\tilde{z}(\tau(s)) - \tilde{\varphi}(\tau(s))|ds \Big\}.$$

Clearly, from the latter inequalites it follows

$$\lim_{i\to\infty} b_i^5(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.52}$$

From the equalities (4.44), (4.45), (4.50)-(4.52) it follows (4.38) for $k = 1$.

Let (4.38) be fulfilled some for $k \geq 1$. Now we will prove the validity of (4.38) for $k + 1$. We have

$$|\zeta_{k+1}^i(t) - \tilde{z}(t)| \leq |x_0^i - \tilde{x}_0| +$$

$$+ \Big| \int\limits_{t_0^i}^{t} Y_1^i(s,t,t_0^i)C_i(s)\dot{\varphi}_i(\eta(s))ds - \int\limits_{\tilde{t}_0}^{t} \tilde{Y}_1(s,t,\tilde{t}_0)\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))ds \Big| +$$

$$+ \Big| \int\limits_{t_0^i}^{t} Y_2^i(s,t,t_0^i)g_i(s,\zeta_k^i(s),h(t_0^i,\varphi_i,\zeta_k^i)(\tau(s)))ds -$$

$$- \int\limits_{\tilde{t}_0}^{t} \tilde{Y}_2(s,t,\tilde{t}_0)\tilde{g}(s,\tilde{z}(s),h(\tilde{t}_0,\tilde{\varphi},\tilde{z})(\tau(s)))ds \Big| = |x_0^i - \tilde{x}_0| + a_i(t) + b_{ik}(t). \tag{4.53}$$

The function $a_i(t)$ has been estimated above (see (4.44)). For $b_{ik}(t)$ we obtain:

$$b_{ik}(t) \leq \|Y_2^i\| \int\limits_J |g_i(s,\tilde{z}(s),h(t_0^i,\varphi_i,\tilde{z})(\tau(s)))ds -$$

$$-g_i(s,\zeta_k^i(s),h(t_0^i,\varphi_i,\zeta_k^i)(\tau(s)))ds| +$$

$$+ \Big| \int\limits_{t_0^i}^{t} Y_2^i(s,t,t_0^i)g_i(s,\tilde{z}(s),h(t_0^i,\varphi_i,\tilde{z})(\tau(s)))ds -$$

$$- \int\limits_{\tilde{t}_0}^{t} \tilde{Y}_2(s,t,\tilde{t}_0)\tilde{g}(s,\tilde{z}(s),h(\tilde{t}_0,\tilde{\varphi},\tilde{z})(\tau(s)))ds \Big| = b_{ik}^1(t) + b_i(t). \tag{4.54}$$

The function $b_i(t)$ has been estimated above (see (4.45), (4.50)-(4.52)). It is easy to see that for $b_{ik}^1(t)$ the inequality

$$b_{ik}^1(t) \le \|Y_2^i\| \int_J L_{g_i,K_1}(s)\big(|\tilde{z}(s) - \zeta_k^i(s)| + |h(t_0^i, \varphi_i, \tilde{z})(\tau(s)) -$$

$$-h(t_0^i, \varphi_i, \zeta_k^i)(\tau(s))|\big)ds \le 2\|Y_2^i\|\,\|\tilde{z} - \zeta_k^i\| \int_J L_{g_i,K_1}(s)ds$$

is valid. By assumption

$$\lim_{i\to\infty} \|\tilde{z} - \zeta_k^i\| = 0.$$

Therefore

$$\lim_{i\to\infty} b_{ik}^1(t) = 0 \quad \text{uniformly for} \quad t \in J. \tag{4.55}$$

For (4.53) on the basis of (4.44), (4.45), (4.50)-(4.52) and (4.55) we obtain

$$\lim_{i\to\infty} \|\zeta_{k+1}^i - \tilde{z}\| = 0.$$

The relation (4.38) is proved for every $k = 1, 2, \ldots$.

Now we use Theorem 1.1, which allows us with an analogous argument (see the proof of Theorem 1.2) to complete the proof of this theorem. □

Introduce a set

$$W_1(\tilde{f}, K_1, \alpha_0) = \Big\{\tilde{f} + \delta f : \delta f \in E(J \times O^2, R^n),\ |\delta f(t, x_1, x_2)| +$$

$$+\int_J L_{\delta f_i,K_1}(s)ds \le \alpha_0, \forall(t,(x_1,x_2)) \in J \times K_1^2\Big\}.$$

Theorem 4.2. *Let $\tilde{y}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A_1$, defined on $[r_1, r_2] \subset (a, b)$; let K_1 contain some neighborhood of the set $\tilde{\varphi}(J_2) \cup \tilde{y}([r_1, r_2])$. Then there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to an arbitrary element*

$$\mu \in V_1(\tilde{\mu}, K_1, \delta_0, \alpha_0) =$$

$$= V(\tilde{t}_0, \delta_0) \times V(\tilde{x}_0, \delta_0) \times V(\tilde{\varphi}, \delta_0) \times V(\tilde{C}, \delta_0) \times V(\tilde{f}, K_1, \delta_0) \cap W_1(\tilde{f}, K_1, \alpha_0)$$

there corresponds $y(t, \mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset J$. Moreover, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V_1(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ the inequality

$$|y(t, \mu) - y(t, \tilde{\mu})| \le \varepsilon, \quad t \in [r_1 - \delta_1, r_2 + \delta_1]m$$

is valid.

This theorem is proved analogously to the previous theorem. In this case instead of Lemma 4.11 we have to use Lemma 4.13.

To every element $\mu \in A_1$ there corresponds the differential equation

$$\dot{x}(t) = C(t)\dot{x}(\eta(t)) + f(t, x(t), x(\tau(t)))$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\rho(t_0), t_0), \qquad x(t_0) = x_0.$$

The solution $x(t) = x(t, \mu)$, $t \in [\rho(t_0), t_1]$ corresponding to the element $\mu \in A_1$ is defined analogously (see Definition 1.2).

Theorem 4.3. *Let $\tilde{x}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A_1$, defined on $[\rho(\tilde{t}_0), \tilde{t}_0] \subset (\rho(a), b)$; let $\tilde{C}(t)$, $t \in J$, be a piecewise-continuous matrix function, K_1 contain some neighborhood of the set $\tilde{\varphi}(J_2) \cup \tilde{x}([\tilde{t}_0, \tilde{t}_1])$. Then there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to an arbitrary element $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds a solution $x(t, \mu)$ defined on $[\rho(t_0), \tilde{t}_1 + \delta_1] \subset [\rho(a), b]$. Moreover, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ the inequality*

$$|x(t, \mu) - x(t, \tilde{\mu})| \leq \varepsilon, \quad t \in [s_2, \tilde{t}_1 + \delta_1],$$

is valid.

Theorem 4.4. *Let $\tilde{x}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A_1$, defined on $[\rho(\tilde{t}_0), \tilde{t}_0] \subset (\rho(a), b)$; let K_1 contain some neighborhood of the set $\tilde{\varphi}(J_2) \cup \tilde{x}([\tilde{t}_0, \tilde{t}_1])$. Then there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to an arbitrary element $\mu \in V_1(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ there corresponds the solution $x(t, \mu)$ defined on $[\rho(t_0), \tilde{t}_1 + \delta_1] \subset [\rho(a), b]$. Moreover, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that for an arbitrary $\mu \in V_1(\tilde{\mu}, K_1, \delta_0, \alpha_0)$ the inequality*

$$|x(t, \mu) - x(t, \tilde{\mu})| \leq \varepsilon, \quad t \in [s_2, \tilde{t}_1 + \delta_1],$$

is valid.

These theorems follow from Theorems 4.1, 4.2, respectively, and are proved analogously (see the proof of Theorem 1.3).

We note that Theorems 4.1, 4.2 and 4.3, 4.4 also are valid, respectively, for the differential equations

$$\dot{y}(t) = \sum_{i=1}^{\nu} C_i(t) h(t_0, \dot{\varphi}, \dot{y})(\eta_i(t)) + f(t, h(t_0, \varphi, y)(\tau_1(t)), \ldots, h(t_0, \varphi, y)(\tau_s(t))),$$

$$\dot{x}(t) = \sum_{i=1}^{\nu} C_i(t)\dot{x}(\eta_i(t)) + f(t, x(\tau_1(t)), \ldots, x(\tau_s(t))),$$

where $C_i \in C(J)$, $\eta_i : R^1 \to R^1$, $i = 1, \ldots, \nu$ are continuousle differentiable functions satisfying $\eta_i(t) < t$, $\dot{\eta}_i(t) > 0$.

Finally note that the Theorems 4.3, 4.4, generally speaking, are not true for the equations whose right-hand sides are non-linear with respect to $\dot{x}(\eta(t))$. For illustration consider

Example. Consider the system

$$\begin{cases} \dot{x} = 0, \\ \dot{y} = \dot{x}^2(t-1), \quad t \in [0,2], \end{cases} \tag{4.56}$$

$$x(t) = 0, \ \ t \in [-1,0], \quad y(0) = 0.$$

It is obvious that the solution of the system (4.56) is $x(t) = y(t) = 0$.

Consider the petrurbed system

$$\begin{cases} \dot{x}_k = f_k(t), \\ \dot{y}_k = \dot{x}_k^2(t-1), \end{cases}$$
$$x_k(t) = 0, \ \ t \in [-1,0], \quad y_k(0) = 0, \tag{4.57}$$

where

$$f_k(t) = \begin{cases} v_k(t), & t \in [0,1], \\ 0, & t \in [1,2]. \end{cases}$$

The function $v_k(t)$ is defined in the following way. For given $k = 2,3,\dots$ we devide the interval $[0,1]$ into the subintervals e_i, $i = 1,\dots,k$, of the length $1/k$. Then $v_k(t) = 1$, $t \in e_1$, $v_k(t) = -1$, $t \in e_2$ and so on.

It is easy to see that

$$\lim_{k\to\infty} \max_{t',t''\in[0,2]} \left| \int_{t'}^{t''} f_k(t)dt \right| = 0.$$

For $t \geq 1$, taking into consideration (4.57) and the structure of the function $f_k(t)$, we get

$$y_k(t) = \int_0^t \dot{x}_k^2(\xi - 1)d\xi = \int_1^t v_k^2(\xi-1)d\xi = t-1.$$

Thus

$$||x_k|| \to 0, \quad ||y_k|| \not\to 0 \ \text{ as } \ k \to \infty.$$

Consequently, for the equation (4.56) Theorems 4.3 and 4.4 are not true.

5. Lemmas on the Estimation of the Increment

Introduce the set

$$V_1 = \{\delta\mu = (\delta t_0, \delta x_0, \delta\varphi, \delta C, \delta f) \in A_1 - \tilde{\mu} : |\delta t_0| \le \alpha_3, |\delta x_0| \le \alpha_3,$$
$$\|\delta\varphi\|_1 \le \alpha_3, \|\delta C\| \le \alpha_3, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,\ |\lambda_i| \le \alpha_3,\ i = 1, \dots, k\}. \tag{5.1}$$

Lemma 5.1. *Let $\tilde{y}(t)$ be the solution corresponding to the element $\tilde{\mu} \in A$ defined on $[r_1, r_2] \subset (a, b)$; let K_1 contain some neighborhood of the set $\tilde{\varphi}(J_2) \cup \tilde{y}([r_1, r_2])$. Then there exist numbers $\delta_2 > 0$, $\varepsilon_2 > 0$ such that, for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V_1$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in A_1$ there corresponds the solution $y(t, \tilde{\mu} + \varepsilon\delta\mu)$ defined on $[r_1 - \delta_2, r_2 + \delta_2] \subset J$. Moreover,*

$$\varphi(t) \in K_1,\ \ t \in J_2,\ y(t, \tilde{\mu} + \varepsilon\delta\mu) \in K_1,\ \ t \in [r_1 - \delta_2, r_2 + \delta_2], \tag{5.2}$$

$$\lim_{\varepsilon\to 0} y(t, \tilde{\mu} + \varepsilon\delta\mu) = y(t, \tilde{\mu}) \ \ \textit{uniformly for}\ \ (t, \delta\mu) \in [r_1 - \delta_2, r_2 + \delta_2] \times V_1, \tag{5.3}$$

$$|\dot{y}(t, \tilde{\mu} + \varepsilon\delta\mu)| \le m(t),\ \ (t, \varepsilon, \delta\mu) \in [r_1 - \delta_2, r_2 + \delta_2] \times [0, \varepsilon_2] \times V_1,$$
$$m(\cdot) \in L_1(J, R_0^+). \tag{5.4}$$

The relations (5.2), (5.3), on the basis of Theorem 4.4, are proved analogously (see the proof of Lemma 2.1). The relation (5.4) taking into account (5.1),(5.2) is easily proved by the method of steps for the left to right with respect to the delay $\eta(t)$ (see(4.23)).

In the sequel we assume that $\tilde{y}(t)$ is defined on $[r_1 - \delta_2, r_2 + \delta_2]$ (see Remark 2.1).

Define the function

$$\Delta y(t) = \Delta y(t, \varepsilon\delta\mu) = y(t, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{y}(t),$$
$$(t, \varepsilon, \delta\mu) \in [r_1 - \delta_2, r_2 + \delta_2] \times [0, \varepsilon_2] \times V_1. \tag{5.5}$$

Lemma 5.2. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(r_2) \ge \tilde{t}_0$ and the conditions*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left\{ \sup_{\delta\mu \in V_1^-} \int_{t_0}^{\tilde{t}_0} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))| dt \right\} < \infty, \tag{5.6}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left\{ \sup_{\delta\mu \in V_1^-} \int_{\gamma(t_0)}^{\gamma_0} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t))) - \right.$$
$$\left. - \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))| dt \right\} < \infty, V_1^- = \{\delta\mu \in V_1 : \delta t_0 \le 0\} \tag{5.7}$$

be fullfilled. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ we have

$$\max_{t \in [\tilde{t}_0, r_2 + \delta_2]} |\Delta y(t)| \le O(\varepsilon), \tag{5.8}$$

$$\int\limits_{\tilde{t}_0}^{r_2+\delta_2} |\dot{\Delta y}(t)|dt \le O(\varepsilon). \tag{5.9}$$

Proof. By assumptions of the lemma there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the conditions

$$\gamma(t_0) > \tilde{t}_0, \quad \sigma(t_0) > \tilde{t}_0, \tag{5.10}$$

$$\int\limits_{t_0}^{\tilde{t}_0} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))|dt \le O(\varepsilon), \tag{5.11}$$

$$\int\limits_{\gamma(t_0)}^{\gamma_0} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t)) + \Delta y(\tau(t))) -$$

$$-\tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))|dt \le O(\varepsilon) \tag{5.12}$$

are fulfilled.

It is easy to see that the function $\Delta y(t)$ on the interval $[\tilde{t}_0, r_2+\delta_2]$ satisfies the equation

$$\dot{\Delta y}(t) = \tilde{C}(t)h(\tilde{t}_0, \varepsilon\dot{\delta\varphi}, \dot{\Delta y})(\eta(t)) + \rho(t, \varepsilon\delta\mu) + a(t, \varepsilon\delta\mu) + b(t, \varepsilon\delta\mu), \tag{5.13}$$

where

$$\rho(t, \varepsilon\delta\mu) = C(t)h(t_0, \dot{\varphi}, \dot{\tilde{y}} + \dot{\Delta y})(\eta(t)) -$$

$$-\tilde{C}(t)h(\tilde{t}_0, \dot{\tilde{\varphi}}, \dot{\tilde{y}})(\eta(t)) - \tilde{C}(t)h(\tilde{t}_0, \varepsilon\dot{\delta\varphi}, \dot{\Delta y})(\eta(t)),$$

$$a(t, \varepsilon\delta\mu) = \tilde{f}(t, \tilde{y}(t) + \Delta y(t), h(t_0, \varphi, \tilde{y} + \Delta y)(\tau(t))) -$$

$$-\tilde{f}(t, \tilde{y}(t), h(\tilde{t}_0, \tilde{\varphi}, \tilde{y})(\tau(t))), \tag{5.14}$$

$$b(t, \varepsilon\delta\mu) = \varepsilon\delta f(t, \tilde{y}(t) + \Delta y(t), h(t_0, \varphi, \tilde{y} + \Delta y)(\tau(t))). \tag{5.15}$$

Rewrite the equation (5.13) in a way analogous to (4.36) in the form of the integral equation

$$\Delta y(t) = \Delta y(\tilde{t}_0) + \varepsilon \int\limits_{\tilde{t}_0}^{t} \tilde{Y}_1(s, t, \tilde{t}_0)\tilde{C}(s)\dot{\delta\varphi}(\eta(s))ds +$$

$$+\int\limits_{\tilde{t}_0}^{t} \tilde{Y}_2(s, t, \tilde{t}_0)[\rho(s, \varepsilon\delta\mu) + a(s, \varepsilon\delta\mu) + b(s, \varepsilon\delta\mu)]ds,$$

where the matrix functions $\tilde{Y}_1(s, t, \tilde{t}_0)$, $\tilde{Y}_2(s, t, \tilde{t}_0)$ correspond to $\tilde{C}(t)$ (see (4.35)).

Hence

$$|\Delta y(t)| \le |\Delta y(\tilde{t}_0)| + O(\varepsilon) + \|\tilde{Y}_2\| \Bigg[\int\limits_{\tilde{t}_0}^{r_2+\delta_2} |\rho(s,\varepsilon\delta\mu)|ds + \int\limits_{\tilde{t}_0}^{t} |a(s,\varepsilon\delta\mu)|ds +$$

$$+ \int\limits_{\tilde{t}_0}^{r_2+\delta_2} |b(s,\varepsilon\delta\mu)|ds \Bigg]. \tag{5.16}$$

We will estimate $|\Delta\, y(\tilde{t}_0)|$. Taking into consideration (5.5), (5.10), (5.1), (5.2) and (5.11), we obtain

$$|\Delta y(\tilde{t}_0)| = |y(\tilde{t}_0, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{x}_0)| = \Bigg| \tilde{x}_0 + \varepsilon\delta x_0 +$$

$$+ \int\limits_{t_0}^{\tilde{t}_0} \left[C(t)\dot{\varphi}(\eta(t)) + \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) + b(t,\varepsilon\delta\mu) \right] dt - \tilde{x}_0 \Bigg| \le \varepsilon|\delta x_0| +$$

$$+ \int\limits_{t_0}^{\tilde{t}_0} \left[|C(t)\dot{\varphi}(\eta(t))| + |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))| \right] dt + O(\varepsilon) \le O(\varepsilon). \tag{5.17}$$

Let $\eta(r_2) \ge \tilde{t}_0$. Then

$$\rho(\varepsilon\delta\mu) = \int\limits_{\tilde{t}_0}^{r_2+\delta_2} |\rho(s,\varepsilon\delta\mu)|ds = \int\limits_{\tilde{t}_0}^{\sigma(t_0)} |\rho(s,\varepsilon\delta\mu)|ds + \int\limits_{\sigma(t_0)}^{\sigma(\tilde{t}_0)} |\rho(s,\varepsilon\delta\mu)|ds +$$

$$+ \int\limits_{\sigma(\tilde{t}_0)}^{r_2+\delta_2} |\rho(s,\varepsilon\delta\mu)|ds = \sum_{i=1}^{3} \rho_i(\varepsilon\delta\mu). \tag{5.18}$$

Now we estimate every term of the exspression (5.18). It is clear,that

$$\rho_1(\varepsilon\delta\mu) = \int\limits_{\tilde{t}_0}^{\sigma(t_0)} |C(t)\dot{\varphi}(\eta(t)) - \tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) - \varepsilon\tilde{C}(t)\delta\dot{\varphi}(\eta(t))|dt =$$

$$= \varepsilon \int\limits_{\tilde{t}_0}^{\sigma(t_0)} |\delta C(t)|\, |\dot{\varphi}(\eta(t))|dt \le O(\varepsilon). \tag{5.19}$$

Further, taking into account (5.1), we have

$$\rho_2(\varepsilon\delta\mu) = \int\limits_{\sigma(t_0)}^{\sigma(\tilde{t}_0)} |C(t)[\dot{\tilde{y}}(\eta(t)) + \dot{\Delta} y(\eta(t))] - \tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) - \varepsilon\tilde{C}(t)\delta\dot{\varphi}(\eta(t)))|dt \le$$

$$\le \|C\| \int_{t_0}^{\tilde{t}_0} \dot{\sigma}(t)|\dot{\tilde{y}}(t) + \dot{\Delta} y(t)|dt + \int_{t_0}^{\tilde{t}_0} \dot{\sigma}(t)|\tilde{C}(\sigma(t))\dot{\varphi}(t)|dt \le$$

$$\le O(\varepsilon) + \|\dot{\sigma}\|\,\|C\| \int_{t_0}^{\tilde{t}_0} |\dot{\tilde{y}}(t) + \dot{\Delta} y(t)|dt.$$

Since $y(t, \tilde{\mu} + \varepsilon\delta\mu) = \tilde{y}(t) + \Delta y(t)$, we have (see (5.11), (2.15))

$$\int_{t_0}^{\tilde{t}_0} |\dot{\tilde{y}}(t) + \dot{\Delta} y(t)|dt =$$

$$= \int_{t_0}^{\tilde{t}_0} |C(t)\dot{\varphi}(\eta(t)) + \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) + b(t, \varepsilon\delta\mu)|dt \le$$

$$\le O(\varepsilon) + \int_{t_0}^{\tilde{t}_0} |\tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))|dt \le O(\varepsilon). \tag{5.20}$$

Thus

$$\rho_2(\varepsilon\delta\mu) \le O(\varepsilon). \tag{5.21}$$

Finally we estimate the last relation of the expression (5.18). Namely, (see (5.4))

$$\rho_3(\varepsilon\delta\mu) = \int_{\sigma(\tilde{t}_0)}^{r_2+\delta_2} |C(t)[\dot{\tilde{y}}(\eta(t)) + \dot{\Delta} y(\eta(t))] - \tilde{C}(t)\dot{\tilde{y}}(\eta(t)) - \tilde{C}(t)\dot{\Delta} y(\eta(t))|dt =$$

$$= \varepsilon \int_{\sigma(\tilde{t}_0)}^{r_2+\delta_2} |\delta C(t)|\,|\dot{y}(t, \tilde{\mu} + \varepsilon\delta\mu)|dt \le \varepsilon\alpha_3 \int_{\sigma(\tilde{t}_0)}^{r_2+\delta_2} m(t)dt = O(\varepsilon). \tag{5.22}$$

Consequently, according to (5.19), (5.21) and (5.22), we get

$$\rho(\varepsilon\delta\mu) \le O(\varepsilon).$$

This inequality also is valid for $\eta(r_2) < \tilde{t}_0$. To see this, it suffices choose numbers δ_3, ε_3 such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ we would have $\sigma(t_0) > r_2 + \delta_2$. After this $\rho(\varepsilon\delta\mu)$ is estimated analogously to $\rho_1(\varepsilon\delta\mu)$.

In an analogous way, using (5.10)-(5.12) (see the proof of Lemma 2.2), it is proved that

$$\int_{\tilde{t}_0}^{t} |a(s, \varepsilon\delta\mu)|ds \le O(\varepsilon) + \int_{\tilde{t}_0}^{t} L(s)|\Delta y(s)|ds, \tag{5.23}$$

$$\int_{\tilde{t}_0}^{r_2+\delta_2} |b(s,\varepsilon\delta\mu)|ds \leq O(\varepsilon). \tag{5.24}$$

On the basis of the obtained estimates, we can write for $\Delta y(t)$ the final estimate

$$|\Delta y(t)| \leq O(\varepsilon) + \int_{\tilde{t}_0}^{t} L(s)|\Delta y(s)|ds.$$

Hence by Gronwall's lemma we obtain (5.8).

Now on the basis of (5.8) we prove the second part of the lemma. We will carry out the proof by the method of steps with rescpect to the delay $\eta(t)$.

After elementary transformations, taking into account (5.23), (5.24) and (5.20), we obtain

$$\int_{\tilde{t}_0}^{\sigma(\tilde{t}_0)} |\dot{\Delta} y(t)|dt = \int_{\tilde{t}_0}^{\sigma(t_0)} |C(t)\dot{\varphi}(\eta(t)) - \tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) + a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu)|dt +$$

$$+ \int_{\sigma(t_0)}^{\sigma(\tilde{t}_0)} |C(t)[\dot{\tilde{y}}(\eta(t)) + \dot{\Delta} y(\eta(t))] - \tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) + a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu)|dt \leq$$

$$\leq \int_{\tilde{t}_0}^{\sigma(\tilde{t}_0)} [|a(t,\varepsilon\delta\mu)| + |b(t,\varepsilon\delta\mu)|dt] + \varepsilon \int_{\tilde{t}_0}^{\sigma(t_0)} |\tilde{C}(t)\delta\dot{\varphi}(\eta(t)) + \delta C(t)\dot{\varphi}(\eta(t))|dt +$$

$$+ \int_{t_0}^{\tilde{t}_0} |C(\sigma(t))|\,|\dot{\tilde{y}}(t) + \dot{\Delta} y(t)|\dot{\sigma}(t)dt + \int_{t_0}^{\tilde{t}_0} \dot{\sigma}(t)|\tilde{C}(\sigma(t))|\,|\dot{\tilde{\varphi}}(t)|dt \leq$$

$$\leq O(\varepsilon)||C||||\dot{\sigma}|| \int_{t_0}^{\tilde{t}_0} |\dot{\tilde{y}}(t) + \dot{\Delta} y(t)|dt \leq O(\varepsilon). \tag{5.25}$$

Further,

$$\int_{\sigma(\tilde{t}_0)}^{\sigma^2(\tilde{t}_0)} |\dot{\Delta} y(t)|dt = \int_{\sigma(\tilde{t}_0)}^{\sigma^2(\tilde{t}_0)} |C(t)[\dot{\tilde{y}}(\eta(t)) + \dot{\Delta} y(\eta(t))] - \tilde{C}(t)\dot{\tilde{y}}(\eta(t)) +$$

$$+a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu)|dt \leq ||\tilde{C}|| \int_{\sigma(\tilde{t}_0)}^{\sigma^2(\tilde{t}_0)} |\dot{\Delta} y(\eta(t))|dt + O(\varepsilon). \tag{5.26}$$

On the basis of the estimate (5.25), we get

$$\int_{\sigma(\tilde{t}_0)}^{\sigma^2(\tilde{t}_0)} |\Delta\dot{y}(\eta(t))|dt = \int_{\tilde{t}_0}^{\sigma(\tilde{t}_0)} \dot{\sigma}(t)|\Delta\dot{y}(t)|dt \le O(\varepsilon).$$

Consequently,

$$\int_{\sigma(\tilde{t}_0)}^{\sigma^2(\tilde{t}_0)} |\Delta\dot{y}(t)|dt \le O(\varepsilon).$$

Continuing this process, we prove that

$$\int_{\sigma(\tilde{t}_0)}^{r_2+\delta_2} |\Delta\dot{y}(t)|dt \le O(\varepsilon).$$

This inequality together with (5.25) yields (5.9). □

Lemma 5.3. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(r_2) \ge \tilde{t}_0$ and the conditions:*

$$\begin{gathered}\lim_{\omega\to\omega_0} f(\omega) = f_0^-, \quad \omega \in R_{\tilde{t}_0}^- \times O^2, \quad \omega_0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0))),\\ \lim_{t\to\tilde{t}_0} \tilde{C}(t) = C_{\tilde{t}_0}^-, \quad t \in R_{\tilde{t}_0}^-,\end{gathered} \tag{5.27}$$

be fulfilled. Let, besides, there exist neighborhoods $V^-(\tilde{t}_0)$, $V^-(\omega_1^0)$, $V^-(\omega_2^0)$, $\omega_1^0 = (\gamma_0, \tilde{y}(\gamma_0), \tilde{x}_0)$, $\omega_2^0 = (\gamma_0, \tilde{y}(\gamma_0), \tilde{\varphi}(\gamma_0))$ such that the functions $\dot{\gamma}(t)$, $t \in V^-(\tilde{t}_0)$, $\tilde{f}(\omega_1) - \tilde{f}(\omega_2)$, $(\omega_1, \omega_2) \in V^-(\omega_1^0) \times V^-(\omega_2^0)$ are bounded. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the relations are fulfilled (5.8), (5.9). *Moreover,*

$$\Delta y(\tilde{t}_0) = \varepsilon\big\{\delta x_0 - [C_{\tilde{t}_0}^- \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f_0^-]\delta t_0\big\} + o(\varepsilon\delta\mu). \tag{5.28}$$

Proof. The first part of the lemma, on the basis of Lemma 5.2, is proved analogously (see the proof of Lemma 2.3).

Now we prove (5.28). It is easy to see that (see (5.17))

$$\begin{aligned}\Delta y(\tilde{t}_0) = y(\tilde{t}_0, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{x}_0 = \varepsilon\big\{\delta x_0 - [C_{\tilde{t}_0}^- \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f_0^-]\delta t_0\big\} + \\ + \int_{t_0}^{\tilde{t}_0} [C(t)\dot{\varphi}(\eta(t)) + \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t))) - C_{\tilde{t}_0}^- \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) - f_0^-]dt + \\ + \int_{t_0}^{\tilde{t}_0} b(t, \varepsilon\delta\mu)dt.\end{aligned} \tag{5.29}$$

It is clear that

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,\tilde t_0]}|C(t)\dot\varphi(\eta(t))+\tilde f(t,\tilde y(t)+\Delta y(t),\varphi(\tau(t)))-C^-_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0))-f^-_0|=0$$

$$\text{uniformly for} \quad \delta\mu\in V^-.$$

Taking into account this and (5.24), from (5.29) we deduce (5.28). □

Lemma 5.4. *Let $\tau(\tilde t_0)<\tilde t_0$, $\tau(r_2)\ge\tilde t_0$ and the conditions*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V_1^+}\left|\int_{\tilde t_0}^{t_0}\tilde f(t,\tilde y(t),\tilde\varphi(\tau(t)))dt\right|<\infty, \tag{5.30}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V_1^+}\left|\int_{\gamma_0}^{\gamma(t_0)}|\tilde f(t,\tilde y(t)+\Delta y(t),\varphi(\tau(t)))-\tilde f(t,\tilde y(t),\tilde y(\tau(t)))]ds\right|<\infty,$$

$$V_1^+=\{\delta\mu\in V_1:\delta t_0\ge 0\} \tag{5.31}$$

be fulfilled. Then there exists a number $\varepsilon_3\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V_1^+$

$$\max_{t\in[t_0,r_2+\delta_2]}|\Delta y(t)|\le O(\varepsilon), \tag{5.32}$$

$$\int_{t_0}^{r_2+\delta_2}|\dot\Delta y(t)|\le O(\varepsilon). \tag{5.33}$$

Proof. By assumptions of the lemma it is guaranteed the existence of a number $\varepsilon_3\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V_1^+$

$$t_0<\gamma_0,\quad \gamma(t_0)<r_2+\delta_2, \tag{5.34}$$

$$\int_{\tilde t_0}^{t_0}|\tilde f(t,\tilde y(t),\tilde\varphi(\tau(t)))dt\le O(\varepsilon), \tag{5.35}$$

$$\int_{\gamma_0}^{\gamma(t_0)}|\tilde f(t,\tilde y(t)+\Delta y(t),\varphi(\tau(t)))-\tilde f(t,\tilde y(t),\tilde y(\tau(t)))|ds\le O(\varepsilon). \tag{5.36}$$

The function $\Delta y(t)$ on the interval $[t_0,r_2+\delta_2]$ satisfies the equation

$$\dot\Delta y(t)=\tilde C(t)h(t_0,\varepsilon\dot\delta\varphi,\dot\Delta y)(\eta(t))+\rho(t,t_0,\varepsilon\delta\mu)+a(t,\varepsilon\delta\mu)+b(t,\varepsilon\delta\mu), \tag{5.37}$$

where $a(t,\varepsilon\delta\mu)$, $b(t,\varepsilon\delta\mu)$, respectively, have the form (5.14), (5.15), while

$$\rho(t,t_0,\varepsilon\delta\mu)=C(t)h(t_0,\dot\varphi,\dot{\tilde y}+\dot\Delta y)(\eta(t))-\tilde C(t)h(\tilde t_0,\dot{\delta\varphi},\dot{\tilde y})(\eta(t))-$$
$$-\tilde C(t)h(t_0,\varepsilon\dot\delta\varphi,\dot\Delta y)(\eta(t)).$$

Rewrite the equation (5.37) in the form of the integral equation

$$\Delta y(t) = \Delta y(t_0) + \int_{t_0}^{t} \tilde{Y}_1(s,t,t_0)\tilde{C}(s)\delta\dot{\varphi}(\eta(s))ds +$$

$$+\int_{t_0}^{t} \tilde{Y}_2(s,t,t_0)[\rho(s,t_0,\varepsilon\delta\mu) + a(s,\varepsilon\delta\mu) + b(s,\varepsilon\delta\mu)]ds.$$

Hence

$$|\Delta y(t)| \le |\Delta y(t_0)| + O(\varepsilon) +$$

$$+||\tilde{Y}_2|| \Bigg[\int_{t_0}^{r_2+\delta_2} |\rho(s,t_0,\varepsilon\delta\mu)|ds + \int_{t_0}^{t} |a(s,\varepsilon\delta\mu)|ds + \int_{t_0}^{r_2+\delta_2} |b(s,\varepsilon\delta\mu)|ds \Bigg]. \quad (5.38)$$

We estimate $\Delta y(t_0)$:

$$|\Delta y(t_0)| = |\tilde{x}_0 + \varepsilon\delta x_0 - \tilde{y}(t_0)| =$$

$$= \Big|\tilde{x}_0 + \varepsilon\delta x_0 - \tilde{x}_0 - \int_{\tilde{t}_0}^{t_0} [\tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) + \tilde{f}(t,\tilde{y}(t),\tilde{\varphi}(\tau(t)))]dt\Big| \le O(\varepsilon). \quad (5.39)$$

In order to estimate

$$\rho(t_0,\varepsilon\delta\mu) = \int_{t_0}^{r_2+\delta_2} |\rho(s,t_0,\varepsilon\delta\mu)|ds,$$

we consider two cases.

Let $\eta(r_2) \ge \tilde{t}_0$ and assume that a number ε_3 is so small that for an arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_3] \times V_1^+$ the inequality $\sigma(t_0) < r_2 + \delta_2$ is fulfilled. We have

$$\rho(t_0,\varepsilon\delta\mu) = \int_{t_0}^{\sigma(\tilde{t}_0)} |\rho(s,t_0,\varepsilon\delta\mu)|ds + \int_{\sigma(\tilde{t}_0)}^{\sigma(t_0)} |\rho(s,t_0,\varepsilon\delta\mu)|ds +$$

$$+\int_{\sigma(t_0)}^{r_2+\delta_2} |\rho(s,t_0,\varepsilon\delta\mu)|ds = \sum_{i=1}^{3}\rho_i(t_0,\varepsilon\delta\mu). \quad (5.40)$$

Now we estimate every term of the expression (5.40).

It is clear that

$$\rho_1(t_0,\varepsilon\delta\mu) = \int_{t_0}^{\sigma(\tilde{t}_0)} |C(t)\dot{\varphi}(\eta(t)) - \tilde{C}(t)\dot{\tilde{\varphi}}(\eta(t)) - \varepsilon\tilde{C}(t)\delta\dot{\varphi}(\eta(t)))|dt =$$

$$= \varepsilon \int\limits_{t_0}^{\sigma(\tilde{t}_0)} |\delta C(t)|\,\dot{\varphi}(\eta(t))|dt \le O(\varepsilon).$$

Next, taking into consideration (5.1), we obtain

$$\rho_2(t_0, \varepsilon\delta\mu) = \int\limits_{\sigma(\tilde{t}_0)}^{\sigma(t_0)} |C(t)\dot{\varphi}(\eta(t)) - \tilde{C}(t)\dot{\tilde{y}}(\eta(t)) - \varepsilon\tilde{C}(t)\dot{\delta\varphi}(\eta(t)))|dt \le$$

$$\le \int\limits_{\tilde{t}_0}^{t_0} \dot{\sigma}(t)|C(\sigma(t))\dot{\varphi}(t) - \varepsilon\tilde{C}(\sigma(t))\dot{\delta\varphi}(t))|dt + \int\limits_{\tilde{t}_0}^{t_0} \dot{\sigma}(t)|\tilde{C}(t)\dot{\tilde{y}}(t)|dt \le$$

$$\le \|\dot{\sigma}\|\,\|\tilde{C}\| \int\limits_{\tilde{t}_0}^{t_0} |\dot{\tilde{y}}(t)|dt.$$

It is obvious that

$$\int\limits_{\tilde{t}_0}^{t_0} |\dot{\tilde{y}}(t)|dt \le O(\varepsilon). \tag{5.41}$$

For the last term we have

$$\rho_3(t_0, \varepsilon\delta\mu) = \int\limits_{\sigma(t_0)}^{r_2+\delta_2} |C(t)[\dot{\tilde{y}}(\eta(t)) + \dot{\Delta}y(\eta(t))] - \tilde{C}(t)\dot{\tilde{y}}(\eta(t)) - \tilde{C}(t)\dot{\Delta}y(\eta(t))|dt =$$

$$= \varepsilon \int\limits_{\sigma(t_0)}^{r_2+\delta_2} |\delta C(t)|\,|\dot{y}(t, \tilde{\mu} + \varepsilon\delta\mu)|dt \le O(\varepsilon).$$

Thus

$$\rho(t_0, \varepsilon\delta\mu) \le O(\varepsilon).$$

This inequality is also valid for $\eta(r_2) < \tilde{t}_0$. To see this, it suffices to choose a number δ_2 such that $\sigma(\tilde{t}_0) > r_2 + \delta_2$. After this $\rho(t_0, \varepsilon\delta\mu)$ is estimated analogously to $\rho_1(t_0, \varepsilon\delta\mu)$.

In an analogous way, using (5.34)-(5.36) (see the proof of Lemma 2.4), it is proved that

$$\int\limits_{t_0}^{t} |a(s, \varepsilon\delta\mu)|ds \le O(\varepsilon) + \int\limits_{t_0}^{t} L(s)|\Delta y(s)|ds, \tag{5.42}$$

$$\int\limits_{t_0}^{r_2+\delta_2} |b(s, \varepsilon\delta\mu)|ds \le O(\varepsilon). \tag{5.43}$$

On the basis of the obtained estimates, for $\Delta y(t)$ the following estimate can be written

$$|\Delta y(t)| \le O(\varepsilon) + \int_{t_0}^{t} L(s)|\Delta y(s)|ds.$$

Hence by Gronwall's lemma (5.32) is obtained.

Now on the basis of (5.32) we prove the second part of the lemma. We carry out the proof by the method of steps with rescpect to the delay $\eta(t)$.

After elementary transformation, taking into account (5.42), (5.43) and (5.41), we obtain

$$\int_{t_0}^{\sigma(t_0)} |\dot{\Delta y}(t)|dt = \int_{t_0}^{\sigma(\tilde{t}_0)} |C(t)\dot\varphi(\eta(t)) - \tilde{C}(t)\dot{\tilde\varphi}(\eta(t)) + a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu)|dt +$$
$$+ \int_{\sigma(\tilde{t}_0)}^{\sigma(t_0)} |C(t)\dot\varphi(\eta(t)) - \tilde{C}(t)\dot{\tilde y}(\eta(t)) + a(t,\varepsilon\delta\mu) + b(t,\varepsilon\delta\mu)|dt \le$$
$$\le \int_{t_0}^{\sigma(\tilde{t}_0)} [|a(t,\varepsilon\delta\mu)| + |b(t,\varepsilon\delta\mu)|]dt + \varepsilon \int_{t_0}^{\sigma(\tilde{t}_0)} |\tilde{C}(t)\dot{\delta\varphi}(\eta(t)) + \delta C(t)\dot\varphi(\eta(t))|dt +$$
$$+ \int_{\tilde{t}_0}^{t_0} \dot\sigma(t)|\tilde{C}(\sigma(t))\dot{\tilde y}(t)|dt + \int_{\tilde{t}_0}^{t_0} \dot\sigma(t)|\tilde{C}(\sigma(t))\dot\varphi(t)|dt \le$$
$$\le O(\varepsilon)\|\tilde{C}\|\,\|\dot\sigma\| \int_{\tilde{t}_0}^{t_0} |\dot{\tilde y}(t)|dt \le O(\varepsilon). \tag{5.44}$$

Further (see (5.26))

$$\int_{\sigma(t_0)}^{\sigma^2(t_0)} |\dot{\Delta y}(t)|dt \le \|\tilde{C}\| \int_{\sigma(t_0)}^{\sigma^2(t_0)} |\dot{\Delta y}(\eta(t))|dt + O(\varepsilon).$$

It is clear that

$$\int_{\sigma(t_0)}^{\sigma^2(t_0)} |\dot{\Delta y}(\eta(t))|dt = \int_{t_0}^{\sigma(t_0)} \dot\sigma(t)|\dot{\Delta y}(t)|dt \le O(\varepsilon).$$

Consequently,

$$\int_{\sigma(t_0)}^{\sigma^2(t_0)} |\dot{\Delta y}(t)|dt \le O(\varepsilon).$$

Continuing this process, we prove that

$$\int_{\sigma(t_0)}^{r_2+\delta_2} |\Delta \dot{y}(t)|dt \le O(\varepsilon).$$

This inequality together with (5.44) yields (5.33). □

Lemma 5.5. *Let $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(r_2) \ge \tilde{t}_0$ and the conditions*

$$\lim_{\omega\to\omega_0} \tilde{f}(\omega) = f_0^+, \quad \omega \in R_{\tilde{t}_0}^+ \times O^2, \quad \lim_{t\to\tilde{t}_0} \tilde{C}(t) = C_{\tilde{t}_0}^+, \quad t \in R_{\tilde{t}_0}^+, \tag{5.45}$$

be fullfilled. Let, moreover, there exist neighborhoods $V^+(\tilde{t}_0)$, $V^+(\omega_1^0)$, $V^+(\omega_2^0)$ such that the functions $\dot{\gamma}(t)$, $t \in V^+(\tilde{t}_0)$, $\tilde{f}(\omega_1) - \tilde{f}(\omega_2)$, $(\omega_1, \omega_2) \in V^+(\omega_1^0) \times V^+(\omega_2^0)$ are bounded. Then there exists a number $\varepsilon_3 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$ the relations (5.32), (5.33) are fulfilled. Moreover,

$$\Delta y(t_0) = \varepsilon\big\{\delta x_0 - [C_{\tilde{t}_0}^+ \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f_0^+]\delta t_0\big\} + o(\varepsilon\delta\mu). \tag{5.46}$$

Proof. The first part of the lemma, by the previous lemma, is proved analogously to Lemma 2.3.

Now we prove (5.46). It is obvious to see that (see (5.39))

$$\Delta y(t_0) = \varepsilon\big\{\delta x_0 - [C_{\tilde{t}_0}^+ \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f_0^+]\delta t_0\big\} +$$

$$+\int_{\tilde{t}_0}^{\tilde{t}_0} [C_{\tilde{t}_0}^+ \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) - \tilde{C}(t)\dot{\varphi}(\eta(t)) + f_0^+ - \tilde{C}(t)\dot{\varphi}(\eta(t)) - \tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))]dt. \tag{5.47}$$

It is clear that

$$\lim_{\varepsilon\to 0} \sup_{t\in[\tilde{t}_0, t_0]} |C_{\tilde{t}_0}^+ \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) - \tilde{C}(t)\dot{\varphi}(\eta(t)) + f_0^+ - \tilde{C}(t)\dot{\varphi}(\eta(t)) -$$

$$-\tilde{f}(t, \tilde{y}(t), \tilde{\varphi}(\tau(t)))| = 0 \quad \text{uniformly for} \quad \delta\mu \in V_1^+.$$

Hence, taking into account (5.47), it follows (5.46). □

Lemma 5.6. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (5.6) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the relations*

$$\max_{t\in[\tilde{t}_0, r_2+\delta_3]} |\Delta y(t)| \le O(\varepsilon), \tag{5.48}$$

$$\int_{\tilde{t}_0}^{r_2+\delta_3} |\Delta \dot{y}(t)|dt \le O(\varepsilon) \tag{5.49}$$

are fulfilled.

Lemma 5.7. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (5.27) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the relations (5.48), (5.49) and (5.28) are fulfilled.*

Lemma 5.8. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (5.30) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$ the relations*

$$\max_{t \in [t_0, r_2 + \delta_3]} |\Delta y(t)| \leq O(\varepsilon), \tag{5.50}$$

$$\int_{t_0}^{r_2 + \delta_3} |\dot{\Delta y}(t)| dt \leq O(\varepsilon) \tag{5.51}$$

are fulfilled.

Lemma 5.9. *Let $\tau(r_2) < \tilde{t}_0$ and the condition (5.45) be fulfilled. Then there exist numbers $\varepsilon_3 \in (0, \varepsilon_2]$, $\delta_3 \in (0, \delta_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$ the relations (5.50), (5.51) and (5.46) are fulfilled.*

These lemmas are proved analogously to Lemmas 2.3-2.9, respectively.

Lemma 5.10. *Let $\tau(\tilde{t}_0) = \tilde{t}_0$ and the condition*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{\delta\mu \in V_1^-} \left\{ \Big| \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt + \right.$$

$$\left. + \int_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t))) + \Delta y(\tau(t)))dt \Big| \right\} < \infty \tag{5.52}$$

be fulfilled. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the inequalites (5.8), (5.9) are fulfilled.

Proof. Let $\varepsilon_3 \in (0, \varepsilon_2]$ be so small that for an arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_3] \times V_1^-$

$$\Big| \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt +$$

$$+ \int_{\gamma(t_0)}^{\tilde{t}_0} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \tilde{y}(\tau(t))) + \Delta y(\tau(t)))dt \Big| \leq O(\varepsilon).$$

It is clear that $\gamma(t_0) \in [t_0, \tilde{t}_0]$. Therefore

$$\Delta y(\tilde{t}_0) = \varepsilon \delta x_0 + \int_{t_0}^{\tilde{t}_0} C(t)\dot{\varphi}(\eta(t))dt + \int_{t_0}^{\gamma(t_0)} \tilde{f}(t, \tilde{y}(t) + \Delta y(t), \varphi(\tau(t)))dt +$$

$$+\int\limits_{\gamma(t_0)}^{\tilde{t}_0}\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\tilde{y}(\tau(t)))+\Delta y(\tau(t)))dt+\int\limits_{t_0}^{\tilde{t}_0}b(t,\varepsilon\tilde{\mu})dt.$$

Hence, on the basis of the previous inequality and from the boundedness of the first integrand (see also (2.15)), we conclude that

$$|\Delta y(\tilde{t}_0)|\le O(\varepsilon).$$

After this, in a standard way (see the proof of Lemmas 5.2, 2.10), (5.8) and (5.9), are obtained □

Lemma 5.11. *Let $\tau(\tilde{t}_0)=\tilde{t}_0$ and there exist the finite limits:*

$$\begin{gathered}\lim_{\omega\to\omega_3^0}\tilde{f}(\omega)=f_2^-,\quad \lim_{\omega\to\omega_4^0}\tilde{f}(\omega)=f_3^-,\quad \omega\in R_{\tilde{t}_0}^-\times O^2,\\ \omega_3^0=(\tilde{t}_0,\tilde{x}_0,\tilde{x}_0),\quad \omega_4^0=(\tilde{t}_0,\tilde{x}_0,\tilde{\varphi}(\tilde{t}_0)),\end{gathered}\tag{5.53}$$

$$\lim_{t\to\tilde{t}_0}\tilde{C}(t)=C_{\tilde{t}_0}^-,\quad \lim_{t\to\tilde{t}_0}\dot{\gamma}(t)=\dot{\gamma}^-,\quad t\in R_{\tilde{t}_0}^-.\tag{5.54}$$

Then there exists a number $\varepsilon_3\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V_1^-$ the inequalites (5.8), (5.9) are valid. Moreover,

$$\Delta y(\tilde{t}_0)=\varepsilon\big\{\delta x_0-[C_{\tilde{t}_0}^-\dot{\tilde{\varphi}}(\eta(\tilde{t}_0))+f_3^-+(f_2^--f_3^-)\dot{\gamma}^-]\delta t_0\big\}+o(\varepsilon\delta\mu).$$

This lemma, by Lemma 5.10, is proved analogously to Lemma 2.11.

Lemma 5.12. *Let $\tau(\tilde{t}_0)=\tilde{t}_0$ and the conditions*

$$\lim_{\omega\to\omega_3^0}\tilde{f}(\omega)=f_2^-,\ \ \omega\in R_{\tilde{t}_0}^-\times O^2,\ \ \lim_{t\to\tilde{t}_0}\tilde{C}(t)=C_{\tilde{t}_0}^-,\ \ \lim_{t\to\tilde{t}_0}\dot{\gamma}(t)=1,\ t\in R_{\tilde{t}_0}^-,\tag{5.55}$$

be fulfilled. Let, moreover, there exist a neighborhood $V_1^-(\omega_4^0)$ such that the function $\tilde{f}(\omega)$, $\omega\in V_1^-(\omega_4^0)$, is bounded. Then there exists a number $\varepsilon_3\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_3]\times V_1^-$ the relations (5.8), (5.9) are valid. Moreover,

$$\Delta y(\tilde{t}_0)=\varepsilon\{\delta x_0-[C_{\tilde{t}_0}^-\dot{\tilde{\varphi}}(\eta(\tilde{t}_0))+f_2^-]\delta t_0\}+o(\varepsilon\delta\mu)$$

holds.

Lemma 5.13. *Let $\tau(t_0)=\tilde{t}_0$ and the conditions*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V_1^+}\int\limits_{\tilde{t}_0}^{t_0}|\tilde{f}(t,\tilde{y}(t),\tilde{y}(\tau(t)))|dt<\infty,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sup_{\delta\mu\in V_1^+}\int\limits_{t_0}^{\gamma(t_0)}|\tilde{f}(t,\tilde{y}(t)+\Delta y(t),\varphi(\tau(t)))-\tilde{f}(t,\tilde{y}(t),\tilde{y}(\tau(t)))|dt<\infty$$

be fulfilled. Then there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$ *the conditions* (5.32), (5.33) *are fulfilled.*

Lemma 5.14. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$ *and the conditions*

$$\lim_{\omega \to \omega_3^0} \tilde{f}(\omega) = f_2^+, \quad \omega \in R_{\tilde{t}_0}^+ \times O^2; \quad \lim_{t \to \tilde{t}_0} \tilde{C}(t) = C_{\tilde{t}_0}^+, \quad t \in R_{\tilde{t}_0}^+, \qquad (5.56)$$

be fulfilled. Let, moreover, there exist neighborhoods $V^+(\tilde{t}_0)$, $V^+(\omega_4^0)$ *such that the functions* $\dot{\gamma}(t)$, $t \in V^+(\tilde{t}_0)$, $\tilde{f}(\omega)$, $\omega \in V^+(\omega_4^0)$ *are bounded. Then there exists a number* $\varepsilon_3 \in (0, \varepsilon_2]$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$ *the conditions* (5.32), (5.33) *are fulfilled. Moreover,*

$$\Delta y(t_0) = \varepsilon\{\delta x_0 - [C_{\tilde{t}_0}^+ \dot{\tilde{\varphi}}(\eta(\tilde{t}_0)) + f_2^+]\delta t_0\} + o(\varepsilon\delta\mu).$$

These lemmas are proved analogously to Lemmas 2.12–2.14, respectively.

6. Differentiability of Solution

Lemma 6.1. *Let* $\tilde{x}(t)$ *be the solution corresponding to the element* $\tilde{\mu} \in A_1$, *defined on* $[\rho(\tilde{t}_0), \tilde{t}_1] \subset (\rho(a), b)$. *Let,* K_1 *contain some neighborhood of the set* $\tilde{\varphi}(J_2) \cup \tilde{x}([\tilde{t}_0, \tilde{t}_1])$. *Then there exist numbers* $\delta_2 > 0$, $\varepsilon_2 > 0$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V_1$ *to the element* $\tilde{\mu} + \varepsilon\delta\mu \in A_1$ *there corresponds the solution* $x(t, \tilde{\mu} + \varepsilon\delta\mu)$, *defined on* $[\rho(t_0), \tilde{t}_1 + \delta_2] \subset (\rho(a), b)$. *Moreover,*

$$\begin{gathered} x(t, \tilde{\mu} + \varepsilon\delta\mu) \in K_1, \quad |\dot{x}(t, \tilde{\mu} + \varepsilon\delta\mu)| \le m(t), \quad t \in [\rho(t_0), \tilde{t}_1 + \delta_2], \\ m(\cdot) \in L_1([\rho(a), b], R_0^+). \end{gathered} \qquad (6.1)$$

This lemma, by Lemma 5.1, is proved analogously to Lemma 3.5.

In the sequel we assume that the trajectory $\tilde{x}(t)$ is defined on the whole interval $[\rho(\tilde{t}_0), \tilde{t}_1 + \delta_2]$ (see Remark 3.1).

We define the function

$$\Delta x(t) = \Delta x(t, \varepsilon\delta\mu) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\rho(a), s_1), \\ x(t, \tilde{\mu} + \varepsilon\delta\mu) - \tilde{x}(t), & t \in [s_1, \tilde{t}_1 + \delta_2], \\ & s_1 = \min\{t_0, \tilde{t}_0\}. \end{cases} \qquad (6.2)$$

It is obvious that

$$\Delta x(t) = \Delta y(t), \quad t \in [s_2, \tilde{t}_1 + \delta_2], \quad s_2 = \max\{t_0, \tilde{t}_0\}. \qquad (6.3)$$

Theorem 6.1. *Let* $\tilde{f} \in E_1(J \times O^2, R^n)$[8], $\tau(\tilde{t}_0) < \tilde{t}_0$, $\tau(\tilde{t}_1) > \tilde{t}_0$, $\eta(\tilde{t}_1) > \tilde{t}_0$, $\gamma_0 < \eta^{m_1}(\tilde{t}_1)$, $\sigma_0 < \eta^{m_2}(\tilde{t}_1)$[9], $\sigma_0 = \sigma(\tilde{t}_0)$. *Let, moreover, there exist the finite limits*

$$\lim_{\omega \to \omega_0} \tilde{f}(\omega) = f_0^-, \quad \omega \in R_{\tilde{t}_0}^- \times O^2, \quad \lim_{t \to \tilde{t}_0} \tilde{C}(t) = C_{\tilde{t}_0}^-, \quad t \in R_{\tilde{t}_0}^-; \qquad (6.4)$$

[8] In all theorems of this section, in we will assume that $\tilde{f} \in E_1(J \times O^2, R^n)$.

[9] Everywhere we assume that $m_1 = m(\gamma_0, \tilde{t}_1)$, $m_2 = m(\sigma_0, \tilde{t}_1)$.

$$\lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^0)}[\tilde f(\omega_1)-\tilde f(\omega_2)] = f_1^-, \ \ \omega_i\in R_{\tilde t_0}^-\times O^2, \ \ i=1,2,$$
$$\lim_{t\to\tilde t_0}\dot\gamma(t)=\dot\gamma^-, \ \ t\in R_{\tilde t_0}^-; \tag{6.5}$$

$$\lim_{t\to\sigma^i(\gamma_0)}\tilde C(t)=C_{\sigma^i(\gamma_0)}^-, \ \ t\in R_{\sigma^i(\gamma_0)}^-, \ \ i=1,\dots,m_1; \tag{6.6}$$
$$\lim_{t\to\sigma^i(\sigma_0)}\tilde C(t)=C_{\sigma^i(\sigma_0)}^-, \ \ t\in R_{\sigma^i(\sigma_0)}^-, \ \ i=0,\dots,m_2. \tag{6.7}$$

Then there exist numbers $\delta_3>0$, $\varepsilon_3>0$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in [\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$

$$\Delta x(t,\varepsilon\delta\mu)=\varepsilon\delta x(t,\delta\mu)+o(t,\varepsilon\delta\mu), \tag{6.8}$$

where

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C_{\tilde t_0}^-\dot{\tilde\varphi}(\eta(\tilde t_0))+f_0^-]+$$
$$+Y_{\sigma_0}^-(t)C_{\sigma_0}^-[C_{\tilde t_0}^-\dot{\tilde\varphi}(\eta(\tilde t_0))+f_0^- -\dot{\tilde\varphi}(\tilde t_0)]+Y_{\gamma_0}^-(t)f_1^-\dot\gamma^-\big\}\delta t_0+\beta(t,\delta\mu), \tag{6.9}$$
$$\beta(t,\delta\mu)=\int_{\tau(\tilde t_0)}^{\tilde t_0}Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\delta\varphi(s)\dot\gamma(s)ds+$$
$$+\int_{\eta(\tilde t_0)}^{\tilde t_0}Y(\sigma(s),t)\tilde C(\sigma(s))\dot\delta\varphi(s)\dot\sigma(s)ds+\int_{\tilde t_0}^{t}Y(s,t)\big(\delta C(s)\dot{\tilde x}(\eta(s))+\delta f[s]\big)ds;$$

$\Phi(s,t)$, $Y(s,t)$ are matrix functions satisfying the system

$$\begin{cases}\dfrac{\partial\Phi(s,t)}{\partial s}=-Y(s,t)\tilde f_{x_1}[s]-Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\dot\gamma(s),\\ Y(s,t)=\Phi(s,t)+Y(\sigma(s),t)\tilde C(\sigma(s))\dot\sigma(s) \quad s\in[\tilde t_0,t].\end{cases}$$

Moreover,

$$Y(s,t)=\begin{cases}E, & s=t,\\ \Theta, & s>t;\end{cases}$$
$$Y_{\gamma_0}^-(t)=\sum_{i=0}^{m_1}\Phi(\sigma^i(\gamma_0),t)C_{i\gamma_0}^-, \ \ C_{i\gamma_0}^-=\prod_{j=1}^{i}C_{\sigma^j(\gamma_0)}^-\dot\sigma(\sigma^{j-1}(\gamma_0)), \ \ i=1,\dots,m_1,$$
$$Y_{\sigma_0}^-(t)=\sum_{i=0}^{m_2}\Phi(\sigma^i(\sigma_0),t)C_{i\sigma_0}^-, \ \ C_{i\sigma_0}^-=\prod_{j=1}^{i}C_{\sigma^j(\sigma_0)}^-\dot\sigma(\sigma^{j-1}(\sigma_0)), \ \ i=1,\dots,m_2,$$
$$C_{0\sigma_0}^-=C_{0\gamma_0}^-=E.$$

Proof. On the basis of Lemma 5.3 there exists a number $\varepsilon_3 \in (0, \varepsilon_2]$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^-$ the relations (see (5.8), (5.9), (5.28), (6.3))

$$\max_{t\in[\tilde{t}_0,\tilde{t}_1+\delta_2]} |\Delta x(t)| \le O(\varepsilon), \tag{6.10}$$

$$\int_{\tilde{t}_0}^{\tilde{t}_1+\delta_2} |\dot{\Delta} x(t)| dt \le O(\varepsilon), \tag{6.11}$$

$$\Delta x(\tilde{t}_0) = \varepsilon\big\{\delta x_0 - [C_{\tilde{t}_0}^- \dot{\varphi}(\eta(\tilde{t}_0)) + f_0^-]\delta t_0\big\} + o(\varepsilon\delta\mu) \tag{6.12}$$

are fulfilled.

Let $\delta_3 \in (0, \delta_2]$ be so small that

$$\gamma_0 < \tilde{t}_1 - \delta_3, \quad \sigma_0 < \tilde{t}_1 - \delta_3.$$

It is easy to see that the function $\Delta x(t)$, $t \in [\rho(\tilde{t}_0), \tilde{t}_1 + \delta_3]$, on the interval $[\tilde{t}_0, \tilde{t}_1 + \delta_3]$ satisfies the equation

$$\dot{\Delta} x(t) = \tilde{f}_{x_1}[t]\Delta x(t) + \tilde{f}_{x_2}[t]\Delta x(\tau(t)) + \tilde{C}(t)\dot{\Delta} x(\eta(t)) +$$
$$+\varepsilon\big(\delta C(t)\dot{\tilde{x}}(\eta(t)) + \delta f[t]\big) + \sum_{i=1}^{3} R_i(t, \varepsilon\delta\mu), \tag{6.13}$$

where $R_i(t, \varepsilon\delta\mu)$, $i = 1, 2$, respectively, have the form (3.26), (3.27), while

$$R_3(t, \varepsilon\delta\mu) = \varepsilon\delta C(t)\dot{\Delta} x(\eta(t)). \tag{6.14}$$

A solution of the equation (6.13), by means of the Cauchy formula (see Lemma 4.1) may be represented as

$$\Delta x(t) = \Phi(\tilde{t}_0, t)\Delta x(\tilde{t}_0) + \varepsilon \int_{\tilde{t}_0}^{t} Y(s, t)\big(\delta C(s)\dot{\tilde{x}}(\eta(s)) + \delta f[s]\big) ds +$$
$$+ \sum_{i=-1}^{3} h_i(t, \tilde{t}_0, \varepsilon\delta\mu), \quad t \in [\tilde{t}_0, \tilde{t}_1 + \delta_3], \tag{6.15}$$

where

$$h_{-1}(t, \tilde{t}_0, \varepsilon\delta\mu) = \int_{\eta(\tilde{t}_0)}^{\tilde{t}_0} Y(\sigma(s), t)\tilde{C}(\sigma(s))\dot{\Delta} x(s)\dot{\sigma}(s) ds, \tag{6.16}$$

$$h_0(t, \tilde{t}_0, \varepsilon\delta\mu) = \int_{\tau(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma(s), t)\tilde{f}_{x_2}[\gamma(s)]\Delta x(s)\dot{\gamma}(s) ds, \tag{6.17}$$

$$h_i(t,\tilde{t}_0,\varepsilon\delta\mu)=\int\limits_{\tilde{t}_0}^{t}Y(s,t)R_i(s,\varepsilon\delta\mu)ds,\quad i=1,2,3. \tag{6.18}$$

It is obvious (see (6.12), Lemma 4.3), that

$$\Phi(\tilde{t}_0,t)\Delta x(\tilde{t}_0)=\varepsilon\Phi(\tilde{t}_0,t)\{\delta x_0-[C^-_{\tilde{t}_0}\dot{\tilde{\varphi}}(\eta(\tilde{t}_0))+f^-_0]\delta t_0\}+o(t,\varepsilon\delta\mu). \tag{6.19}$$

Now we transform $h_{-1}(t,\tilde{t}_0,\varepsilon\delta\mu)$:

$$h_{-1}(t,\tilde{t}_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{\eta(\tilde{t}_0)}^{t_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\dot{\delta}\varphi(s)\dot{\sigma}(s)ds+$$

$$+\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\dot{\Delta}x(s)\dot{\sigma}(s)ds=\varepsilon\int\limits_{\eta(\tilde{t}_0)}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\dot{\delta}\varphi(s)\dot{\sigma}(s)ds+$$

$$+\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\dot{\Delta}x(s)\dot{\sigma}(s)ds+o(t,\varepsilon\delta\mu).$$

Further, using the equality $x(t,\tilde{\mu}+\varepsilon\delta\mu)=y(t,\tilde{\mu}+\varepsilon\delta\mu),\ t\in[t_0,\tilde{t}_0]$, we get

$$\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\dot{\Delta}x(s)\dot{\sigma}(s)ds=$$

$$=\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\big[\dot{y}(t,\tilde{\mu}+\varepsilon\delta\mu)-\dot{\tilde{\varphi}}(s)\big]\dot{\sigma}(s)ds=$$

$$=\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\big[C(s)\dot{\varphi}(\eta(s))+\tilde{f}(s,\tilde{y}(s)+\Delta y(s),\varphi(\tau(s)))+b(s,\varepsilon\delta\mu)-$$

$$-\dot{\tilde{\varphi}}(s)\big]\dot{\sigma}(s)ds=\int\limits_{t_0}^{\tilde{t}_0}Y(\sigma(s),t)\tilde{C}(\sigma(s))\big[\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))+$$

$$+\tilde{f}(s,\tilde{y}(s)+\Delta y(s),\varphi(\tau(s)))-\dot{\tilde{\varphi}}(s)\big]\dot{\sigma}(s)ds+o(t,\varepsilon\delta\mu).$$

From assumptions of the theorem (see Lemma 4.4) it follows

$$\lim_{\substack{\varepsilon\to 0\\ s\in[t_0,\tilde{t}_0]}}Y(\sigma(s),t)\tilde{C}(\sigma(s))\big[\tilde{C}(s)\dot{\tilde{\varphi}}(\eta(s))+\tilde{f}(s,\tilde{y}(s)+$$

$$+\Delta y(s),\varphi(\tau(s)))-\dot{\tilde{\varphi}}(s)\big]\dot{\sigma}(s)=Y^-_{\sigma_0}(t)C^-_{\sigma_0}\big[C^-_{\tilde{t}_0}\dot{\tilde{\varphi}}(\eta(\tilde{t}_0))+f^-_0-\dot{\tilde{\varphi}}(\tilde{t}_0)\big]\dot{\sigma}(\tilde{t}_0).$$

After this, in a standard way, we prove

$$\int_{t_0}^{\tilde t_0} Y(\sigma(s),t)\tilde C(\sigma(s))\dot\Delta x(s)\dot\sigma(s)ds =$$

$$= -\varepsilon Y^-_{\sigma_0}(t)C^-_{\sigma_0}\left[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0)) + f_0^- - \dot\varphi(\tilde t_0)\right]\dot\sigma(\tilde t_0)\delta t_0 + o(t,\varepsilon\delta\mu).$$

Consequently,

$$h_{-1}(t,\tilde t_0,\varepsilon\delta\mu) = -\varepsilon Y^-_{\sigma_0}(t)C^-_{\sigma_0}\left[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0)) + f_0^- - \dot\varphi(\tilde t_0)\right]\dot\sigma(\tilde t_0)\delta t_0 +$$

$$+\varepsilon\int_{\eta(\tilde t_0)}^{\tilde t_0} Y(\sigma(s),t)\tilde C(\sigma(s))\dot\delta\varphi(s)\dot\sigma(s)ds + o(t,\varepsilon\delta\mu). \tag{6.20}$$

For $h_i(t,\tilde t_0,\varepsilon\delta\mu),\ i=0,1,2$, using (6.1), (6.5) and (6.11) we obtain (see the proof of Theorem 3.1)

$$h_0(t,\tilde t_0,\varepsilon\delta\mu) = \varepsilon\int_{\tau(\tilde t_0)}^{\tilde t_0} Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\delta\varphi(s)\dot\gamma(s)ds +$$

$$+\int_{\gamma(t_0)}^{\gamma_0} Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds + o(t,\varepsilon\delta\mu), \tag{6.21}$$

$$h_1(t,\tilde t_0,\varepsilon\delta\mu) = -\varepsilon Y^-_{\gamma_0}(t)f_1^-\dot\gamma^-\delta t_0 -$$

$$-\int_{\gamma(t_0)}^{\gamma_0} Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds + o(t,\varepsilon\delta\mu), \tag{6.22}$$

$$h_2(t,\tilde t_0,\varepsilon\delta\mu) = o(t,\varepsilon\delta\mu). \tag{6.23}$$

It remains to estimate $h_3(t,\tilde t_0,\varepsilon\delta\mu)$ (see (6.14))

$$|h_3(t,\tilde t_0,\varepsilon\delta\mu)| = \varepsilon\alpha_3\|Y\|\int_{\tilde t_0}^{\tilde t_1+\delta_3}|\dot\Delta x(\eta(t))|dt.$$

It is clear that (see (5.10), (6.1), (6.11))

$$\int_{\tilde t_0}^{\tilde t_1+\delta_3}|\dot\Delta x(\eta(t))|dt = \varepsilon\int_{\tilde t_0}^{\sigma(t_0)}|\dot\delta\varphi(\eta(t))|dt +$$

$$+\int_{\sigma(t_0)}^{\sigma(\tilde t_0)}\left[|\dot x(\eta(t),\tilde\mu+\varepsilon\delta\mu)| + |\dot\varphi(\eta(t))|\right]dt +$$

$$+\int\limits_{\sigma(\tilde t_0)}^{\tilde t_1+\delta_3}|\dot\Delta x(\eta(t))|dt \le O(\varepsilon)+\int\limits_{\sigma(t_0)}^{\sigma(\tilde t_0)} m(\eta(t))dt.$$

Thus

$$h_3(t,\tilde t_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu). \tag{6.24}$$

From (6.15), taking into account (6.19)-(6.24), the formula (6.8) is obtained, where $\delta x(t,\delta\mu)$ has the form (6.9). □

Theorem 6.2. *Let $\tau(\tilde t_0)<\tilde t_0,\ \tau(\tilde t_1)>\tilde t_0,\ \eta(\tilde t_1)>\tilde t_0;\ \gamma_0<\eta^{m_1}(\tilde t_1),\ \sigma_0<\eta^{m_2}(\tilde t_1)$. Let, moreover, there exist the finite limits*

$$\lim_{\omega\to\omega_0}\tilde f(\omega)=f_0^+,\ \ \omega\in R^+_{\tilde t_0}\times O^2,\ \ \lim_{t\to\tilde t_0}\tilde C(t)=C^+_{\tilde t_0},\ \ t\in R^+_{\tilde t_0}; \tag{6.25}$$

$$\lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^0)}[\tilde f(\omega_1)-\tilde f(\omega_2)]=f_1^+,\ \ \omega_i\in R^+_{\tilde t_0}\times O^2,\ \ i=1,2,$$

$$\lim_{t\to\tilde t_0}\dot\gamma(t)=\dot\gamma^+,\ \ t\in R^+_{\tilde t_0}; \tag{6.26}$$

$$\lim_{t\to\sigma^i(\gamma_0)}\tilde C(t)=C^+_{\sigma^i(\gamma_0)},\ \ t\in R^+_{\sigma^i(\gamma_0)},\ \ i=1,\dots,m_1; \tag{6.27}$$

$$\lim_{t\to\sigma^i(\sigma_0)}\tilde C(t)=C^+_{\sigma^i(\sigma_0)},\ \ t\in R^+_{\sigma^i(\sigma_0)},\ \ i=0,\dots,m_2. \tag{6.28}$$

Then there exist numbers $\delta_3>0$, $\varepsilon_3>0$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^+$ the equality (6.8) is valid, where

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^+]+$$
$$+Y^+_{\sigma_0}(t)C^+_{\sigma_0}[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^+-\dot\varphi(\tilde t_0)]\dot\sigma(\tilde t_0)+Y^+_{\gamma_0}(t)f_1^+\dot\gamma^+\big\}\delta t_0+\beta(t,\delta\mu), \tag{6.29}$$

$$Y^+_{\gamma_0}(t)=\sum_{i=0}^{m_1}\Phi(\sigma^i(\gamma_0),t)C^+_{i\gamma_0},\ \ C^+_{i\gamma_0}=\prod_{j=1}^{i}C^+_{\sigma^j(\gamma_0)}\dot\sigma(\sigma^{j-1}(\gamma_0)),\ i=1,\dots,m_1,$$

$$Y^+_{\sigma_0}(t)=\sum_{i=0}^{m_2}\Phi(\sigma^i(\sigma_0),t)C^+_{i\sigma_0},\ \ C^+_{i\sigma_0}=\prod_{j=1}^{i}C^+_{\sigma^j(\sigma_0)}\dot\sigma(\sigma^{j-1}(\sigma_0)),\ i=1,\dots,m_2,$$

$$C^+_{0\sigma_0}=C^+_{0\gamma_0}=E.$$

Proof. By assumptions of the theorem the conditions of Lemma 5.5 hold. Therefore there exists a number $\bar\varepsilon\in(0,\varepsilon_2]$ such that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\bar\varepsilon]\times V_1^+$ the conditions

$$\max_{t\in[t_0,\tilde t_1+\delta_2]}|\Delta x(t)|\le O(\varepsilon), \tag{6.30}$$

$$\int\limits_{t_0}^{\tilde t_1+\delta_2}|\dot\Delta x(t)|dt\le O(\varepsilon), \tag{6.31}$$

$$\Delta x(t_0) = \varepsilon\big\{\delta x_0 - [C^+_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0)) + f^+_0]\delta t_0\big\} + o(\varepsilon\delta\mu) \tag{6.32}$$

are fulfilled.

Let the numbers $\varepsilon_3 \in (0, \bar\varepsilon]$, $\delta_3 \in (0, \delta_2]$ be so small that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_3] \times V_1^+$

$$\gamma(t_0) < \tilde t_1 - \delta_3, \quad \sigma(t_0) < \tilde t_1 - \delta_3, \quad \gamma_0 > t_0, \quad \sigma_0 > t_0.$$

The function $\Delta x(t)$, $t \in [\rho(t_0), \tilde t_1 + \delta_3]$, on the interval $[t_0, \tilde t_1 + \delta_3]$ satisfies the equation (6.15), therefore it may be represented as (see (6.16)–(6.18))

$$\Delta x(t) = \Phi(t_0, t)\Delta x(t_0) + \varepsilon\int_{t_0}^{t} Y(s,t)\big(\delta C(s)\dot{\tilde x}(\eta(s)) + \delta f[s]\big)ds +$$

$$+\sum_{i=-1}^{3} h_i(t, t_0, \varepsilon\delta\mu), \quad t \in [t_0, \tilde t_1 + \delta_3]. \tag{6.33}$$

Since $t_0 \in [\tilde t_0, \tau(\tilde t_1 + \delta_3)]$, the function $\Phi(s,t)$ is continuous on $[\tilde t_0, \tau(\tilde t_1 + \delta_3)] \times [\tilde t_1 - \delta_3, \tilde t_1 + \delta_3]$ (see Lemma 4.3). This allows us to write

$$\Phi(t_0, t)\Delta x(t_0) = \varepsilon\Phi(\tilde t_0, t)\big\{\delta x_0 - [C^+_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0)) + f^+_0]\delta t_0\big\} + o(t, \varepsilon\delta\mu). \tag{6.34}$$

Now we transform $h_{-1}(t, t_0, \varepsilon\delta\mu)$ (see (6.16), (6.26), (6.30)):

$$h_{-1}(t, t_0, \varepsilon\delta\mu) = \varepsilon\int_{\eta(t_0)}^{\tilde t_0} Y(\sigma(s), t)\tilde C(\sigma(s))\delta\dot\varphi(s)\dot\sigma(s)ds +$$

$$+\int_{\tilde t_0}^{t_0} Y(\sigma(s), t)\tilde C(\gamma(s))\dot\Delta x(s)\dot\sigma(s)ds = \varepsilon\int_{\eta(\tilde t_0)}^{\tilde t_0} Y(\sigma(s), t)\tilde C(\sigma(s))\delta\dot\varphi(s)\dot\sigma(s)ds +$$

$$+\int_{\gamma_0}^{\gamma(t_0)} Y(s,t)\tilde C(s)\dot\Delta x(\eta(s))ds + o(t, \varepsilon\delta\mu).$$

It is easy to see that

$$\int_{\tilde t_0}^{t_0} Y(\sigma(s), t)\tilde C(\sigma(s))\dot\Delta x(s)\dot\sigma(s)ds =$$

$$= \int_{\tilde t_0}^{t_0} Y(\sigma(s), t)\tilde C(\sigma(s))\big[\dot\varphi(s) - \dot{\tilde x}(s)\big]\dot\sigma(s)ds =$$

$$= \int_{\tilde t_0}^{t_0} Y(\sigma(s), t)\tilde C(\sigma(s))\big[\dot\varphi(s) - \tilde C(s)\dot{\tilde\varphi}(\eta(s)) - \tilde f(s, \tilde x(s), \tilde\varphi(\tau(s)))\big]\dot\sigma(s)ds.$$

The assumptions of the theorem allow us to conclude (see Lemma 4.4)

$$\lim_{\substack{\varepsilon\to 0\\ s\in[\tilde t_0,t_0]}} Y(\sigma(s),t)\tilde C(\sigma(s))\big[\dot\varphi(s)-\tilde C(s)\dot{\tilde\varphi}(\eta(s))-\tilde f(s,\tilde x(s),\tilde\varphi(\tau(s)))\big]\dot\sigma(s)=$$

$$=Y^+_{\sigma_0}(t)C^+_{\sigma_0}\big[\dot{\tilde\varphi}(\tilde t_0)-C^+_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0))-f^+_0\big]\dot\sigma(\tilde t_0).$$

After this, in a standard way, we prove

$$\int_{\tilde t_0}^{t_0} Y(\sigma(s),t)\tilde C(\sigma(s))\dot\Delta x(s)\dot\sigma(s)ds=$$

$$=-\varepsilon Y^+_{\sigma_0}(t)C^+_{\sigma_0}\big[C^+_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0))+f^+_0-\dot{\tilde\varphi}(\tilde t_0)\big]\dot\sigma(\tilde t_0)\delta t_0+o(t,\varepsilon\delta\mu).$$

Thus

$$h_{-1}(t,t_0,\varepsilon\delta\mu)=-\varepsilon Y^+_{\sigma_0}(t)C^+_{\sigma_0}\big[C^+_{\tilde t_0}\dot{\tilde\varphi}(\eta(\tilde t_0))+f^+_0-\dot{\tilde\varphi}(\tilde t_0)\big]\dot\sigma(\tilde t_0)\delta t_0+$$

$$+\varepsilon\int_{\eta(\tilde t_0)}^{\tilde t_0} Y(\sigma(s),t)\tilde C(\sigma(s))\dot\delta_\varphi(s)\dot\sigma(s)ds+o(t,\varepsilon\delta\mu).$$

For $h_i(t,t_0,\varepsilon\delta\mu)$, $i=0,1,2$, using (6.1), (6.26) and (6.30), we obtain

$$h_0(t,t_0,\varepsilon\delta\mu)=\varepsilon\int_{\tau(\tilde t_0)}^{\tilde t_0} Y(\gamma(s),t)\tilde f_{x_2}[\gamma(s)]\delta\varphi(s)\dot\gamma(s)ds+$$

$$+\int_{\gamma_0}^{\gamma(t_0)} Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds+o(t,\varepsilon\delta\mu),\qquad(6.35)$$

$$h_1(t,t_0,\varepsilon\delta\mu)=-\varepsilon Y^+_{\gamma_0}(t)f^+_1\dot\gamma^+\delta t_0-$$

$$-\int_{\gamma_0}^{\gamma(t_0)} Y(s,t)\tilde f_{x_2}[s]\Delta x(\tau(s))ds+o(t,\varepsilon\delta\mu),\qquad(6.36)$$

$$h_2(t,t_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu).\qquad(6.37)$$

It remains to estimate $h_3(t,t_0,\varepsilon\delta\mu)$:

$$|h_3(t,t_0,\varepsilon\delta\mu)|=\varepsilon\alpha_3\|Y\|\int_{t_0}^{\tilde t_1+\delta_3}|\dot\Delta x(\eta(t))|dt.$$

It is clear that (see (6.1), (6.31))

$$\int_{t_0}^{\tilde t_1+\delta_3}|\dot\Delta x(\eta(t))|dt\le\varepsilon\int_{t_0}^{\sigma_0}|\dot\delta\varphi(\eta(t))|dt+\int_{\sigma_0}^{\sigma(t_0)}\big[|\dot{\tilde x}(\eta(t))|+|\dot\varphi(\eta(t))|\big]dt+$$

$$+\int\limits_{\sigma(t_0)}^{\tilde{t}_1+\delta_3}|\dot{\Delta}x(\eta(t))|dt \le O(\varepsilon)+\int\limits_{\sigma_0}^{\sigma(t_0)} m(\eta(t))dt.$$

Thus

$$h_3(t,t_0,\varepsilon\delta\mu)=o(t,\varepsilon\delta\mu). \tag{6.38}$$

From (6.33), taking into account (6.34)-(6.38), we obtain the formula (6.8), where $\delta x(t,\delta\mu)$ has the form (6.29). □

Theorem 6.3. *Let* $\tau(\tilde{t}_0)<\tilde{t}_0$, $\tau(\tilde{t}_1)>\tilde{t}_0$, $\eta(\tilde{t}_1)>\tilde{t}_0$; $\gamma_0<\eta^{m_1}(\tilde{t}_1)$, $\sigma_0<\eta^{m_2}(\tilde{t}_1)$; *the function* $\dot{\tau}(t)$ *is continuous at point* $\tilde{t}_0$; *the function* $\tilde{f}(\omega)$ *is continuous at points* ω_0, ω_1^0, ω_2^0; *the function* $\tilde{C}(t)$ *is continuous at points* $\tilde{t}_0$, $\sigma^i(\gamma_0)$, $i=1,\dots,m_1$, $\sigma^i(\sigma_0)$, $i=0,\dots,m_2$. *Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\times[0,\varepsilon_3]\times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde{t}_0,t)\delta x_0-\big\{\Phi(\tilde{t}_0,t)[\tilde{C}(\tilde{t}_0)\dot{\varphi}(\eta(\tilde{t}_0))+\tilde{f}(\omega_0)]+$$

$$+Y(\sigma_0,t)\tilde{C}(\sigma_0)[\tilde{C}(\tilde{t}_0)\dot{\varphi}(\eta(\tilde{t}_0))+\tilde{f}(\omega_0)-\dot{\varphi}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0)+$$

$$+Y(\gamma_0,t)[\tilde{f}(\omega_1^0-\tilde{f}(\omega_2^0)]\dot{\gamma}(\tilde{t}_0)\big\}\delta t_0+\beta(t,\delta\mu).$$

Finally we note that the proof of the theorems given below on the basis of Lemmas 5.6–5.9, 5.11, 5.12, 5.14 are carried out analogously (see § 3, the proof of theorems 6.1, 6.2).

Theorem 6.4. *Let* $\tau(\tilde{t}_1)<\tilde{t}_0,\eta(\tilde{t}_1)>\tilde{t}_0$, $\sigma_0<\eta^{m_2}(\tilde{t}_1)$ *and the conditions* (6.4), (6.7) *be fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde{t}_0,t)\delta x_0-\big\{\Phi(\tilde{t}_0,t)[C_{\tilde{t}_0}^-\dot{\varphi}(\eta(\tilde{t}_0))+f_0^-]+$$

$$+Y_{\sigma_0}^-(t)C_{\sigma_0}^-[C_{\tilde{t}_0}^-\dot{\varphi}(\eta(\tilde{t}_0))+f_0^--\dot{\varphi}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0)\big\}\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.5. *Let* $\tau(\tilde{t}_1)<\tilde{t}_0,\eta(\tilde{t}_1)>\tilde{t}_0$, $\sigma_0<\eta^{m_2}(\tilde{t}_1)$ *and the conditions* (6.25), (6.28) *be fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_3,\tilde{t}_1+\delta_3]\times[0,\varepsilon_3]\times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde{t}_0,t)\delta x_0-\big\{\Phi(\tilde{t}_0,t)[C_{\tilde{t}_0}^+\dot{\varphi}(\eta(\tilde{t}_0))+f_0^+]+$$

$$+Y_{\sigma_0}^+(t)C_{\sigma_0}^+[C_{\tilde{t}_0}^+\dot{\varphi}(\eta(\tilde{t}_0))+f_0^+-\dot{\varphi}(\tilde{t}_0)]\dot{\sigma}(\tilde{t}_0)\big\}\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.6. *Let* $\tau(\tilde t_1) < \tilde t_0,\ \eta(\tilde t_1) > \tilde t_0,\ \sigma_0 < \eta^{m_2}(\tilde t_1)$*, the function* $\dot\tau(t)$ *be continuous at the point* $\tilde t_0$*, the function* $\tilde f(\omega)$ *be continuous at the point* ω_0*, the function* $\tilde C(t)$ *be continuous at the points* $\tilde t_0,\ \sigma^i(\sigma_0),\ i = 0,\dots,m_2$*. Then there exist numbers* $\delta_3 > 0,\ \varepsilon_3 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu) \in [\tilde t_1 - \delta_3, \tilde t_1 + \delta_3] \times [0,\varepsilon_3] \times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu) = \Phi(\tilde t_0,t)\delta x_0 - \big\{\Phi(\tilde t_0,t)[\tilde C(\tilde t_0)\dot{\tilde\varphi}(\eta(\tilde t_0)) + \tilde f(\omega_0)] +$$

$$+Y(\sigma_0,t)\tilde C(\sigma_0)[\tilde C(\tilde t_0)\dot{\tilde\varphi}(\eta(\tilde t_0)) + \tilde f(\omega_0) - \dot{\tilde\varphi}(\tilde t_0)]\dot\sigma(\tilde t_0)\big\}\delta t_0 + \beta(t,\delta\mu).$$

Theorem 6.7. *Let* $\tau(\tilde t_0) = \tilde t_0, \eta(\tilde t_1) > \tilde t_0,\ \sigma_0 < \eta^{m_2}(\tilde t_1)$ *and the conditions* (5.53), (5.54), (6.7) *be fulfilled. Then there exist numbers* $\delta_3 > 0,\ \varepsilon_3 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu) \in [\tilde t_1 - \delta_3, \tilde t_1 + \delta_3] \times [0,\varepsilon_3] \times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu) = \Phi(\tilde t_0,t)\delta x_0 - \big\{\Phi(\tilde t_0,t)[C_{\tilde t_0}^-\dot{\tilde\varphi}(\eta(\tilde t_0)) + f_3^- + (f_2^- - f_3^-)\dot\gamma^-] +$$

$$+Y_{\sigma_0}^-(t)C_{\sigma_0}^-[C_{\tilde t_0}^-\dot{\tilde\varphi}(\eta(\tilde t_0)) + f_2^- - \dot{\tilde\varphi}(\tilde t_0)]\dot\sigma(\tilde t_0)\big\}\delta t_0 + \beta_1(t,\delta\mu),$$

$$\beta_1(t,\delta\mu) = \int_{\eta(\tilde t_0)}^{\tilde t_0} Y(\sigma(s),t)\tilde C(\sigma(s))\dot{\delta\varphi}(s)\dot\sigma(s)ds +$$

$$+\int_{\tilde t_0}^{t} Y(s,t)\big(\delta C(s)\dot{\tilde x}(\eta(s)) + \delta f[s]\big)ds.$$

Theorem 6.8. *Let* $\tau(\tilde t_0) = \tilde t_0, \eta(\tilde t_1) > \tilde t_0,\ \tilde t_0 < \eta^{m_1}(\tilde t_1),\ \sigma_0 < \eta^{m_2}(\tilde t_1)$ *and the condition* (6.28) *be fulfilled. Let, moreover, there exist the finite limits*

$$\begin{gathered}\lim_{\omega\to\omega_3^0}\tilde f(\omega) = f_2^+,\ \ \lim_{\omega\to\omega_4^0}\tilde f(\omega) = f_3^+,\ \ \omega \in R_{\tilde t_0}^+ \times O^2,\\ \lim_{t\to\tilde t_0}\dot\gamma(t) = \dot\gamma^+,\ \ \lim_{t\to\tilde t_0}\tilde C(t) = C_{\tilde t_0}^+,\ \ t \in R_{\tilde t_0}^+; \\ \lim_{t\to\sigma^i(\tilde t_0)}\tilde C(t) = C_{\sigma^i(\tilde t_0)}^+,\ \ t \in R_{\sigma^i(\tilde t_0)}^+,\ \ i = 1,\dots,m_1.\end{gathered} \tag{6.39}$$

Then there exist numbers $\delta_3 > 0,\ \varepsilon_3 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu) \in [\tilde t_1 - \delta_3, \tilde t_1 + \delta_3] \times [0,\varepsilon_3] \times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu) = \Phi(\tilde t_0,t)\delta x_0 - \big\{\Phi(\tilde t_0,t)[C_{\tilde t_0}^+\dot{\tilde\varphi}(\eta(\tilde t_0)) + f_2^+] +$$

$$+Y_{\sigma_0}^+(t)C_{\sigma_0}^+[C_{\tilde t_0}^+\dot{\tilde\varphi}(\eta(\tilde t_0)) + f_2^+ - \dot{\tilde\varphi}(\tilde t_0)]\dot\sigma(\tilde t_0) +$$

$$+Y_{\gamma_0}^+(t)(f_3^+ - f_2^+)(1 - \dot\gamma^+)\big\}\delta t_0 + \beta_1(t,\delta\mu).$$

Theorem 6.9. *Let* $\tau(\tilde t_0) = \tilde t_0,\ \eta(\tilde t_1) > \tilde t_0,\ \sigma_0 < \eta^{m_2}(\tilde t_1)$ *and the conditions* (5.55), (6.7) *be fulfilled. Let, moreover, there exist a neighborhood* $V_1^-(\omega_4^0)$ *such that the function* $\tilde f(\omega),\ \omega \in V_1^-(\omega_4^0)$*, is bounded. Then there exist* *numbers* $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_2^-]+$$

$$+Y^-_{\sigma_0}(t)C^-_{\sigma_0}[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_2^- -\dot\varphi(\tilde t_0)]\dot\sigma(\tilde t_0)\big\}\delta t_0+\beta_1(t,\delta\mu).$$

Theorem 6.10. *Let* $\tau(\tilde t_0)=\tilde t_0$, $\eta(\tilde t_1)>\tilde t_0$, $\sigma_0<\eta^{m_2}(\tilde t_1)$ *and the conditions* (5.56), (6.28) *be fulfilled. Let, moreover,*

$$\lim_{t\to\tilde t_0}\dot\gamma(t)=1,\quad t\in R^+_{\tilde t_0},\tag{6.40}$$

and there exist a neighborhood $V_1^+(\omega_4^0)$ *such that the function* $\tilde f(\omega)$, $\omega\in V_1^+(\omega_4^0)$, *is bounded. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_2^+]+$$

$$+Y^+_{\sigma_0}(t)C^+_{\sigma_0}[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_2^+ -\dot\varphi(\tilde t_0)]\dot\sigma(\tilde t_0)\big\}\delta t_0+\beta_1(t,\delta\mu).$$

Theorem 6.11. *Let* $\tau(\tilde t_0)=\tilde t_0$, $\eta(\tilde t_1)>\tilde t_0$, $\sigma_0<\eta^{m_2}(\tilde t_1)$, *the function* $\dot\tau(t)$ *be continuous at the point* $\tilde t_0$, *the function* $\tilde C(t)$ *be continuous at the points* $\tilde t_0$, $\sigma^i(\sigma_0)$, $i=0,\dots,m_2$; *the function* $\tilde f(\omega)$ *be continuous at the point* ω_3^0, *the function* $\tilde f(\omega)$ *be bounded in some neighborhood of the point* ω_4^0. *Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[\tilde C(\tilde t_0)\dot\varphi(\eta(\tilde t_0))+\tilde f(\omega_3^0)]+$$

$$+Y(\sigma_0,t)\tilde C(\sigma_0)[\tilde C(\tilde t_0)\dot\varphi(\eta(\tilde t_0))+\tilde f(\omega_3^0)-\dot\varphi(\tilde t_0)]\dot\sigma(\tilde t_0)\big\}\delta t_0+\beta_1(t,\delta\mu).$$

Theorem 6.12. *Let* $\tau(\tilde t_0)<\tilde t_0$, $\tau(\tilde t_1)>\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$ *and the conditions* (6.4), (6.5) *are fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^-]+$$

$$+\Phi(\gamma_0,t)f_1^-\dot\gamma^-\big\}\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.13. *Let* $\tau(\tilde t_0)<\tilde t_0$, $\tau(\tilde t_1)>\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$ *and the conditions* (6.25), (6.26) *be fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that* *for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^+]+$$

$$+\Phi(\gamma_0,t)f_1^+\dot\gamma^+\big\}\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.14. *Let* $\tau(\tilde t_0)<\tilde t_0$, $\tau(\tilde t_1)>\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$, *the functions* $\tilde C(t)$, $\dot\tau(t)$ *be continuous at the point* $\tilde t_0$, *the function* $\tilde f(\omega)$ *be continuous at the points* ω_0, ω_1^0, ω_2^0. *Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\big\{\Phi(\tilde t_0,t)[\tilde C(\tilde t_0)\dot\varphi(\eta(\tilde t_0))+\tilde f(\omega_0)]+$$

$$+\Phi(\gamma_0,t)[\tilde f(\omega_1^0)-\tilde f(\omega_2^0)]\dot\gamma(\tilde t_0)\big\}\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.15. *Let* $\tau(\tilde t_1)<\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$ *and the condition* (6.4) *be fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-$$

$$-\Phi(\tilde t_0,t)[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^-]+\beta(t,\delta\mu).$$

Theorem 6.16. *Let* $\tau(\tilde t_1)<\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$ *and the condition* (6.25) *be valid. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-$$

$$-\Phi(\tilde t_0,t)[C^+_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_0^+]+\beta(t,\delta\mu).$$

Theorem 6.17. *Let* $\tau(\tilde t_1)<\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$, *the function* $\tilde C(t)$ *be continuous at the point* $\tilde t_0$, *the function* $\tilde f(\omega)$ *be continuous at the point* ω_0. *Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\Phi(\tilde t_0,t)\big[\tilde C(\tilde t_0)\dot\varphi(\eta(\tilde t_0))+\tilde f(\omega_0)\big]\delta t_0+\beta(t,\delta\mu).$$

Theorem 6.18. *Let* $\tau(\tilde t_0)=\tilde t_0$, $\eta(\tilde t_1)<\tilde t_0$ *and the conditions* (5.53), (5.54) *be fulfilled. Then there exist numbers* $\delta_3>0$, $\varepsilon_3>0$ *such that for an arbitrary* $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_3,\tilde t_1+\delta_3]\times[0,\varepsilon_3]\times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t,\delta\mu)=\Phi(\tilde t_0,t)\delta x_0-\Phi(\tilde t_0,t)\big[C^-_{\tilde t_0}\dot\varphi(\eta(\tilde t_0))+f_3^-+(f_2^--f_3^-)\dot\gamma^-\big]\delta t_0+\beta_1(t,\delta\mu).$$

Theorem 6.19. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$, $\eta(\tilde{t}_1) < \tilde{t}_0$ *and the condition* (6.39) *be fulfilled. Then there exist numbers* $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t, \delta\mu) = \Phi(\tilde{t}_0, t)\delta x_0 - \Phi(\tilde{t}_0, t)\big[C_{\tilde{t}_0}^+ \dot{\varphi}(\eta(\tilde{t}_0)) + f_3^+ + (f_2^+ - f_3^+)\dot{\gamma}^+\big]\delta t_0 + \beta_1(t, \delta\mu).$$

Theorem 6.20. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$, $\eta(\tilde{t}_1) < \tilde{t}_0$ *and the conditions* (5.53), (5.54), (6.39) *be fulfilled. Moreover, let* $\tilde{C}(t)$ *be continuous at the point* $\tilde{t}_0$ *and*

$$f_3^- + (f_2^- - f_3^-)\dot{\gamma}^- = f_3^+ + (f_2^+ - f_3^+)\dot{\gamma}^+ = \hat{f}.$$

Then there exist numbers $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t, \delta\mu) = \Phi(\tilde{t}_0, t)\delta x_0 - \Phi(\tilde{t}_0, t)\big[\tilde{C}(\tilde{t}_0)\dot{\varphi}(\eta(\tilde{t}_0)) + \hat{f}\big]\delta t_0 + \beta_1(t, \delta\mu).$$

Theorem 6.21. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$, $\eta(\tilde{t}_1) < \tilde{t}_0$ *and the condition* (5.55) *be fulfilled. Let, moreover, there exist a neighborhood* $V_1^-(\omega_4^0)$ *such that the function* $\tilde{f}(\omega)$, $\omega \in V_1^-(\omega_4^0)$, *is bounded. Then there exist numbers* $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V_1^-$ *the formula* (6.8) *is valid, where*

$$\delta x(t, \delta\mu) = \Phi(\tilde{t}_0, t)\delta x_0 - \Phi(\tilde{t}_0, t)\big[C_{\tilde{t}_0}^- \dot{\varphi}(\eta(\tilde{t}_0)) + f_2^-\big]\delta t_0 + \beta_1(t, \delta\mu).$$

Theorem 6.22. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$, $\eta(\tilde{t}_1) < \tilde{t}_0$ *and the conditions* (5.56), (6.40) *be fulfilled. Let, moreover, there exist a neighborhood* $V_1^+(\omega_4^0)$ *such that the function* $\tilde{f}(\omega)$, $\omega \in V_1^+(\omega_4^0)$ *is bounded. Then there exist numbers* $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V_1^+$ *the formula* (6.8) *is valid, where*

$$\delta x(t, \delta\mu) = \Phi(\tilde{t}_0, t)\delta x_0 - \Phi(\tilde{t}_0, t)\big[C_{\tilde{t}_0}^+ \dot{\varphi}(\eta(\tilde{t}_0)) + f_2^+\big]\delta t_0 + \beta_1(t, \delta\mu).$$

Theorem 6.23. *Let* $\tau(\tilde{t}_0) = \tilde{t}_0$, $\eta(\tilde{t}_1) < \tilde{t}_0$, *the functions* $\tilde{C}(t)$, $\dot{\tau}(t)$ *be continuous at the point* $\tilde{t}_0$, *the function* $\tilde{f}(\omega)$ *be continuous at the point* ω_3^0, *the function* $\tilde{f}(\omega)$ *be bounded in a some neighborhood of the point* ω_4^0. *Then there exist numbers* $\delta_3 > 0$, $\varepsilon_3 > 0$ *such that for an arbitrary* $(t, \varepsilon, \delta\mu) \in [\tilde{t}_1 - \delta_3, \tilde{t}_1 + \delta_3] \times [0, \varepsilon_3] \times V_1$ *the formula* (6.8) *is valid, where*

$$\delta x(t, \delta\mu) = \Phi(\tilde{t}_0, t)\delta x_0 - \Phi(\tilde{t}_0, t)\big[\tilde{C}(\tilde{t}_0)\dot{\varphi}(\eta(\tilde{t}_0)) + \tilde{f}(\omega_3^0)\big]\delta t_0 + \beta_1(t, \delta\mu).$$

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