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THREE-DIMENSIONAL MATHEMATICAL PROBLEMS OF THERMOELASTICITY OF ANISOTROPIC BODIES, II

Dedicated to V. D. Kupradze
on the occasion of his 95th birthday

Abstract. A wide class of basic, mixed, and crack type boundary value and interface problems for the steady state and pseudo-oscillation equations of the thermoelasticity theory of anisotropic bodies are considered. The generalized Sommerfeld–Kupradze type thermo-radiation conditions are formulated and uniqueness and existence theorems are proved by the potential method and the theory of pseudodifferential equations on manifold. The almost best regularity properties of solutions are established.

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CHAPTER VI
MIXED AND CRACK TYPE PROBLEMS

In this chapter we study the basic mixed BVPs, the crack type problems, and the mixed interface problems formulated in Chapter II. Applying the boundary integral equation method we prove the existence theorems in Sobolev spaces and establish the almost best regularity results for solutions near the boundary of cracks and at the collision curves of changing boundary and transmission conditions.

Throughout this chapter the interface surfaces, the collision curves and the crack boundaries are assumed to be C^∞-smooth. Moreover, the parameters r and ω in the steady state oscillation problems are subjected to the requirement (15.3).

16. Basic Mixed BVPs

16.1 In this subsection we present some results from the theory of elliptic pseudodifferential equations on manifolds with boundary in Bessel-potential and Besov spaces. They will be the main tools for proving existence theorems for the above mentioned mixed and crack type problems. All the results outlined below in this subsection can be found, for example, in [4], [20], [43], [69], [15], [70], [71], [72].

Let $\mathcal{S} \in C^\infty$ be a compact n-dimensional manifold with the boundary $\partial\mathcal{S} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $m \times m$ matrix pseudodifferential operator of order $\kappa \in I\!R$ on $\overline{\mathcal{S}}$. Denote by $\sigma(\mathcal{A})(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system. Here $x \in \mathcal{S}$, $\xi \in I\!R^n \setminus \{0\}$. Consider the following $m \times m$ matrix function

$$\mathcal{A}_\eta^{(0)}(x,\xi) = |\xi|^{-\kappa}\sigma(\mathcal{A})(x,|\xi'|\eta,\xi_n), \tag{16.1}$$

where $\xi' = (\xi_1, ..., \xi_{n-1})$ and η belongs to the unit sphere $\Sigma^{(n-2)}$ in $I\!R^{n-1}$.

It is known that the matrix $\mathcal{A}_\eta^{(0)}$ in (16.1) admits the factorization

$$\mathcal{A}_\eta^{(0)}(x,\xi) = A_\eta^-(x,\xi)\mathcal{D}(\eta,x,\xi)A_\eta^+(x,\xi) \quad \text{for } x \in \partial\mathcal{S},$$

where $[A_\eta^-(x,\xi)]^{\pm 1}$ and $[A_\eta^+(x,\xi)]^{\pm 1}$ are matrices, which are homogeneous of degree 0 in ξ and admit analytic bounded continuations with respect to ξ_n into the lower and upper complex half-planes, respectively. Moreover, $\mathcal{D}(\eta,x,\xi)$ is a bounded lower triangular matrix with entries of the form

$$\Big(\frac{\xi_n - \mathrm{i}|\xi'|}{\xi_n + \mathrm{i}|\xi'|}\Big)^{\delta_j(x)}, \;\; j = 1, ...m,$$

on the main diagonal; here

$$\delta_j(x) = (2\pi \mathrm{i})^{-1}\ln\lambda_j(x), \;\; j = 1, ..., m,$$

where $\lambda_1(x), ..., \lambda_m(x)$ are the eigenvalues of the matrix

$$A(x) = [\sigma(\mathcal{A})(x,0,...0,-1)]^{-1}[\sigma(\mathcal{A})(x,0,...,0,+1)].$$

The branch in the logarithmic function is chosen with regard to the inequality $1/p - 1 < \operatorname{Re}\delta_j(x) < 1/p,\ j = 1, ..., m,\ \ p > 1$. The numbers $\delta_j(x)$ do not depend on the choice of the local co-ordinate system.

Note that, if $\sigma(\mathcal{A})(x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{S}}$ and $\xi \in I\!R^n \setminus \{0\}$, then

$$\operatorname{Re}\delta_j(x) = 0 \quad \text{for} \quad j = 1, ..., m, \tag{16.2}$$

since, in this case, the eigenvalues of the matrix $A(x)$ are positive numbers for any $x \in \overline{\mathcal{S}}$.

The Fredholm properties of such operators are characterized by the following lemma.

Lemma 16.1. *Let $1 < p < \infty$, $s \in I\!R$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator having a positive definite principal homogeneous symbol matrix, i.e., $\sigma(\mathcal{A})(x, \xi)\zeta \cdot \zeta \ge c\,|\zeta|^2$ for $x \in \overline{\mathcal{S}}$, $\xi \in I\!R^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^m$,*
where c is a positive constant.

Then the operators

$$\mathcal{A} \ \ : \ \ \widetilde{H}^s_p(\mathcal{S}) \to H^{s-\kappa}_p(\mathcal{S}), \tag{16.3}$$

$$: \ \ \widetilde{B}^s_{p,q}(\mathcal{S}) \to B^{s-\kappa}_{p,q}(\mathcal{S}), \tag{16.4}$$

are bounded Fredholm operators of index zero if and only if

$$1/p - 1 < s - \kappa/2 < 1/p. \tag{16.5}$$

Moreover, the null-spaces and indices of the operators (16.3), (16.4) *are the same for all values of the parameter $q \in [1, +\infty]$, and for all values of the parameters $p \in (1, \infty)$ and $s \in I\!R$ satisfying the inequality* (16.5).

16.2. First we consider the basic mixed BVP $(\mathcal{P}_{mix})^+_\tau$ for the pseudo-oscillation equations of thermoelasticity (see (5.9)–(5.10)).

We assume that the boundary data meet the following conditions

$$f^{(1)}_j \in B^{1-1/p}_{p,p}(S_1),\ \ F^{(2)}_j \in B^{-1/p}_{p,p}(S_2),\ j = \overline{1,4},\ 1 < p < \infty, \tag{16.6}$$

and look for the solution U in the space $W^1_p(\Omega^+)$.

Let $f_0 = (f_{01}, \cdots, f_{04})^\top \in B^{1-1/p}_{p,p}(S)$ be some fixed extention of the given vector function $f^{(1)} = (f^{(1)}_1, \cdots, f^{(1)}_4)^\top \in B^{1-1/p}_{p,p}(S_1)$ onto the whole surface $S = \partial\Omega^+$. Then an arbitrary extention, preserving the functional space, is represented as

$$f = f_0 + \varphi \in B^{1-1/p}_{p,p}(S), \tag{16.7}$$

where $\varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$. Clearly, $f|_{S_1} = f_0|_{S_1} = f^{(1)}$.

Let us seek the solution of the mixed BVP $(\mathcal{P}_{mix})^+_\tau$ in the form of a single layer potential

$$U(x) = V_\tau(\mathcal{H}^{-1}_\tau f)(x), \quad x \in \Omega^+, \tag{16.8}$$

where V_τ is given by (11.1), $\mathcal{H}^{-1}_\tau$ is the operator inverse to $\mathcal{H}_\tau$ (see (11.3) and Remark 12.13), and f is given by formula (16.7).

Applying Theorem 11.3 we can easily see that the conditions (5.9) are automatically satisfied, while the conditions (5.10) lead to the ΨDE for the unknown vector function φ

$$[B(D,n)U]^+ = [-2^{-1}I_4 + \mathcal{K}_{1,\tau}]\mathcal{H}_\tau^{-1}(f_0+\varphi) = F^{(2)} \text{ on } S_2, \tag{16.9}$$

where f_0 and $F^{(2)} = (F_1^{(2)}, \cdots, F_4^{(2)})^\top \in B_{p,p}^{-1/p}(S_2)$ are given vector-functions, and where the operator $\mathcal{K}_{1,\tau}$ is defined by (11.4).

Let

$$\mathcal{N}^+_{\tau,mix} := [-2^{-1}I_4 + \mathcal{K}_{1,\tau}]\mathcal{H}_\tau^{-1}. \tag{16.10}$$

Then the equation (16.9) is written as

$$r_{S_2}\,\mathcal{N}^+_{\tau,mix}\,\varphi \;=\; g \text{ on } S_2, \tag{16.11}$$

where r_{S_2} is the restriction operator on S_2, and

$$g = F^{(2)} - r_{S_2}\,\mathcal{N}^+_{\tau,mix}\,f_0 \in B_{p,p}^{-1/p}(S_2). \tag{16.12}$$

The properties of the operators $\mathcal{N}^+_{\tau,mix}$ and $r_{S_2}\,\mathcal{N}^+_{\tau,mix}$ are described by the following lemmata.

Lemma 16.2. *The principal homogeneous symbol matrix of the ΨDO $\mathcal{N}^+_{\tau,mix}$ is positive definite for arbitrary $x \in S$ and $\tilde{\xi} \in I\!R^2 \setminus \{0\}$.*

Proof. It is verbatim the proof of Lemma 14.2 for the operator $\mathcal{N}_{1,\tau}$. □

Lemma 16.3. *The operators*

$$r_{S_2}\,\mathcal{N}^+_{\tau,mix} \;:\; [\widetilde{B}^{s+1}_{p,q}(S_2)]^4 \to [B^s_{p,q}(S_2)]^4, \tag{16.13}$$

$$: \; [\widetilde{H}^{s+1}_p(S_2)]^4 \to [H^s_p(S_2)]^4, \tag{16.14}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition

$$1/p - 3/2 < s < 1/p - 1/2 \tag{16.15}$$

holds.

Proof. The boundedness and Fredholmity of the operators (16.13) and (16.14) under the restriction (16.15) follow from Lemmata 16.2 and 16.1 with $s+1$ and 1 in the place of s and κ. Due to these lemmata the Fredholm indices of the operators (16.13) and (16.14) are equal to zero and the dominant singular part of the operator $\mathcal{N}^+_{\tau,mix}$ is formally self-adjoint.

It remains to prove that the operators under consideration have the trivial null-spaces. Obviously, if we are able to find two numbers $s_1 \in I\!R$ and $p_1 \in (1,\infty)$ satisfying the inequalities (16.15) such that the homogeneous equation

$$r_{S_2}\,\mathcal{N}^+_{\tau,mix}\,\varphi \;=\; 0 \tag{16.16}$$

has no nontrivial solutions in the space $\widetilde{B}^{s_1+1}_{p_1,p_1}(S_2)$ $[\widetilde{H}^{s_1+1}_{p_1}(S_2)]$, then due to Lemma 16.1 we can conclude that the null-spaces of the operators (16.13), (16.14) are trivial for all values of the parameters s and p subjected to the condition (16.15).

To this end let us take

$$s_1 = -1/2, \quad p_1 = 2, \quad q = 2, \tag{16.17}$$

which satisfy inequalities (16.15). We recall that $\widetilde{B}^{\pm 1/2}_{2,2}(S_2) = \widetilde{H}^{\pm 1/2}_{2}(S_2)$.

Let some vector function $\varphi_0 \in \widetilde{B}^{1/2}_{2,2}(S_2)$ solve the homogeneous equation (16.16) and let us construct the single layer potential

$$U_0(x) = V_\tau(\mathcal{H}_\tau^{-1}\,\varphi_0)(x), \quad x \in \Omega^+. \tag{16.18}$$

By Theorem 11.3 and Remark 12.13 we have

$$U_0(x) \in H^1_2(\Omega^+) = W^1_2(\Omega^+), \tag{16.19}$$

and, moreover, U_0 satisfies the conditions corresponding to the homogeneous mixed BVP $(\mathcal{P}_{mix})^+_\tau$ due to the the homogeneous equation (16.16) and the inclusion $\varphi_0 \in \widetilde{B}^{1/2}_{2,2}(S_2)$. With regard to Theorem 8.3 we then infer that $U_0 = 0$ in Ω^+, and, consequently, $[U_0]^+ = \varphi_0 = 0$. This completes the proof. □

Now we can formulate the following existence result.

Theorem 16.4. *Let* $4/3 < p < 4$ *and conditions* (16.6) *be fulfilled. Then the nonhomogeneous mixed problem* $(\mathcal{P}_{mix})^+_\tau$ *is uniquely solvable in the space* $W^1_p(\Omega^+)$ *and the solution is representable in the form of the single layer potential* (16.8)*, where the density* f *is given by* (16.7) *and where* φ *is the unique solution of the* ΨDE (16.11)*.*

Proof. First we note that, in accordance with Lemma 16.3, the ΨDE (16.11) is uniquely solvable for $s = -1/p$ and $4/3 < p < 4$, where the last inequality follows from the condition (16.15). This implies the solvability of the problem $(\mathcal{P}_{mix})^+_\tau$ in the space $W^1_p(\Omega^+)$ with p as above. Next we show that this problem is uniquely solvable in the space $W^1_p(\Omega^+)$ for arbitrary $p \in (4/3, 4)$ (for $p = 2$ it has been proved in Theorem 8.3).

We proceed as follows. Let $U \in W^1_p(\Omega^+)$ be some solution of the homogeneous problem $(\mathcal{P}_{mix})^+_\tau$. Clearly, then

$$[U]^+ \in \widetilde{B}^{1-1/p}_{p,p}(S_2). \tag{16.20}$$

By Remark 12.13 we have the following representation for the vector U (see (12.55))

$$U(x) = V_\tau(\mathcal{H}_\tau^{-1}\,[U]^+)(x), \quad x \in \Omega^+. \tag{16.21}$$

Since U satisfies the homogeneous conditions (5.10), from (16.21) we get

$$r_{S_2}\,\mathcal{N}^+_{\tau,mix}\,[U]^+ = 0 \ \text{ on } \ S_2. \tag{16.22}$$

Whence $[U]^+ = 0$ on S follows due to the inclusion (16.20), Lemma 16.3, and the inequality $4/3 < p < 4$. Therefore, $U = 0$ in Ω^+. □

Now we can prove the main regularity result for the solution to the mixed BVP $(\mathcal{P}_{mix})^+_\tau$.

Theorem 16.5. *Let the conditions* (16.6) *be fulfilled,*

$$4/3 < p < 4, \ 1 < t < \infty, \ 1 \le q \le \infty, \ 1/t - 3/2 < s < 1/t - 1/2, \quad (16.23)$$

and let $U \in W^1_p(\Omega^+)$ *be the unique solution to the mixed problem* $(\mathcal{P}_{mix})^+_\tau$.
In addition to (16.6),

i) *if*

$$f^{(1)} \in B^{s+1}_{t,t}(S_1), \ \ F^{(2)} \in B^s_{t,t}(S_2), \quad (16.24)$$

then

$$U \in H^{s+1+1/t}_t(\Omega^+); \quad (16.25)$$

ii) *if*

$$f^{(1)} \in B^{s+1}_{t,q}(S_1), \ \ F^{(2)} \in B^s_{t,q}(S_2), \quad (16.26)$$

then

$$U \in B^{s+1+1/t}_{t,q}(\Omega^+); \quad (16.27)$$

iii) *if*

$$f^{(1)} \in \mathrm{C}^\alpha(S_1), \ \ F^{(2)} \in B^{\alpha-1}_{\infty,\infty}(S_2), \ \text{ for some } \alpha > 0, \quad (16.28)$$

then

$$U \in \mathrm{C}^\nu(\overline{\Omega^+}) \ \text{ with any } \nu \in (0, \alpha_0), \ \ \alpha_0 := \min\{\alpha, 1/2\}. \quad (16.29)$$

Proof. Theorem 11.3 and Remark 12.13 (see (12.53)) together with the conditions (16.24) [(16.26)] imply $g \in B^s_{t,t}(S_2)$ $[B^s_{t,q}(S_2)]$, where g is defined by (16.12). Note that $f_0 \in B^{s+1}_{t,t}(S)$ $[B^{s+1}_{t,q}(S)]$ is some extension of the vector $f^{(1)}$ onto the whole of S.

Next, by Lemma 16.3 and conditions (16.23) we conclude that the equation (16.11) is uniquely solvable in the space $\widetilde{B}^{s+1}_{t,t}(S_2)$ $[\widetilde{B}^{s+1}_{t,q}(S_2)]$. Therefore, we have that in the representation (16.8) of the unique solution U to the problem $(\mathcal{P}_{mix})^+_\tau$ in the space $W^1_p(\Omega^+)$ the density vector $f = f_0 + \varphi$ satisfies inclusion

$$f = f_0 + \varphi \in B^{s+1}_{t,t}(S) \ [B^{s+1}_{t,q}(S)] \quad (16.30)$$

as well (together with the inclusion (16.7)).

Applying again Theorem 11.3 and Remark 12.13 concerning the mapping properties of the single layer operator V_τ and the ΨDO $\mathcal{H}^{-1}_\tau$ we find that (16.25) [(16.27)] holds.

For the last assertion (item iii)) we use the following embeddings (see, e.g., [78], [79])

$$\mathrm{C}^\alpha(\mathcal{S}) = B^\alpha_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{S}) \subset B^{\alpha-\varepsilon}_{\infty,q}(\mathcal{S}) \subset$$
$$\subset B^{\alpha-\varepsilon}_{t,q}(\mathcal{S}) \subset \mathrm{C}^{\alpha-\varepsilon-k/t}(\mathcal{S}), \quad (16.31)$$

where ε is an arbitrary small positive number, $\mathcal{S} \subset I\!R^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < t < \infty$, $\alpha - \varepsilon - k/t > 0$, α and $\alpha - \varepsilon - k/t$ are not integer numbers. From the assumption iii) of the theorem and the embeddings (16.31), it is easily seen that the condition (16.26) follows with any $s \le \alpha - \varepsilon - 1$.

Bearing in mind (16.23), and taking t sufficiently large and ε sufficiently small, we are able to put $s = \alpha - \varepsilon - 1$ if

$$1/t - 3/2 < \alpha - \varepsilon - 1 < 1/t - 1/2, \tag{16.32}$$

and $s \in (1/t - 3/2, 1/t - 1/2)$ if

$$1/t - 1/2 < \alpha - \varepsilon - 1. \tag{16.33}$$

By (16.27) the solution U belongs then to $B^{s+1+1/t}_{t,q}(\Omega^+)$ with $s + 1 + 1/t = \alpha - \varepsilon + 1/t$ if there holds (16.32), and with $s+1+1/t \in (2/t-1/2, 2/t+1/2)$ if there holds (16.33). In the last case we can take $s + 1 + 1/t = 2/t + 1/2 - \varepsilon$. Therefore, we have either $U \in B^{\alpha-\varepsilon+1/t}_{t,q}(\Omega^+)$ or $U \in B^{2/t+1/2-\varepsilon}_{t,q}(\Omega^+)$ in accordance with inequalities (16.32) and (16.33). Now the last embedding in (16.31) (with $k = 3$) yields that either $U \in \mathrm{C}^{\alpha-\varepsilon-2/t}(\overline{\Omega^+})$ or $U \in \mathrm{C}^{1/2-\varepsilon-1/t}(\overline{\Omega^+})$, which lead to the inclusion

$$U \in \mathrm{C}^{\alpha_0-\varepsilon-2/t}(\overline{\Omega^+}), \tag{16.34}$$

where $\alpha_0 := \min\{\alpha, 1/2\}$. Since t is sufficiently large and ε is sufficiently small, the embedding (16.34) completes the proof. □

16.3. The basic mixed exterior BVP $(\mathcal{P}_{mix})^-_\tau$ (see (5.9)–(5.10)) can be considered by applying quite the same approach and by the word for word arguments. Therefore, in this subsection we formulate only the basic results concerning the existence and regularity of solutions.

Let the boundary data $f^{(1)}_j$ and $F^{(2)}_j$ $(j = \overline{1,4})$ of the BVP $(\mathcal{P}_{mix})^-_\tau$ satisfy the conditions (16.6), and f_0, f, and φ be as in the previous subsection. We again look for the solution in the form of the single layer potential

$$U(x) = V_\tau(\mathcal{H}^{-1}_\tau f)(x), \quad x \in \Omega^-, \tag{16.35}$$

where

$$f = f_0 + \varphi \in B^{1-1/p}_{p,p}(S), \;\; f_0 \in B^{1-1/p}_{p,p}(S), \;\; \varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2). \tag{16.36}$$

As above f_0 is the given vector function satisfying the condition $f_0|_{S_1} = f^{(1)}$, while φ is the unknown vector function which has to be defined by the ΨDE

$$r_{S_2}\, \mathcal{N}^-_{\tau,mix}\, \varphi \;=\; g \;\text{ on }\; S_2, \tag{16.37}$$

where r_{S_2} is again the restriction operator on S_2, and

$$\mathcal{N}^-_{\tau,mix} = [2^{-1} I_4 + \mathcal{K}_{1,\tau}]\mathcal{H}^{-1}_\tau, \tag{16.38}$$

$$g = F^{(2)} - r_{S_2}\, \mathcal{N}^-_{\tau,mix}\, f_0 \in B^{-1/p}_{p,p}(S_2). \tag{16.39}$$

Lemma 16.6. *The principal homogeneous symbol matrix of the ΨDO $\mathcal{N}^-_{\tau,mix}$ is positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$.*

Lemma 16.7. *The operators*

$$\begin{aligned} r_{S_2}\, \mathcal{N}^-_{\tau,mix} \;\; &: \;\; [\widetilde{B}^{s+1}_{p,q}(S_2)]^4 \to [B^s_{p,q}(S_2)]^4, \\ &: \;\; [\widetilde{H}^{s+1}_p(S_2)]^4 \to [H^s_p(S_2)]^4, \end{aligned}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Theorem 16.8. *Let* $4/3 < p < 4$ *and let the conditions* (16.6) *be fulfilled. Then the nonhomogeneous mixed problem* $(\mathcal{P}_{mix})^-_\tau$ *is uniquely solvable in the space* $W^1_p(\Omega^-)$ *and the solution is representable in the form* (16.35), *where the density* f *is given by* (16.36) *and where* φ *is the unique solution of the* ΨDE (16.37).

Theorem 16.9. *Let the conditions* (16.6) *and* (16.23) *be fulfilled, and let* $U \in W^1_p(\Omega^-)$ *be the unique solution to the mixed problem* $(\mathcal{P}_{mix})^-_\tau$.

In addition to (16.6),

i) *if there hold the inclusions* (16.24), *then*

$$U \in H^{s+1+1/t}_t(\Omega^-);$$

ii) *if there hold the inclusions* (16.26), *then*

$$U \in B^{s+1+1/t}_{t,q}(\Omega^-);$$

iii) *if there hold the inclusions* (16.28), *then*

$$U \in \mathrm{C}^\nu(\overline{\Omega^-}) \text{ with any } \nu \in (0,\alpha_0), \ \ \alpha_0 := \min\{\alpha, 1/2\}.$$

The proofs of these propositions are verbatim the proofs of Lemmata 16.2, 16.3, and Theorems 16.4, 16.5.

16.4. In this subsection we shall study the basic mixed exterior BVP $(\mathcal{P}_{mix})^-_\omega$ for the steady state oscillation equations of the thermoelasticity theory formulated in Section 5 (see (5.9)–(5.10)). Again let $f^{(1)}$, $F^{(2)}$, f_0, f, and φ be the same as in Subsection 16.2.

We look for a solution to the BVP $(\mathcal{P}_{mix})^-_\omega$ in the form

$$U(x) = (W + p_0 V)\,([\mathcal{N}^-_1]^{-1} f)(x), \quad x \in \Omega^-, \tag{16.40}$$

where V and W are the single and double layer potentials given by formulae (10.1) and (10.2), respectively, p_0 is defined by (13.5),

$$f = f_0 + \varphi \in B^{1-1/p}_{p,p}(S), \ \ f_0 \in B^{1-1/p}_{p,p}(S), \ \ \varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2), \tag{16.41}$$

and $[\mathcal{N}^-_1]^{-1}$ is an elliptic SIO inverse to the operator (cf. (13.6))

$$\mathcal{N}^-_1 := -2^{-1} I_4 + \mathcal{K}_2 + p_0 \mathcal{H}. \tag{16.42}$$

Note that $[\mathcal{N}^-_1]^{-1}$ is an elliptic SIO due to Lemma 10.2. Moreover, the mapping

$$[\mathcal{N}^-_1]^{-1} \,:\, [B^s_{p,q}(S)]^4 \to [B^s_{p,q}(S)]^4, \ 1 < p < \infty, \ 1 \le q \le \infty, \ s \in \mathbb{R}, \tag{16.43}$$

is an isomorphism according to Lemma 13.13.

Applying Theorem 10.8, item i), one can easily see that the vector U represented by formula (16.40) automatically satisfies the boundary conditions (5.9) on S_1 since $[U]^- = f$ on S and $f|_{S_1} = f_0|_{S_1} = f^{(1)}$. It remains to fulfil the conditions (5.10) on S_2 which lead to the ΨDE for the unknown vector φ

$$[B(D,n)U]^- = [\mathcal{L} + p_0(2^{-1} I_4 + \mathcal{K}_1)][\mathcal{N}^-_1]^{-1}(f_0 + \varphi) = F^{(2)} \text{ on } S_2, \tag{16.44}$$

where $\mathcal{L}$ is defined by (10.36) and (10.6), while $\mathcal{K}_1$ is given by (10.4).

Next we set

$$\mathcal{N}_{mix}^{-} := -[\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)]\,[\mathcal{N}_1^{-}]^{-1}, \tag{16.45}$$

and rewrite the equation (16.44) as

$$r_{S_2}\,\mathcal{N}_{mix}^{-}\,\varphi = q \ \text{ on } \ S_2, \tag{16.46}$$

where r_{S_2} is again the restriction operator on S_2, and

$$q = -F^{(2)} + r_{S_2}\,\mathcal{N}_{mix}^{-}\,f_0 \in B_{p,p}^{-1/p}(S_2). \tag{16.47}$$

The inclusion (16.47) for the right-hand side vector function q follows from Theorem 10.8 and the mapping property (16.43). Further, we present the properties of the operators $\mathcal{N}_{mix}^{-}$ and $r_{S_2}\,\mathcal{N}_{mix}^{-}$.

Lemma 16.10. *The principal homogeneous symbol matrix of the ΨDO $\mathcal{N}_{mix}^{-}$ is positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$.*

Proof. First we note that the principal homogeneous symbol matrix of the operator $\mathcal{N}_{mix}^{-}$ reads as

$$\sigma(\mathcal{N}_{mix}^{-}) = -\sigma(\mathcal{L})\sigma([\mathcal{N}_1^{-}]^{-}) =$$

$$= -\begin{bmatrix} [\sigma(\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4} \begin{bmatrix} [\sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)})]_{3\times 3}^{-1} & [0]_{3\times 1} \\ [0]_{1\times 3} & -2 \end{bmatrix}_{4\times 4} =$$

$$= \begin{bmatrix} [\,-\sigma(\mathcal{L}^{(0)})[\sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)})]^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2\sigma(\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4},$$

due to formulae (10.25), (10.30), (10.49). As we have already mentioned in the proof of Lemma 15.5, the matrix $[\,-\sigma(\mathcal{L}^{(0)})[\sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)})]^{-1}]_{3\times 3}$ is positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$ (for details see [59], [41], [34], [57]), while the function $2\sigma(\mathcal{L}_4^{(0)})$ is positive in accordance with the inequality (10.50). $\sigma(\mathcal{N}_{mix}^{-})$ is positive definite. □

Lemma 16.11. *The operators*

$$r_{S_2}\,\mathcal{N}_{mix}^{-} \ : \ [\widetilde{B}_{p,q}^{s+1}(S_2)]^4 \to [B_{p,q}^{s}(S_2)]^4, \tag{16.48}$$

$$: \ [\widetilde{H}_{p}^{s+1}(S_2)]^4 \to [H_{p}^{s}(S_2)]^4, \tag{16.49}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. It is quite similar to the proof of Lemma 16.3. Indeed, the boundedness and Fredholmity of the operators in question and that the Fredholm indices are equal to zero follow from Lemma 16.10 and Lemma 16.1 with $s+1$ and 1 in the place of s and κ.

Further, due to Lemma 16.10 the dominant singular part of the operator $\mathcal{N}_{mix}^{-}$ is formally self-adjoint.

To prove that their null-spaces are trivial, as in the proof of Lemma 16.3, we concider the homogeneous ΨDE

$$r_{S_2}\,\mathcal{N}^-_{mix}\,\varphi = 0 \ \text{ on } \ S_2, \tag{16.50}$$

and prove that it has only the trivial solution in the space $\widetilde{B}^{1/2}_{2,2}(S_2) = \widetilde{H}^{1/2}_2(S_2)$. It corresponds to the particular values of the parameters s and p (and q) given by (16.17).

Let some vector function $\varphi_0 \in \widetilde{B}^{1/2}_{2,2}(S_2)$ solve the equation (16.50), and construct the vector

$$U_0(x) = (W + p_0 V)\,([\mathcal{N}^-_1]^{-1}\,\varphi_0)(x), \quad x \in \Omega^-. \tag{16.51}$$

By Theorem 10.8, Lemma 13.13 and the mapping property (16.43) we conclude

$$U_0(x) \in W^1_{2,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-). \tag{16.52}$$

Moreover, U_0 satisfies the boundary conditions of the homogeneous mixed BVP $(\mathcal{P}_{mix})^-_\omega$ due to the homogeneous equation (16.50) and the inclusion $\varphi_0 \in \widetilde{B}^{1/2}_{2,2}(S_2)$. By virtue of the uniqueness results (see Theorem 9.6) the vector function (16.51) then vanish in Ω^-, and, consequently, $[U_0]^- = \varphi_0 = 0$ on S. The proof is completed. □

These lemmata imply the foolowing existence results.

Theorem 16.12. *Let* $4/3 < p < 4$ *and let the conditions* (16.6) *be fulfilled. Then the nonhomogeneous mixed exterior problem* $(\mathcal{P}_{mix})^-_\omega$ *is uniquely solvable in the class* $W^1_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)$ *and the solution is representable in the form* (16.40), *where the density* f *is given by* (16.41) *and where* φ *is the unique solution of the* Ψ*DE* (16.46).

Proof. Again it is quite similar to the proof of Theorem 16.4. If we fix $s = -1/p$, then the nonhomogeneous equation (16.46) is uniquely solvable in the space $\widetilde{B}^{1-1/p}_{p,p}(S_2)$ for arbitrary $p \in (4/3, 4)$ which follows from Lemma 16.11 and the inequality (16.15) (with $s = -1/p$). This implies the solvability of the nonhomogeneous mixed exterior problem $(\mathcal{P}_{mix})^-_\omega$ in the class $W^1_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)$, indicated in the theorem.

Now we show that this problem is uniquely solvable for arbitrary $p \in (4/3, 4)$ (for $p = 2$ it has already been proved in Theorem 9.6).

To this end let us consider the homogeneous problem $(\mathcal{P}_{mix})^-_\omega$ in the class $W^1_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)$ with $p \in (4/3, 4)$, and let a vector function U be its arbitrary solution. Since $[U]^- \in B^{1-1/p}_{p,p}(S)$ we conclude that U is uniquely representable in the form

$$U(x) = (W + p_0 V)\,([\mathcal{N}^-_1]^{-1}\,[U]^-)(x), \quad x \in \Omega^-, \tag{16.53}$$

due to Theorem 13.14.

Moreover, $[U]^- \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$ and

$$[B(D,n)U]^-_{S_2} = r_{S_2}\,\mathcal{N}^-_{mix}\,[U]^- = 0 \ \text{ on } \ S_2, \tag{16.54}$$

inasmuch as U is a solution to the homogeneous problem $(\mathcal{P}_{mix})^-_\omega$. Further, Lemma 16.11 together with the conditions $s = -1/p$ and $p \in (4/3, 4)$ implies that $[U]^- = 0$ on S. Now the representation formula (16.53) completes the proof. □

Finally, we formulate the following regularity results.

Theorem 16.13. *Let the conditions* (16.6) *and* (16.23) *be fulfilled, and let the vector-function* $U \in W^1_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)$ *be the unique solution to the mixed problem* $(\mathcal{P}_{mix})^-_\omega$.

In addition to (16.6),

i) *if there hold the inclusions* (16.24)*, then*

$$U \in H^{s+1+1/t}_{t,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-); \tag{16.55}$$

ii) *if there hold the inclusions* (16.26)*, then*

$$U \in B^{s+1+1/t}_{t,q,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-); \tag{16.56}$$

iii) *if there hold the inclusions* (16.28)*, then*

$$U \in \mathrm{C}^\nu(\overline{\Omega^-}) \cap \mathrm{SK}^m_r(\Omega^-) \text{ with any } \nu \in (0, \alpha_0),\ \alpha_0 := \min\{\alpha, 1/2\}. \tag{16.57}$$

The proof of these propositions is verbatim the proof of Theorem 16.5. We only emphasize here that every solution of the equation (1.10) in Ω^- in the distributional sence, actually, is C^∞-regular in the domain Ω^-. Therefore, the inclusions (16.55)-(16.56) should be established in some compact (exterior) neigbourhood of the boundary S where we can apply the embeddings (16.31) and the arguments employed in the proof of Theorem 16.5.

17. Crack Type Problems

In this section we shall investigate the crack type problems $(\mathcal{CR.D})_\omega$ and $(\mathcal{CR.N})_\omega$ for the steady state oscillation equations of the thermoelasticity theory formulated in Section 6. We note that the crack type problems $(\mathcal{CR.D})_\tau$ and $(\mathcal{CR.N})_\tau$ for the pseudo-oscillation equations of the thermoelasticity theory are considered in detail in the reference [16].

17.1. First we treat the problem $(\mathcal{CR.D})_\omega$ (see (6.1)). Let S_1, ∂S_1, $f^{(\pm)}$, $\widetilde{f}^{(\pm)}$, $f^{(\pm)}_j$ $(j = \overline{1,4})$, be the same as in Section 6. Here we again assume that

$$f^{(\pm)}_j \in B^{1-1/p}_{p,p}(S_1),\ \ f^{(+)}_j - f^{(-)}_j \in \widetilde{B}^{1-1/p}_{p,p}(S_1),\ \ j = \overline{1,4},\ \ p > 1. \tag{17.1}$$

We recall that S_1 is a submanifold of the closed C^∞-regular surface S surrounding the bounded domain Ω^+, $\mathbb{R}^3_{S_1} = \mathbb{R}^3 \setminus \overline{S}_1$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Let $U \in W^1_{p,\mathrm{loc}}(\mathbb{R}^3_{S_1}) \cap \mathrm{SK}^m_r(\mathbb{R}^3_{S_1})$ be some solution to the steady state oscillation equations (1.10). Then $U \in \mathrm{C}^\infty(\mathbb{R}^3_{S_1}) \cap \mathrm{SK}^m_r(\mathbb{R}^3_{S_1})$ and, moreover,

$$[U]^+_{S_2} = [U]^-_{S_2},\ \ [B(D,n)U]^+_{S_2} = [B(D,n)U]^-_{S_2}, \tag{17.2}$$

where $S_2 = S \setminus \overline{S}_1$.

Due to Theorem 10.8 and the representations (3.2)–(3.3) we have the following formulae

$$W\left([U]_S^+\right)(x) - V\left([B(D,n)U]_S^+\right)(x) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-, \end{cases} \tag{17.3}$$

$$-W\left([U]_S^-\right)(x) + V\left([B(D,n)U]_S^-\right)(x) = \begin{cases} 0 & \text{for} \quad x \in \Omega^+, \\ U(x) & \text{for} \quad x \in \Omega^-, \end{cases} \tag{17.4}$$

since $U|_{\Omega^+} \in W_p^1(\Omega^+)$ and $U|_{\Omega^-} \in W^1_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}_r^m(\Omega^-)$ and $A(D,-\mathrm{i}\omega)U = 0$ in $I\!R^3_{S_1}$. Here V and W are single and double layer potentials defined by (10.1) and (10.2), respectively.

By adding these equations term by term and using the conditions (17.2), we obtain the following general integral representation of the above vector function U:

$$U(x) \;=\; W(\varphi)(x) - V(\psi)(x), \quad x \in I\!R^3_{S_1}, \tag{17.5}$$

where

$$\varphi \;=\; [U]^+_{S_1} - [U]^-_{S_1} \in \widetilde{B}^{1-1/p}_{p,p}(S_1), \tag{17.6}$$

$$\psi \;=\; [B(D,n)U]^+_{S_1} - [B(D,n)U]^-_{S_1} \in \widetilde{B}^{-1/p}_{p,p}(S_1). \tag{17.7}$$

We remark that the double and single layer potentials in (17.5) with densities (17.6) and (17.7) are C^∞-regular vector functions in $I\!R^3_{S_1}$ and belong to the class $W^1_{p,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1})$ in accordance with Theorem 10.8. Furthermore, if the representation (17.5) holds for some vector function $U \in W^1_{p,\mathrm{loc}}(I\!R^3_{S_1})$ with $\varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_1)$ and $\psi \in \widetilde{B}^{-1/p}_{p,p}(S_1)$, then automatically $U \in \mathrm{SK}^m_r(I\!R^3_{S_1})$, and the densities φ and ψ are related to the vector U by the equations (17.6) and (17.7) (which follow from the jump relations of the surface potentials involved in (17.5)).

Next, we transform the boundary conditions of the problem $(\mathcal{CR.D})_\omega$ to the equivalent equations on S_1:

$$[U]^+_{S_1} - [U]^-_{S_1} = f^{(+)} - f^{(-)}, \tag{17.8}$$

$$[U]^+_{S_1} + [U]^-_{S_1} = f^{(+)} + f^{(-)}. \tag{17.9}$$

Now, we look for the solution in the form (17.5), where φ and ψ are unknown densities having the mechanical sense described by the equations (17.6)-(17.7) due to the above remark.

It is evident that φ is then represented explicitly by formula

$$\varphi = f^{(+)} - f^{(-)} \in \widetilde{B}^{1-1/p}_{p,p}(S_1) \tag{17.10}$$

in accordance with (17.8), while the second boundary condition (17.9) leads to the ΨDE for ψ on S_1:

$$-r_{S_1}\,\mathcal{H}\,\psi \;=\; g \;\text{ on }\; S_1; \tag{17.11}$$

here the operator $\mathcal{H}$ is given by (10.3), r_{S_1} is the restriction operator to S_1, and

$$g = 2^{-1}(f^{(+)} + f^{(-)}) - r_{S_1}\,\mathcal{K}_2\,(f^{(+)} - f^{(-)}) \in \widetilde{B}^{1-1/p}_{p,p}(S_1), \qquad (17.12)$$

where the SIO $\mathcal{K}_2$ is defined by (10.5).

The inclusion (17.12) follows from Theorem 10.8.

The operator $r_{S_1}\,\mathcal{H}$ possesses the following properties.

Lemma 17.1. *The principal homogeneous symbol matrix of the pseudodifferential operator $-\mathcal{H}$ is positive definite for arbitrary $x \in \overline{S}_1$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$.*

Proof. It follows from Remark 10.4. □

Lemma 17.2. *The operators*

$$r_{S_1}\,\mathcal{H} \quad : \quad [\widetilde{B}^{s}_{p,q}(S_1)]^4 \to [B^{s+1}_{p,q}(S_1)]^4, \qquad (17.13)$$

$$: \quad [\widetilde{H}^{s}_{p}(S_1)]^4 \to [H^{s+1}_{p}(S_1)]^4, \qquad (17.14)$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. The mapping properties, boundedness, and Fredholmity of the operators (17.13)-(17.14) follow from Theorem 10.8 and Lemma 16.1 (with $\kappa = -1$). Further, by Lemma 17.1 we conclude that the Fredholm indices of the operators in question are equal to zero.

To prove that the null-spaces are trivial, we take again $s = -1/2$ and $p = q = 2$ (which satisfy the inequalities (16.15)) and consider the homogeneous equation

$$-r_{S_1}\,\mathcal{H}\,\psi \;=\; 0 \;\text{ on }\; S_1 \qquad (17.15)$$

in the space $\widetilde{B}^{-1/2}_{2,2}(S_1) = \widetilde{H}^{-1/2}_{2}(S_1)$.

Let $\psi_0 \in \widetilde{B}^{-1/2}_{2,2}(S_1)$ be some solution to the equation (17.15) and construct the vector function

$$U_0(x) \;=\; -V(\psi_0)(x), \quad x \in I\!R^3_{S_1}. \qquad (17.16)$$

Obviously, $U_0 \in W^1_{2,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1})$. Moreover, U_0 solves the homogeneous crack problem $(\mathcal{CR}.\mathcal{D})_\omega$ in $I\!R^3_{S_1}$ due to the choice of the density ψ_0 and the continuity of the single layer potential (see Theorem 10.8). By Theorem 9.7 we then infer that $U_0 = 0$ in $I\!R^3_{S_1}$, and, consequently, by Theorem 10.8 we have $[B(D,n)U_0]^+_{S_1} - [B(D,n)U_0]^-_{S_1} \;=\; -\psi_0 = 0$. This shows that $\ker[r_{S_1}\,\mathcal{H}]$ is trivial in $\widetilde{B}^{-1/2}_{2,2}(S_1)$. Now by Lemma 16.1 we conclude that, if s and p satisfy inequality (16.15), the operators (17.13) and (17.14) have trivial kernels and, therefore, are invertible. □

This lemma implies the following existence theorem.

Theorem 17.3. *Let $4/3 < p < 4$ and let the conditions* (17.1) *be fulfilled. Then the nonhomogeneous crack type problem $(\mathcal{CR}.\mathcal{D})_\omega$ is uniquely solvable in the class $W^1_{p,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1})$ and the solution is representable in* *the form* (17.5)*, where* φ *is given by* (17.10) *and* ψ *is the unique solution of the* ΨDE (17.11)*.*

Proof. If we set $s=-1/p$, then the condition (16.15) yields the inequalities for p: $4/3<p<4$. Therefore, due to Lemma 17.2, the nonhomogeneous equation (17.11) with the right-hand side q given by (17.12) is uniquely solvable. This shows that the nonhomogeneous crack type problem $(\mathcal{CR}.\mathcal{D})_\omega$ is solvable in the class $W^1_{p,\mathrm{loc}}(\mathbb{R}^3_{S_1})\cap \mathrm{SK}^m_r(\mathbb{R}^3_{S_1})$, and the vector U defined by (17.5) represents a solution to the problem in question.

Next, we prove that the problem is uniquely solvable for arbitrary $p\in(4/3,4)$.

Let $4/3<p<4$ and let U be any solution to the homogeneous problem $(\mathcal{CR}.\mathcal{D})_\omega$ from the class indicated in the theorem. Due to the above mentioned results, U is then representable by the formula (17.5) where φ and ψ are defined by (17.6) and (17.7). Therefore, $\varphi=0$, and

$$U(x)\;=\;-V(\psi)(x),\quad x\in\mathbb{R}^3_{S_1}. \tag{17.17}$$

Further, the homogeneous boundary conditions on S_1 yield that

$$-r_{S_1}\,\mathcal{H}\,\psi\;=\;0\;\;\text{on}\;\;S_1, \tag{17.18}$$

where $\psi\in\widetilde{B}^{-1/p}_{p,p}(S_1)$ with $4/3<p<4$. From this equation by Lemma 17.2 it follows that $\psi=0$ on S_1, since for $s=-1/p$ and $p\in(4/3,4)$ the condition (16.15) holds and the homogeneous equation (17.18) does not possess nontrivial solutions. Now by (17.17) we get $U=0$ in $\mathbb{R}^3_{S_1}$ which completes the proof. □

As in the case of the basic mixed BVPs here we have the following regularity results.

Theorem 17.4. *Let the conditions* (17.1) *and* (16.23) *be fulfilled, and let the vector function* $U\in W^1_{p,\mathrm{loc}}(\mathbb{R}^3_{S_1})\cap\mathrm{SK}^m_r(\mathbb{R}^3_{S_1})$ *be the unique solution to the problem* $(\mathcal{CR}.\mathcal{D})_\omega$*.*

In addition to (17.1)*,*

i) *if*

$$f^{(\pm)}\in B^{s+1}_{t,t}(S_1),\;\; f^{(+)}-f^{(-)}\in\widetilde{B}^{s+1}_{t,t}(S_1), \tag{17.19}$$

then

$$U\in H^{s+1+1/t}_{t,\mathrm{loc}}(\mathbb{R}^3_{S_1})\cap\mathrm{SK}^m_r(\mathbb{R}^3_{S_1}); \tag{17.20}$$

ii) *if*

$$f^{(\pm)}\in B^{s+1}_{t,q}(S_1),\;\; f^{(+)}-f^{(-)}\in\widetilde{B}^{s+1}_{t,q}(S_1), \tag{17.21}$$

then

$$U\in B^{s+1+1/t}_{t,q,\mathrm{loc}}(\mathbb{R}^3_{S_1})\cap\mathrm{SK}^m_r(\mathbb{R}^3_{S_1}); \tag{17.22}$$

iii) *if*

$$f^{(\pm)}\in\mathrm{C}^\alpha(S_1),\;\;[f^{(+)}-f^{(-)}]_{\partial S_1}=0,\;\;\text{for some}\;\;\alpha>0, \tag{17.23}$$

then

$$\begin{aligned}&U|_{\overline{\Omega^+}}\in\mathrm{C}^\nu(\overline{\Omega^+}),\\&U|_{\overline{\Omega^-}}\in\mathrm{C}^\nu(\overline{\Omega^-})\cap\mathrm{SK}^m_r(\Omega^-)\end{aligned} \tag{17.24}$$

with any $\nu \in (0, \alpha_0)$, $\alpha_0 := \min\{\alpha, 1/2\}$.

Proof. It is again verbatim the proof of Theorem 16.5 (see also the remark after Theorem 16.13). □

17.2. In this subsection we consider the problem $(\mathcal{CR}.\mathcal{N})_\omega$ (see (6.2)). The corresponding boundary conditions (6.2) we transform to the equivalent equations on the crack surface S_1:

$$[B(D,n)U]^+_{S_1} - [B(D,n)U]^-_{S_1} = F^{(+)} - F^{(-)}, \tag{17.25}$$

$$[B(D,n)U]^+_{S_1} + [B(D,n)U]^-_{S_1} = F^{(+)} + F^{(-)}, \tag{17.26}$$

where we assume that

$$F_j^{(\pm)} \in B^{-1/p}_{p,p}(S_1), \;\; F_j^{(+)} - F_j^{(-)} \in \widetilde{B}^{-1/p}_{p,p}(S_1), \;\; j = \overline{1,4}, \;\; p > 1. \tag{17.27}$$

We look for a solution

$$U \in W^1_{p,\mathrm{loc}}(I\!\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!\!R^3_{S_1}) \tag{17.28}$$

in the form (17.5), where the densities φ and ψ are related to the sought for vector U again by the realations (17.6) and (17.7). Therefore, we can define ψ explicitly

$$\psi = F^{(+)} - F^{(-)} \in \widetilde{B}^{-1/p}_{p,p}(S_1), \tag{17.29}$$

while the boundary condition (17.26) implies the ΨDE (of order 1) for the unknown vector-function φ

$$r_{S_1} \mathcal{L}\, \varphi \;=\; g \;\;\text{on}\;\; S_1; \tag{17.30}$$

here the ΨDO $\mathcal{L}$ is given by (10.6) and

$$g = 2^{-1}(F^{(+)} + F^{(-)}) + r_{S_1} \mathcal{K}_1 \, (F^{(+)} - F^{(-)}) \in B^{-1/p}_{p,p}(S_1), \tag{17.31}$$

where the SIO $\mathcal{K}_1$ is defined by (10.4). Note that the inclusion (17.31) for the right-hand side vector g follows again from Theorem 10.8 and conditions (17.27).

Now we show that the equation (17.30) is uniquely solvable in the space $\widetilde{B}^{1-1/p}_{p,p}(S_1)$. To this end we remark that the principal homogeneous symbol matrix of the operator $\mathcal{L}$ is positive definite for arbitrary $x \in \overline{S}_1$ and $\widetilde{\xi} \in I\!\!R^2 \setminus \{0\}$ due to Lemma 10.7. The basic invertibility property of the operator $r_{S_1} \mathcal{L}$ is described by the following proposition.

Lemma 17.5. *The operators*

$$r_{S_1} \mathcal{L} \quad : \quad [\widetilde{B}^{s+1}_{p,q}(S_1)]^4 \to [B^s_{p,q}(S_1)]^4, \tag{17.32}$$

$$: \quad [\widetilde{H}^{s+1}_p(S_1)]^4 \to [H^s_p(S_1)]^4, \tag{17.33}$$

are bounded for any $s \in I\!\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. It is quite similar to the proof of Lemma 17.2. □

With the help of this lemma and by the arguments employed in the proofs of Theorems 17.3 and 16.5 one can easily derive the following existence and uniqueness results and establish the regularity of solutions.

Theorem 17.6. *Let* $4/3 < p < 4$ *and let the conditions* (17.27) *be fulfilled. Then the nonhomogeneous crack type problem* $(\mathcal{CR.N})_\omega$ *is uniquely solvable in the class* $W^1_{p,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1})$ *and the solution is representable in the form* (17.5), *where* ψ *is given by* (17.29) *and* φ *is the unique solution of the* ΨDE (17.30).

Theorem 17.7. *Let the conditions* (17.27) *and* (16.23) *be fulfilled, and let the vector-function* $U \in W^1_{p,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1})$ *be the unique solution to the problem* $(\mathcal{CR.N})_\omega$.

In addition to (17.27),

i) *if*

$$F^{(\pm)} \in B^s_{t,t}(S_1), \;\; F^{(+)} - F^{(-)} \in \widetilde{B}^s_{t,t}(S_1),$$

then

$$U \in H^{s+1+1/t}_{t,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1});$$

ii) *if*

$$F^{(\pm)} \in B^s_{t,q}(S_1), \;\; F^{(+)} - F^{(-)} \in \widetilde{B}^s_{t,q}(S_1),$$

then

$$U \in B^{s+1+1/t}_{t,q,\mathrm{loc}}(I\!R^3_{S_1}) \cap \mathrm{SK}^m_r(I\!R^3_{S_1});$$

iii) *if*

$$F^{(\pm)} \in B^{\alpha-1}_{\infty,\infty}(S_1), \;\; F^{(+)} - F^{(-)} \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(S_1), \;\; \text{for some } \alpha > 0,$$

then

$U|_{\overline{\Omega^+}} \in \mathrm{C}^\nu(\overline{\Omega^+})$,
$U|_{\overline{\Omega^-}} \in \mathrm{C}^\nu(\overline{\Omega^-}) \cap \mathrm{SK}^m_r(\Omega^-)$ with any $\nu \in (0, \alpha_0)$, $\alpha_0 := \min\{\alpha, 1/2\}$.

Remark 17.8. For an arbitrary solution $U \in W^1_p(I\!R^3_{S_1})$ of the pseudo-oscillation equation (1.9) there also holds the representation formula by potential type integrals similar to (17.5) with the densities φ and ψ related to the vector U by relations (17.6) and (17.7). Therefore, for the crack type problems $(\mathcal{CR.D})_\tau$ and $(\mathcal{CR.N})_\tau$ the existence and uniqueness theorems, and the regularity results analogous to the above ones can be proved with quite the same arguments (for details see [16]).

18. Mixed Interface Problems of Steady State Oscillations

In this section first we shall prove the existence and uniqueness theorems for the mixed interface problems for the steady state oscillation equations of the thermoelasticity theory formulated in Section 7. Afterwards, as in the previous sections, we shall establish the smoothness properties of solutions. Throughout this section we shall keep and employ the notations of Section 15.

18.1. Problem $(\mathcal{C} - \mathcal{DD})_\omega$**.** To examine the existence of solutions to the problem in question (see (7.13)–(7.14)) we shell exploit the representation formulae (15.61)–(15.62), and use again the Fredholm properties of ΨDOs on manifold with boundary described by Lemma 16.1. First, let us note that the conditions (7.14) on S_2 are equivalent to the following equations

$$[U^{(1)}]^+ - [U^{(2)}]^- = \varphi^{(+)} - \varphi^{(-)}, \;\; [U^{(1)}]^+ + [U^{(2)}]^- = \varphi^{(+)} + \varphi^{(-)}, \;\text{ on }\; S_2.$$

According to (7.21) and (7.23) we require that

$$f^{(1)} \in B^{1-1/p}_{p,p}(S_1), \;\; \varphi^{(\pm)} \in B^{1-1/p}_{p,p}(S_2), \;\; F^{(1)} \in B^{-1/p}_{p,p}(S_1), \tag{18.1}$$

and, moreover,

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \in B^{1-1/p}_{p,p}(S),$$
$$\text{where } f = \begin{cases} f^{(1)} & \text{on} \quad S_1, \\ \varphi^{(+)} - \varphi^{(-)} & \text{on} \quad S_2. \end{cases} \tag{18.2}$$

Clearly, this last inclusion is the necessary compatibility condition for the problem $(\mathcal{C} - \mathcal{D}\mathcal{D})_\omega$.

In view of the third inclusion in (18.1), the vector $F^{(1)}$ can be extended from S_1 onto S_2 preserving the functional space $B^{-1/p}_{p,p}(S)$. Denote some fixed extension by F^0,

$$F^0 \in B^{-1/p}_{p,p}(S), \;\; F^0|_{S_1} = F^{(1)}. \tag{18.3}$$

Evidently, any arbitrary extension F of $F^{(1)}$ onto the whole of S which preserves the functional space can be represented as

$$F = F^0 + \varphi \in B^{-1/p}_{p,p}(S), \;\text{ where }\; \varphi \in \widetilde{B}^{-1/p}_{p,p}(S_2). \tag{18.4}$$

Now we can reformulate the interface problem $(\mathcal{C} - \mathcal{D}\mathcal{D})_\omega$ in the following equivalent form: Find a pair of vector functions

$$(U^{(1)}, U^{(2)}) \;=\; (W^1_p(\Omega^1)\,,\, W^1_{p,\mathrm{loc}}(\Omega^2) \cap \mathrm{SK}^m_r(\Omega^2)) \tag{18.5}$$

satisfying the differential equations (7.2) and the interface conditions

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \;\text{ on }\; S, \tag{18.6}$$
$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \;\text{ on }\; S_1, \tag{18.7}$$
$$[U^{(1)}]^+ + [U^{(2)}]^- = \varphi^{(+)} + \varphi^{(-)} \;\text{ on }\; S_2, \tag{18.8}$$

where $B^{(\mu)}(D,n)$ is defined by (1.25), f and F are given by (18) and (18.4), respectively. Let us note that f and F^0 are considered now as the known vector functions on the whole of S, while F is given only on S_1 ($F|_{S_1} = F^0|_{S_1} = F^{(1)}$), and $\varphi^{(\pm)}$ are given vector functions on S_2.

We look for the solution to the problem $(\mathcal{C} - \mathcal{D}\mathcal{D})_\omega$ in the form (cf. (15.61)–(15.62))

$$U^{(1)}(x) = W^{(1)}\left(\Psi\,[F^0 + \varphi] - \Psi\,\Psi_2\,\Phi_2^{-1}\,f\right)(x), \tag{18.9}$$
$$U^{(2)}(x) = \left(W^{(2)} + p_0\,V^{(2)}\right)\left(\Phi_2^{-1}\,\Phi_1\,\Psi\,[F^0 + \varphi] - \right.$$
$$\left. -\Phi_2^{-1}\,[\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1} \;+\; I]\,f\right)(x), \tag{18.10}$$

where $\varphi \in \widetilde{B}_{p,p}^{-1/p}(S_2)$ is the unknown vector-function, and F^0 and f are as above. Furthermore, $W^{(\mu)}$ and $V^{(\mu)}$ are the double and single layer potentials of steady state oscillations, the complex number p_0 and the boundary operators Ψ, Ψ_j, Φ_j are defined by equations (13.5) and (15.58), (15.9), (15.10).

It is easy to verify that the interface conditions (18.6) and (18.7) are satisfied automatically, since from (18.9) and (18.10) it follows that

$$[U^{(1)}]^+ - [U^{(2)}]^- = f, \;\; [B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^0 + \varphi \;\text{ on } S.$$

It remains only to satisfy the condition (18.8) which leads to the ΨDE for φ

$$[U^{(1)}]^+ + [U^{(2)}]^- = \Phi_1 \,\Psi\, [F^0 + \varphi] - \Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1}\, f + \Phi_1\, \Psi\, [F^0 + \varphi] - \\ -[\Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1} \, + \, I]\, f = \varphi^{(+)} + \varphi^{(-)} \;\text{ on }\; S_2, \tag{18.11}$$

which can be rewritten as

$$r_{S_2}\, [\Phi_1\, \Psi\, \varphi] = r_{S_2}\, \mathcal{K}_H\, \varphi = q \;\text{ on }\; S_2, \tag{18.12}$$

where r_{S_2} is the restriction operator on S_2, the ΨDO (of order -1) $\mathcal{K}_H$ has been defined by (15.105), while the given right-hand side q reads as follows

$$q = 2^{-1}(\varphi^{(+)} + \varphi^{(-)}) - r_{S_2}\, \{\Phi_1\, \Psi\, F^0 - \\ - [\Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1} \, + 2^{-1}\, I]\, f\} \in B_{p,p}^{1-1/p}(S_2). \tag{18.13}$$

Due to Lemma 15.14 the principal homogeneous symbol matrix of the operator $\mathcal{K}_H = \Phi_1\, \Psi$ is positive definite. Therefore, we can apply Lemma 16.1 to study the equation (18.12).

Lemma 18.1. *The operators*

$$r_{S_2}\, \mathcal{K}_H \quad : \quad [\widetilde{B}_{p,q}^{s}(S_2)]^4 \to [B_{p,q}^{s+1}(S_2)]^4, \tag{18.14}$$

$$: \quad [\widetilde{H}_p^{s}(S_2)]^4 \to [H_p^{s+1}(S_2)]^4, \tag{18.15}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. The mapping properties (18.14) and (18.15), boundedness and Fredholmity of the above operators follow from equations $\mathcal{K}_H = \Phi_1\, \Psi$, $\Phi_1 = 2^{-1}\, I_4 + \mathcal{K}_2^{(1)}$, $\Psi = [\Psi_1 - \Psi_2\, \Phi_2^{-1}\, \Phi_1]^{-1}$, and Corollary 15.6, Theorem 10.8 and Lemma 16.1 (with $\kappa = -1$). From the positive definiteness of the principal homogeneous symbol matrix $\sigma(\mathcal{K}_H)$ it follows that the Fredholm indices of the operators (18.14) and (18.15) are equal to zero.

It remains to prove that the corresponding null-spaces are trivial. To this end, let us take $s = -1/2$ and $p = q = 2$, which meet inequalities (16.15), and show that the homogeneous equation

$$r_{S_2}\, \mathcal{K}_H\, \varphi = 0 \;\text{ on }\; S_2 \tag{18.16}$$

has no nontrivial solutions in the space $\widetilde{B}_{2,2}^{-1/2}(S_2) = \widetilde{H}_2^{-1/2}(S_2)$.

Let $\varphi \in \widetilde{B}_{2,2}^{-1/2}(S_2)$ be any solution to the equation (18.16) and construct the vector functions

$$U_0^{(1)}(x) = W^{(1)}\,(\Psi\,\varphi)\,(x), \quad x \in \Omega^1, \tag{18.17}$$

$$U_0^{(2)}(x) = \Big(W^{(2)} + p_0\,V^{(2)}\Big)\,\big(\Phi_2^{-1}\,\Phi_1\,\Psi\,\varphi\big)\,(x), \quad x \in \Omega^2. \tag{18.18}$$

Clearly, $\Psi\,\varphi \in B_{2,2}^{1/2}(S)$ and $\Phi_2^{-1}\,\Phi_1\,\Psi\,\varphi \in B_{2,2}^{1/2}(S)$. Therefore, by Theorem 10.8 we have

$$(U_0^{(1)}, U_0^{(2)}) \in (W_2^1(\Omega^1)\,,\, W_{2,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2)). \tag{18.19}$$

Moreover, these vectors satisfy homogeneous differential equations of steady state oscillations (7.2) in the corresponding domains Ω^1 and Ω^2, and the homogeneous interface conditions of the problem $(\mathcal{C}-\mathcal{DD})_\omega$ on S, since

$$[U_0^{(1)}]_S^+ = [U_0^{(2)}]_S^-, \quad [B^{(1)}(D,n)U_0^{(1)}]_{S_1}^+ - [B^{(2)}(D,n)U_0^{(2)}]_{S_1}^- = \varphi|_{S_1} = 0,$$
$$[U_0^{(1)}]_{S_2}^+ + [U_0^{(2)}]_{S_2}^- = r_{S_2}\,\mathcal{K}_H\,\varphi = 0 \quad \text{on} \quad S_2.$$

These conditions follow from the formulae (18.17), (18.18), definition of the operator Ψ (see (15.58)) and the fact that φ solves the homogeneous equation (18.16).

Therefore, by Theorem 9.12 we conclude that $U_0^{(1)} = 0$ in Ω^1 and $U_0^{(2)} = 0$ in Ω^2. Whence $\varphi = 0$ on S follows. Thus, the null-spaces of the operators (18.14) and (18.15) are trivial in the space $\widetilde{B}_{2,2}^{-1/2}(S_2) = \widetilde{H}_2^{-1/2}(S_2)$. Now, Lemma 16.1 completes the proof for arbitrary p and s satisfying the inequalities (16.15), and arbitrary $q \in [1,\infty]$. □

This lemma implies the following existence theorems.

Theorem 18.2. *Let* $4/3 < p < 4$ *and let the conditions* (18.1)–(18) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C}-\mathcal{DD})_\omega$ *is uniquely solvable in the class* $(W_p^1(\Omega^1)\,,\, W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ (*with the parameters* r *and* ω *as in* (15.3)) *and the solution is representable in the form* (18.9)–(18.10), *where* φ *is the unique solution of the* ΨDE (18.12).

Proof. First we observe that, if $s = -1/p$, then the inequality (16.15) yields $4/3 < p < 4$. Therefore, by Lemma 18.1 the nonhomogeneous ΨDE (18.12) with the right-hand side q given by (18.13) is uniquely solvable in the space $\widetilde{B}_{p,p}^{1-1/p}(S_2)$. This shows that the nonhomogeneous problem $(\mathcal{C}-\mathcal{DD})_\omega$ is solvable under the conditions indicated in the theorem, and the pair $(U^{(1)}, U^{(2)})$ defined by (18.9)-(18.10) represents a solution to the problem in question.

Further, we prove that the problem is uniquely solvable for any $p \in (4/3, 4)$.

Let some pair $(U^{(1)}, U^{(2)}) \in (W_p^1(\Omega^1)\,,\, W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ (with the parameters p, r, and ω as in the theorem) represents a solution to the homogeneous problem $(\mathcal{C}-\mathcal{DD})_\omega$. In accordance with (18.6)-(18.7) then we have

$$[U^{(1)}]_S^+ - [U^{(2)}]_S^- = 0,$$
$$[B^{(1)}(D,n)U^{(1)}]_S^+ - [B^{(2)}(D,n)U^{(2)}]_S^- = F \in \widetilde{B}_{p,p}^{-1/p}(S_2), \tag{18.20}$$
$$[U^{(1)}]^+ + [U^{(2)}]^- = 0 \ \text{ on } \ S_2.$$

Clearly, F may differ from zero only on the submanifold $\overline{S}_2$ due to the homogeneous condition (18.7).

Further, by Theorem 15.8 we conclude that the vector functions $U^{(1)}$ and $U^{(2)}$ are uniquely representable in the form

$$U^{(1)}(x) = W^{(1)}\,(\Psi\, F)\,(x), \ \ x \in \Omega^1,$$
$$U^{(2)}(x) = \Big(W^{(2)} + p_0\, V^{(2)}\Big)\,\big(\Phi_2^{-1}\,\Phi_1\,\Psi\, F\big)\,(x), \ \ x \in \Omega^2,$$

where F is defined by the second equation in (18.20).

The third equation in (18.20) then yields

$$r_{S_2}\,\mathcal{K}_H\, F = 0 \ \text{ on } \ S_2,$$

where $F \in \widetilde{B}_{p,p}^{-1/p}(S_2)$ and $p \in (4/3, 4)$. Therefore, $F = 0$ on S due to Lemma 18.1 (with $s = -1/p$) which implies $U^{(\mu)} = 0$ in Ω^μ $(\mu = 1, 2)$. □

Now we can formulate the following regularity results.

Theorem 18.3. *Let the conditions* (18.1), (18)*, and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) = (W_p^1(\Omega^1)\,,\, W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C} - \mathcal{DD})_\omega$.

In addition to (18.1)–(18),

i) *if*

$$f^{(1)} \in B_{t,t}^{s+1}(S_1), \ \varphi^{(\pm)} \in B_{t,t}^{s+1}(S_2), \ F^{(1)} \in B_{t,t}^{s}(S_1), \ f \in B_{t,t}^{s+1}(S), \tag{18.21}$$

then

$$(U^{(1)}, U^{(2)}) \in (H_t^{s+1+1/t}(\Omega^1)\,,\, H_{t,\mathrm{loc}}^{s+1+1/t}(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2)); \tag{18.22}$$

ii) *if*

$$f^{(1)} \in B_{t,q}^{s+1}(S_1), \ \varphi^{(\pm)} \in B_{t,q}^{s+1}(S_2), \ F^{(1)} \in B_{t,q}^{s}(S_1), \ f \in B_{t,q}^{s+1}(S), \tag{18.23}$$

then

$$(U^{(1)}, U^{(2)}) \in (B_{t,q}^{s+1+1/t}(\Omega^1)\,,\, B_{t,q,\mathrm{loc}}^{s+1+1/t}(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2)); \tag{18.24}$$

iii) *if*

$$f^{(1)} \in \mathrm{C}^\alpha(S_1), \ \varphi^{(\pm)} \in \mathrm{C}^\alpha(S_2), \ F^{(1)} \in B_{\infty,\infty}^{\alpha-1}(S_1), \ f \in \mathrm{C}^\alpha(S), \tag{18.25}$$

for some $\alpha > 0$*, then*

$$\begin{aligned}&(U^{(1)}, U^{(2)}) \in (\mathrm{C}^\nu(\overline{\Omega^1})\,,\, \mathrm{C}^\nu(\overline{\Omega^2}) \cap \mathrm{SK}_r^m(\Omega^2))\\ &\textit{with any} \ \ \nu \in (0, \alpha_0), \ \ \alpha_0 := \min\{\alpha, 1/2\}.\end{aligned} \tag{18.26}$$

Proof. Here it is again verbatim the proof of Theorem 16.5 (see also the remark after Theorem 16.13). □

18.2. Problem $(\mathcal{C}-\mathcal{N}\mathcal{N})_\omega$. As in the previous subsection we start with the reformulation of the problem. In particular, the conditions (7.13) and (7.15) are equivalent to the following equations

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \quad \text{on} \quad S, \tag{18.27}$$

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \quad \text{on} \quad S_1, \tag{18.28}$$

$$[B^{(1)}(D,n)U^{(1)}]^+ + [B^{(2)}(D,n)U^{(2)}]^- = \Phi^{(+)} + \Phi^{(-)} \quad \text{on} \quad S_2, \tag{18.29}$$

where

$$F := \begin{cases} F^{(1)} & \text{on} \quad S_1, \\ \Phi^{(+)} - \Phi^{(-)} & \text{on} \quad S_2. \end{cases} \quad F \in B^{-1/p}_{p,p}(S), \ \Phi^{(\pm)} \in B^{-1/p}_{p,p}(S_2), \tag{18.30}$$

$$f := f^0 + \varphi \in B^{1-1/p}_{p,p}(S), \ \ f^0 \in B^{1-1/p}_{p,p}(S), \ \ \varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2); \tag{18.31}$$

here f^0 is some fixed extension of the vector $f^{(1)}$ from S_1 onto S_2 preserving the functional space: $f^0|_{S_1} = f^{(1)}$, and, therefore, $f = f^0 + \varphi$ with φ as in (18.31), represents an arbitrary extension of $f^{(1)}$ onto the whole of S: $f|_{S_1} = f^0|_{S_1} = f^{(1)}$.

Obviously, the inclusion $F \in B^{-1/p}_{p,p}(S)$ is the necessary compatibility condition for the problem under consideration.

Let us now look for the solution to the problem $(\mathcal{C}-\mathcal{N}\mathcal{N})_\omega$ in the form (cf. (15.61)–(15.62))

$$U^{(1)}(x) = W^{(1)}\left(\Psi\, F - \Psi\, \Psi_2\, \Phi_2^{-1}\, [f^0 + \varphi]\right)(x), \tag{18.32}$$

$$U^{(2)}(x) = \Big(W^{(2)} + p_0\, V^{(2)}\Big)\left(\Phi_2^{-1}\, \Phi_1\, \Psi\, F - \right.$$
$$\left. -\Phi_2^{-1}\, [\Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1} \ + \ I]\, [f^0 + \varphi]\right)(x), \tag{18.33}$$

where f^0 and F are the given vector functions on S, while φ is the unknown vector function.

It can be easily seen that the conditions (18.27) and (18.28) are satisfied automatically, since

$$[U^{(1)}]^+ - [U^{(2)}]^- = f^0 + \varphi,$$
$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \quad \text{on} \quad S.$$

due to the above representations.

It remains only to fulfil the condition (18.29) which yields the following ΨDE on S_2 for the unknown vector φ:

$$[B^{(1)}(D,n)U^{(1)}]^+ + [B^{(2)}(D,n)U^{(2)}]^- =$$
$$= \Psi_1\, \Psi\, F - \Psi_1\, \Psi\, \Psi_2\, \Phi_2^{-1}\, [f^0 + \varphi] + \Psi_2\, \Phi_2^{-1}\, \Phi_1\, \Psi\, F -$$
$$-\Psi_2\, \Phi_2^{-1}\, [\Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1} \ + \ I]\, [f^0 + \varphi] = \Phi^{(+)} + \Phi^{(-)}, \tag{18.34}$$

With the help of equations (15.9), (15.10), (15.58) we can simplify this equation:

$$r_{S_2}\,[-\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,\varphi] = r_{S_2}\,\mathcal{K}_G\,\varphi = q \quad\text{on}\quad S_2, \tag{18.35}$$

where the ΨDE (of order $+1$) $\mathcal{K}_G$ has been defined by (15.86), while the right-hand side vector function q reads as follows

$$q = 2^{-1}(\Phi^{(+)} + \Phi^{(-)}) - r_{S_2}\{\Psi_1\Psi - 2^{-1}I]F + \mathcal{K}_G f^0\} \in B^{-1/p}_{p,p}(S_2). \tag{18.36}$$

According to Lemma 15.9 the principal homogeneous symbol matrix of the operator $\mathcal{K}_G$ is positive definite. Therefore, we can again apply Lemma 16.1 to examine the equation (18.35), and employ the same arguments as in the previous section to prove the following propositions.

Lemma 18.4. *The operators*

$$r_{S_2}\,\mathcal{K}_G \;:\; [\widetilde{B}^{s+1}_{p,q}(S_2)]^4 \to [B^{s}_{p,q}(S_2)]^4, \tag{18.37}$$

$$:\; [\widetilde{H}^{s+1}_{p}(S_2)]^4 \to [H^{s}_{p}(S_2)]^4, \tag{18.38}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Theorem 18.5. *Let* $4/3 < p < 4$ *and let the conditions* (18.30)–(18.31) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C}-\mathcal{N}\mathcal{N})_\omega$ *is uniquely solvable in the class of vector functions* $(W^1_p(\Omega^1)\,,\,W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ (*with the parameters* r *and* ω *as in* (15.3)) *and the solution is representable in the form* (18.32)–(18.33), *where* φ *is the unique solution of the* Ψ*DE* (18.35).

Theorem 18.6. *Let the conditions* (18.30), (18.31), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1)\,,\,W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C}-\mathcal{N}\mathcal{N})_\omega$.

In addition to (18.30)–(18.31),

i) *if*

$$f^{(1)} \in B^{s+1}_{t,t}(S_1),\;\; F^{(1)} \in B^{s}_{t,t}(S_1),\;\; \Phi^{(\pm)} \in B^{s}_{t,t}(S_2),\;\; F \in B^{s}_{t,t}(S), \tag{18.39}$$

then there holds the inclusion (18.22);

ii) *if*

$$f^{(1)} \in B^{s+1}_{t,q}(S_1),\;\; F^{(1)} \in B^{s}_{t,q}(S_1),\;\; \Phi^{(\pm)} \in B^{s}_{t,q}(S_2),\;\; F \in B^{s}_{t,q}(S), \tag{18.40}$$

then there holds the inclusion (18.24);

iii) *if*

$$f^{(1)} \in \mathrm{C}^{\alpha}(S_1),\;\; F^{(1)} \in B^{\alpha-1}_{\infty,\infty}(S_1),\;\; \Phi^{(\pm)} \in B^{\alpha-1}_{\infty,\infty}(S_2),\;\; F \in B^{\alpha-1}_{\infty,\infty}(S), \tag{18.41}$$

for some $\alpha > 0$, *then there holds the inclusion* (18.26).

The proofs of the above assertions are verbatim the proofs of Lemma 18.1 and Theorems 18.2 and 16.5.

18.3. Problem $(\mathcal{C}-\mathcal{DC})_\omega$**.** In this case the interface conditions read as follows (see Subsection 7.2):

$$\begin{aligned}
&[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4,\\
&[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4 \ \text{ on } \ S,
\end{aligned} \tag{18.42}$$

$$\begin{aligned}
&[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}^{(1)},\\
&[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}^{(1)} \ \text{ on } \ S_1,
\end{aligned} \tag{18.43}$$

$$[u^{(1)}]^+ = \widetilde{\varphi}^{(+)}, \ \ [u^{(2)}]^- = \widetilde{\varphi}^{(-)} \ \text{ on } \ S_2, \tag{18.44}$$

where

$$\begin{aligned}
&f_4 \in B_{p,p}^{1-1/p}(S), \ \ F_4 \in B_{p,p}^{-1/p}(S),\\
&\widetilde{\varphi}^{(\pm)} = (\varphi_1^{(\pm)}, \varphi_2^{(\pm)}, \varphi_3^{(\pm)})^\top \in [B_{p,p}^{1-1/p}(S_2)]^3,\\
&\widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^\top \in [B_{p,p}^{1-1/p}(S_1)]^3,\\
&\widetilde{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [B_{p,p}^{-1/p}(S_1)]^3.
\end{aligned} \tag{18.45}$$

Let $\widetilde{F}^0 = (F_1^0, F_2^0, F_3^0)^\top$ be some fixed extension of the vector $\widetilde{F}^{(1)}$ from S_1 onto S_2 preserving the functional space, i.e.,

$$\widetilde{F}^0 \in [B_{p,p}^{-1/p}(S)]^3, \ \ \widetilde{F}^0|_{S_1} = \widetilde{F}^{(1)}. \tag{18.46}$$

Then an arbitrary extension of $\widetilde{F}^{(1)}$ onto the whole of S preserving the functional space can be written as follows

$$\widetilde{F} = (F_1, F_2, F_3)^\top = \widetilde{F}^0 + \widetilde{\varphi} \in [B_{p,p}^{-1/p}(S)]^3, \tag{18.47}$$

where $\widetilde{\varphi}$ is an arbitrary vector function with the support in $\overline{S}_2$, i.e.,

$$\widetilde{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^\top \in [\widetilde{B}_{p,p}^{-1/p}(S_2)]^3. \tag{18.48}$$

Next we set

$$F = (F_1, \cdots, F_4)^\top := F^0 + \varphi \in [B_{p,p}^{-1/p}(S)]^4, \tag{18.49}$$

where

$$F^0 = (\widetilde{F}^0, F_4)^\top \in [B_{p,p}^{-1/p}(S)]^4 \tag{18.50}$$

is the given vector function, and

$$\varphi = (\widetilde{\varphi}, 0)^\top \in [\widetilde{B}_{p,p}^{-1/p}(S_2)]^4 \tag{18.51}$$

with $\widetilde{\varphi}$ subjected to the condition (18.48).

It is easily seen that the conditions (18.42)-(18.44) are equivalent to the equations

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \ \text{ on } \ S, \tag{18.52}$$

$$[B^{(1)}(D,n)U^{(1)}]_k^+ - [B^{(2)}(D,n)U^{(2)}]_k^- = F_k \ \text{ on } \ S_1, \ \ k=1,2,3, \tag{18.53}$$

$$[B^{(1)}(D,n)U^{(1)}]_4^+ - [B^{(2)}(D,n)U^{(2)}]_4^- = F_4 \ \text{ on } \ S, \tag{18.54}$$

$$[U^{(1)}]_k^+ + [U^{(2)}]_k^- = \varphi_k^{(+)} + \varphi_k^{(-)} \ \text{ on } \ S_2, \ \ k=1,2,3, \tag{18.55}$$

where f is the given vector function

$$f=(f_1,\cdots,f_4)^\top=:\begin{cases}(\widetilde{f}^{(1)},f_4)^\top & \text{on}\quad S_1,\\ (\widetilde{\varphi}^{(+)}-\widetilde{\varphi}^{(-)},f_4)^\top & \text{on}\quad S_2,\end{cases}\tag{18.56}$$

satisfying the following necessary compatibility condition (cf. (7.25))

$$f\in[B^{1-1/p}_{p,p}(S)]^4,\tag{18.57}$$

and F_k and $\widetilde{\varphi}^{\pm}$ are as above.

After this reformulation of the problem in question let us look for the solution in the form (18.9)-(18.10), where f, F^0, and φ are defined by formulae (18.56), (18.50), and (18.51), respectively. These representations imply

$$\begin{aligned}&[U^{(1)}]^+-[U^{(2)}]^-=f,\\ &[B^{(1)}(D,n)U^{(1)}]^+-[B^{(2)}(D,n)U^{(2)}]^-=F^0+\varphi.\end{aligned}\tag{18.58}$$

Therefore, the conditions (18.52), (18.53), and (18.54) are satisfied automatically. It remains to meet the conditions (18.55) which, by virtue of (18.11) and (18.12), lead to the system of ΨDEs for the vector function $\varphi=(\widetilde{\varphi},0)^\top$ on S_2:

$$r_{S_2}[\Phi_1\,\Psi\,\varphi]_k=r_{S_2}[(\mathcal{K}_H)_{kj}\,\varphi_j]=q_k\quad\text{on}\quad S_2,\quad k=1,2,3,\tag{18.59}$$

where the summation over the repeated index j is meant from 1 to 3, and (see (18.13))

$$\begin{aligned}q_k=2^{-1}(\varphi_k^{(+)}+\varphi_k^{(-)})-r_{S_2}\{\Phi_1\,\Psi\,F^0-\\ -[\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+2^{-1}I]\,f\}_k\in B^{1-1/p}_{p,p}(S_2);\end{aligned}\tag{18.60}$$

here $\mathcal{K}_H$ is again the ΨDO of order -1 defined by (15.105) with properties described by Lemmata 15.14 and 18.1.

Let

$$\widetilde{\mathcal{K}}_H:=[(\mathcal{K}_H)_{kj}]_{3\times3},\quad k,j=1,2,3,\qquad \widetilde{q}:=(q_1,q_2,q_3)^\top.\tag{18.61}$$

Then (18.59) can be written in the matrix form as

$$r_{S_2}\,\widetilde{\mathcal{K}}_H\,\widetilde{\varphi}\;=\;\widetilde{q}\tag{18.62}$$

where $\widetilde{\varphi}=(\varphi_1,\varphi_2,\varphi_3)^\top\in[\widetilde{B}^{-1/p}_{p,p}(S_2)]^3$ is the sought for vector.

The following properties of the ΨDO $\widetilde{\mathcal{K}}_H$ are immediate consequences of Lemmata 15.14 and 18.1.

Lemma 18.7. *The principal homogeneous symbol matrix of the operator* $\widetilde{\mathcal{K}}_H$ *is positive definite for arbitrary* $x\in S$ *and* $\widetilde{\xi}\in I\!R^2\setminus\{0\}$. *The following operators*

$$\begin{aligned}r_{S_2}\,\widetilde{\mathcal{K}}_H\quad &:\quad [\widetilde{B}^s_{p,q}(S_2)]^3\to[B^{s+1}_{p,q}(S_2)]^3, &(18.63)\\ &:\quad [\widetilde{H}^s_p(S_2)]^3\to[H^{s+1}_p(S_2)]^3, &(18.64)\end{aligned}$$

are bounded for any $s\in I\!R$, $1<p<\infty$, $1\le q\le\infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. The first assertion of the lemma follows from the proof of Lemma 15.14 (see (15.106)–(15.107)), since $\sigma(\widetilde{\mathcal{K}}_H) = X$, where X is the positive definite 3×3 matrix given by formula (15.107) (for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$).

The boundedness of the operators (18.63)-(18.64) is a consequence of Lemma 18.1.

It is evident that the Fredholm indices of these operators are equal to zero. This follows from the positive definiteness of the principal symbol matrix $\sigma(\widetilde{\mathcal{K}}_H)$. Therefore, to prove the last proposition of the lemma, we have to show that the corresponding null-spaces are trivial for any s and p satisfying the inequalities (16.15).

Again, we take $s = -1/p$ and $p = q = 2$ to prove that the homogeneous ΨDE

$$r_{S_2}\,\widetilde{\mathcal{K}}_H\,\widetilde{\varphi} \,=\, 0 \tag{18.65}$$

has no nontrivial solutions. Let $\widetilde{\varphi}_0 = (\varphi_{01},\varphi_{02},\varphi_{03})^\top \in [\widetilde{B}^{-1/p}_{p,p}(S_2)]^3$ be any solution to the equation (18.65) and using the formulae (18.17) and (18.18) construct the vector functions $U_0^{(1)}$ and $U_0^{(2)}$, where the density φ is represented as follows

$$\varphi = (\widetilde{\varphi}_0, 0)^\top \in [B^{-1/2}_{2,2}(S_2)]^4.$$

Therefore, the inclusion (18.19) remains valid, and, moreover, $U_0^{(1)}$ and $U_0^{(2)}$ satisfy the homogeneous interface conditions (18.52)-(18.55):

$$\begin{aligned}
&[U_0^{(1)}]^+ = [U_0^{(2)}]^- \quad \text{on} \quad S,\\
&[B^{(1)}(D,n)U_0^{(1)}]^+_k - [B^{(2)}(D,n)U_0^{(2)}]^-_k = \varphi_{0k} \ \text{ on } \ S_1, \ \ k=1,2,3,\\
&[B^{(1)}(D,n)U_0^{(1)}]^+_4 - [B^{(2)}(D,n)U_0^{(2)}]^-_4 = 0 \ \text{ on } \ S,\\
&[U_0^{(1)}]^+_k + [U_0^{(2)}]^-_k = [\,r_{S_2}\,\Phi_1\,\Psi\,\varphi\,]_k = [\,r_{S_2}\,\widetilde{\mathcal{K}}_H\,\widetilde{\varphi}\,]_k = 0 \ \text{ on } \ S_2, \ \ k=1,2,3.
\end{aligned}$$

Due to Theorem 9.12 we infer $U_0^{(\mu)}$ in Ω^μ $(\mu = 1,2)$, which, in turn, yields that $\varphi_{0k} = 0$, $k = 1,2,3$. Thus the null-spaces of the operators (18.63)-(18.64) are trivial in the spaces $\widetilde{B}^{-1/2}_{2,2}(S_2) = \widetilde{H}^{-1/2}_2(S_2)$. Now Lemma 16.1 completes the proof. □

This lemma implies the following existence and regularity results.

Theorem 18.8. *Let* $4/3 < p < 4$ *and let the conditions* (18.45), (18.57) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C}-\mathcal{DC})_\omega$ *is uniquely solvable in the class of vector functions* $(W^1_p(\Omega^1)\,,\, W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ (*with the parameters* r *and* ω *as in* (15.3)) *and the solution is representable by formulae* (18.9)–(18.10), *where* f, F^0, *and* φ *are given by* (18.56), (18.50) *and* (18.51), *respectively, and* $\widetilde{\varphi}$ *is the unique solution of the* ΨDE (18.62).

Theorem 18.9. *Let the conditions* (18.45), (18.57), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1), W^1_{p,\mathrm{loc}}(\Omega^2) \cap \mathrm{SK}^m_r(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C} - \mathcal{DC})_\omega$.

In addition to (18.45), (18.57),

i) *if*

$$\begin{aligned} &f_4 \in B^{s+1}_{t,t}(S), \;\; F_4 \in B^s_{t,t}(S), \;\; \widetilde{\varphi}^{(\pm)} \in [B^{s+1}_{t,t}(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [B^{s+1}_{t,t}(S_1)]^3, \;\; \widetilde{F}^{(1)} \in [B^s_{t,t}(S_1)]^3, \;\; f \in [B^{s+1}_{t,t}(S)]^4, \end{aligned} \tag{18.66}$$

then there holds the inclusion (18.22);

ii) *if*

$$\begin{aligned} &f_4 \in B^{s+1}_{t,q}(S), \;\; F_4 \in B^s_{t,q}(S), \;\; \widetilde{\varphi}^{(\pm)} \in [B^{s+1}_{t,q}(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [B^{s+1}_{t,q}(S_1)]^3, \;\; \widetilde{F}^{(1)} \in [B^s_{t,q}(S_1)]^3, \;\; f \in [B^{s+1}_{t,q}(S)]^4, \end{aligned} \tag{18.67}$$

then there holds the inclusion (18.24);

iii) *if*

$$\begin{aligned} &f_4 \in \mathrm{C}^\alpha(S), \;\; F_4 \in B^{\alpha-1}_{\infty,\infty}(S), \;\; \widetilde{\varphi}^{(\pm)} \in [\mathrm{C}^\alpha(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [\mathrm{C}^\alpha(S_1)]^3, \;\; \widetilde{F}^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_1)]^3, \;\; f \in [\mathrm{C}^\alpha(S)]^4, \end{aligned} \tag{18.68}$$

for some $\alpha > 0$, *then there holds the inclusion* (18.26).

The proofs of these theorems are again verbatim the proofs of Theorems 18.2 and 16.5.

18.4. Problem $(\mathcal{C} - \mathcal{NC})_\omega$**.** The investigation of this problem can be carried out by quite the same approach as in the previous subsection. The interface conditions of the problem now have the following form:

$$\begin{aligned} &[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \\ &[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4 \quad \text{on} \;\; S, \end{aligned} \tag{18.69}$$

$$\begin{aligned} &[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}^{(1)}, \\ &[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}^{(1)} \quad \text{on} \;\; S_1, \end{aligned} \tag{18.70}$$

$$[P^{(1)}(D,n)U^{(1)}]^+ = \widetilde{\Phi}^{(+)}, \;\; [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{\Phi}^{(-)}, \quad \text{on} \;\; S_2, \tag{18.71}$$

where

$$\begin{aligned} &f_4 \in B^{1-1/p}_{p,p}(S), \;\; F_4 \in B^{-1/p}_{p,p}(S), \\ &\widetilde{\Phi}^{(\pm)} = (\Phi_1^{(\pm)}, \Phi_2^{(\pm)}, \Phi_3^{(\pm)})^\top \in [B^{-1/p}_{p,p}(S_2)]^3, \\ &\widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^\top \in [B^{1-1/p}_{p,p}(S_1)]^3, \\ &\widetilde{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [B^{-1/p}_{p,p}(S_1)]^3. \end{aligned} \tag{18.72}$$

Let $\widetilde{f}^0 = (f_1^0, f_2^0, f_3^0)^\top$ be some fixed extension of the vector $\widetilde{f}^{(1)}$ from S_1 onto S_2 preserving the functional space, i.e.,

$$\widetilde{f}^0 \in [B^{1-1/p}_{p,p}(S)]^3, \;\; \widetilde{f}^0|_{S_1} = \widetilde{f}^{(1)}. \tag{18.73}$$

Again an arbitrary extension of $\widetilde{f}^{(1)}$ onto the whole of S preserving the functional space can be represented as the sum

$$\widetilde{f} = (f_1, f_2, f_3)^\top := \widetilde{f}^0 + \widetilde{\varphi} \in [B^{1-1/p}_{p,p}(S)]^3, \ \ \widetilde{f}|_{S_1} = \widetilde{f}^0|_{S_1} = \widetilde{f}^{(1)}, \quad (18.74)$$

where $\widetilde{\varphi}$ is an arbitrary vector function supported on $\overline{S}_2$

$$\widetilde{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^\top \in [\widetilde{B}^{1-1/p}_{p,p}(S_2)]^3. \quad (18.75)$$

Further, let us introduce the notations

$$f = (f_1, \cdots, f_4)^\top := f^0 + \varphi \in [B^{1-1/p}_{p,p}(S)]^4, \quad (18.76)$$

where

$$f^0 = (\widetilde{f}^0, f_4)^\top \in [B^{1-1/p}_{p,p}(S)]^4 \quad (18.77)$$

is the given vector function, and

$$\varphi := (\widetilde{\varphi}, 0)^\top \in [\widetilde{B}^{1-1/p}_{p,p}(S_2)]^4 \quad (18.78)$$

with $\widetilde{\varphi}$ subjected to the condition (18.75).

Next we reduce the conditions (18.69)-(18.71) to the following equivalent equations

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \ \text{ on } \ S, \quad (18.79)$$

$$[U^{(1)}]^+_4 - [U^{(2)}]^-_4 = f_4 \ \text{ on } \ S, \quad (18.80)$$

$$[U^{(1)}]^+_k - [U^{(2)}]^-_k = f_k, \ \text{ on } \ S_1, \ \ k = 1, 2, 3, \quad (18.81)$$

$$[B^{(1)}(D,n)U^{(1)}]^+_k + [B^{(2)}(D,n)U^{(2)}]^-_k = \\ = \Phi^{(+)}_k + \Phi^{(-)}_k \ \text{ on } \ S_2, \ \ k = 1, 2, 3, \quad (18.82)$$

where F is the given vector function

$$F = (F_1, \cdots, F_4)^\top := \begin{cases} (\widetilde{F}^{(1)}, F_4)^\top & \text{on } \ S_1, \\ (\widetilde{\Phi}^{(+)} - \widetilde{\Phi}^{(-)}, F_4)^\top & \text{on } \ S_2, \end{cases} \quad (18.83)$$

satisfying the necessary compatibility condition (cf. (7.26))

$$F \in [B^{-1/p}_{p,p}(S)]^4, \quad (18.84)$$

and f_k and $\widetilde{\Phi}^{\pm}$ are as above.

Now we look for a solution to the reformulated problem (18.79)-(18.82) in the form (18.32)-(18.33), where the density vectors f^0, F, and φ are defined by formulae (18.77), (18.83), and (18.78), respectively. By virtue of these representations we have

$$\begin{aligned} &[U^{(1)}]^+ - [U^{(2)}]^- = f^0 + \varphi, \\ &[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F. \end{aligned} \quad (18.85)$$

Therefore, the conditions (18.79), (18.80), and (18.81) are fulfilled automatically. The remaining conditions (18.82), in accordance with the equation (18.34), lead to the system of ΨDEs for the unknown vector function

$\varphi = (\widetilde{\varphi}, 0)^\top$ on S_2:

$$r_{S_2}\,[-\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,\varphi]_k = r_{S_2}\,[(\mathcal{K}_G)_{kj}\,\varphi_j] = q_k \ \text{ on } \ S_2, \ \ k = 1,2,3, \quad (18.86)$$

where $\mathcal{K}_G = -\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}$ is the same ΨDO of order $+1$ as in Subsection 16.2 (see also (15.86)), the summation over the repeated index j is again meant from 1 to 3, and (see (18.36))

$$q_k = 2^{-1}(\Phi_k^{(+)} + \Phi_k^{(-)}) - r_{S_2}\,\{[\Psi_1\,\Psi\, - 2^{-1}\,I]\,F + \mathcal{K}_G\,f^0\}_k \in$$
$$\in B_{p,p}^{-1/p}(S_2), \ \ k = 1,2,3. \quad (18.87)$$

Next we set

$$\widetilde{\mathcal{K}}_G := [(\mathcal{K}_G)_{kj}]_{3\times 3}, \ \ k,j = 1,2,3, \qquad \widetilde{q} := (q_1, q_2, q_3)^\top. \quad (18.88)$$

The system (18.86) can be then rewritten in the matrix form as follows

$$r_{S_2}\,\widetilde{\mathcal{K}}_G\,\widetilde{\varphi} \,=\, \widetilde{q} \quad (18.89)$$

where $\widetilde{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^\top \in [\widetilde{B}_{p,p}^{1-1/p}(S_2)]^3$ is the sought for vector function.

Lemma 18.10. *The principal homogeneous symbol matrix of the operator $\widetilde{\mathcal{K}}_G$ is positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$. The operators*

$$r_{S_2}\,\widetilde{\mathcal{K}}_G \ \ : \ \ [\widetilde{B}_{p,q}^{s+1}(S_2)]^3 \to [B_{p,q}^{s}(S_2)]^3, \quad (18.90)$$
$$: \ \ [\widetilde{H}_p^{s+1}(S_2)]^3 \to [H_p^{s}(S_2)]^3, \quad (18.91)$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. It is quite similar to the proof of Lemma 18.7 and follows from Lemmata 15.9, 18.4, and 16.1. □

With the help of this lemma one can easily derive the following exictence and regularity results.

Theorem 18.11. *Let $4/3 < p < 4$ and let the conditions* (18.72) *and* (18.84) *be fulfilled. Then the nonhomogeneous problem $(\mathcal{C}-\mathcal{NC})_\omega$ is uniquely solvable in the class of vector functions $(W_p^1(\Omega^1)\,,\,W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$* (*with the parameters r and ω as in* (15.3)) *and the solution is representable by formulae* (18.32)–(18.33)*, where F, f^0, and φ are given by* (18.83)*,* (18.77) *and* (18.78)*, respectively, and $\widetilde{\varphi}$ is the unique solution of the ΨDE* (18.89).

Theorem 18.12. *Let the conditions* (18.72)*,* (18.84)*, and* (16.23) *be fulfilled, and let the pair $(U^{(1)}, U^{(2)}) \in (W_p^1(\Omega^1)\,,\,W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ be the unique solution to the problem $(\mathcal{C} - \mathcal{NC})_\omega$.*

In addition to (18.72), (18.84),

i) *if*

$$\begin{aligned} &f_4 \in B_{t,t}^{s+1}(S), \ \ F_4 \in B_{t,t}^{s}(S), \ \ \widetilde{\Phi}^{(\pm)} \in [B_{t,t}^{s}(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [B_{t,t}^{s+1}(S_1)]^3, \ \ \widetilde{F}^{(1)} \in [B_{t,t}^{s}(S_1)]^3, \ \ F \in [B_{t,t}^{s}(S)]^4, \end{aligned} \quad (18.92)$$

then there holds the inclusion (18.22);

ii) *if*

$$\begin{aligned} &f_4 \in B^{s+1}_{t,q}(S),\ \ F_4 \in B^{s}_{t,q}(S),\ \ \widetilde{\Phi}^{(\pm)} \in [B^{s}_{t,q}(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [B^{s+1}_{t,q}(S_1)]^3,\ \ \widetilde{F}^{(1)} \in [B^{s}_{t,q}(S_1)]^3,\ \ F \in [B^{s}_{t,q}(S)]^4, \end{aligned} \tag{18.93}$$

then there holds the inclusion (18.24);

iii) *if*

$$\begin{aligned} &f_4 \in \mathrm{C}^{\alpha}(S),\ \ F_4 \in B^{\alpha-1}_{\infty,\infty}(S),\ \ \widetilde{\Phi}^{(\pm)} \in [B^{\alpha-1}_{\infty,\infty}(S_2)]^3, \\ &\widetilde{f}^{(1)} \in [\mathrm{C}^{\alpha}(S_1)]^3,\ \ \widetilde{F}^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_1)]^3,\ \ F \in [B^{\alpha-1}_{\infty,\infty}(S)]^4, \end{aligned} \tag{18.94}$$

for some $\alpha > 0$*, then there holds the inclusion* (18.26).

The proofs of these propositions are again word for word of the proofs of Theorems 18.2 and 16.5.

18.5. Problem $(\mathcal{C}-\mathcal{G})_\omega$**.** The interface conditions of the problem $(\mathcal{C}-\mathcal{G})_\omega$ read as (see Subsection 7.2):

$$\begin{aligned} &[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \\ &[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4 \quad \text{on} \quad S, \end{aligned} \tag{18.95}$$

$$\begin{aligned} &[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}^{(1)}, \\ &[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}^{(1)} \quad \text{on} \quad S_1, \end{aligned} \tag{18.96}$$

$$\left.\begin{aligned} &[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}_n^{(2)}, \\ &[P^{(1)}(D,n)U^{(1)}\cdot n]^+ - [P^{(2)}(D,n)U^{(2)}\cdot n]^- = \widetilde{F}_n^{(2)}, \\ &[P^{(1)}(D,n)U^{(1)}\cdot l]^+ = \widetilde{\Phi}_l^{(+)},\ \ [P^{(1)}(D,n)U^{(1)}\cdot m]^+ = \widetilde{\Phi}_m^{(+)}, \\ &[P^{(2)}(D,n)U^{(2)}\cdot l]^- = \widetilde{\Phi}_l^{(-)},\ \ [P^{(2)}(D,n)U^{(2)}\cdot m]^- = \widetilde{\Phi}_m^{(-)}, \end{aligned}\right\} \text{on } S_2, \tag{18.97}$$

where the boundary data belong to the following natural spaces

$$\begin{aligned} &\widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^\top \in [B^{1-1/p}_{p,p}(S_1)]^3,\ \ f_4 \in B^{1-1/p}_{p,p}(S), \\ &\widetilde{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [B^{-1/p}_{p,p}(S_1)]^3,\ \ F_4 \in B^{-1/p}_{p,p}(S), \\ &\widetilde{\Phi}_l^{(\pm)},\ \widetilde{\Phi}_m^{(\pm)},\ \widetilde{F}_n^{(2)} \in B^{-1/p}_{p,p}(S_2),\ \ \widetilde{f}_n^{(2)} \in B^{1-1/p}_{p,p}(S_2), \end{aligned} \tag{18.98}$$

These interface conditions imply that the vector function

$$F := \begin{cases} (\widetilde{F}^{(1)}, F_4)^\top & \text{on} \ \ S_1, \\ \Big([\widetilde{\Phi}_l^{(+)} - \widetilde{\Phi}_l^{(-)}]l + [\widetilde{\Phi}_m^{(+)} - \widetilde{\Phi}_m^{(-)}]m + \widetilde{F}_n^{(2)}n, F_4\Big)^\top & \text{on} \ \ S_2, \end{cases} \tag{18.99}$$

represents the difference $[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^-$ on S, and, therefore, we assume the following natural compatibility condition (cf. (7.28))

$$F = (F_1, \cdots, F_4)^\top \in [B^{-1/p}_{p,p}(S)]^4. \tag{18.100}$$

Analogously, the function

$$\widetilde{f}_n := \begin{cases} \widetilde{f}^{(1)}\cdot n & \text{on} \quad S_1, \\ \widetilde{f}_n^{(2)} & \text{on} \quad S_2, \end{cases} \tag{18.101}$$

represents the difference $[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^-$ on S, and, we again provide the natural compatibility condition

$$\widetilde{f}_n \in B^{1-1/p}_{p,p}(S). \tag{18.102}$$

Further, let us represent the boundary vector functions $\widetilde{f}^{(1)}$ in the form

$$\widetilde{f}^{(1)} = \widetilde{f}^{(1)}_l\, l + \widetilde{f}^{(1)}_m\, m + \widetilde{f}^{(1)}_n\, n \ \text{ on} S_1, \tag{18.103}$$

where

$$\widetilde{f}^{(1)}_l = \widetilde{f}^{(1)}\cdot l, \ \ \widetilde{f}^{(1)}_m = \widetilde{f}^{(1)}\cdot m, \ \ \widetilde{f}^{(1)}_n = \widetilde{f}^{(1)}\cdot n. \tag{18.104}$$

We denote by $\widetilde{f}^{(0)}_l$ and $\widetilde{f}^{(0)}_m$ some fixed extensions of the functions $\widetilde{f}^{(1)}_l$ and $\widetilde{f}^{(1)}_m$ from S_1 onto S_2 preserving the functional space. Then arbitrary extensions can be represented as

$$\widetilde{f}_l = \widetilde{f}^{(0)}_l + \varphi_l, \ \ \widetilde{f}_m = \widetilde{f}^{(0)}_m + \varphi_m, \tag{18.105}$$

where

$$\begin{aligned} &\widetilde{f}_l,\, \widetilde{f}^{(0)}_l,\, \widetilde{f}_m,\, \widetilde{f}^{(0)}_m \in B^{1-1/p}_{p,p}(S), \ \ \varphi_l,\, \varphi_m \in \widetilde{B}^{1-1/p}_{p,p}(S_2),\\ &\widetilde{f}_l|_{S_1} = \widetilde{f}^{(0)}_l|_{S_1} = \widetilde{f}^{(1)}_l \ \ \widetilde{f}_m|_{S_1} = \widetilde{f}^{(0)}_m|_{S_1} = \widetilde{f}^{(1)}_m. \end{aligned} \tag{18.106}$$

Clearly, here φ_l and φ_m are arbitrary scalar functions of the space $\widetilde{B}^{1-1/p}_{p,p}(S_2)$.

Finally, let us set

$$f = (f_1, \cdots, f_4)^\top := f^0 + \varphi \in [B^{1-1/p}_{p,p}(S)]^4, \ \ \widetilde{f} = (f_1, f_2, f_3)^\top, \tag{18.107}$$

where f_4 is the same function as in (18.95), while

$$f^0 = (\widetilde{f}^{(0)}_l\, l + \widetilde{f}^{(0)}_m\, m + \widetilde{f}_n\, n,\ f_4)^\top \in [B^{1-1/p}_{p,p}(S)]^4, \tag{18.108}$$

$$\varphi = (\varphi_l\, l + \varphi_m\, m,\ 0)^\top \in [\widetilde{B}^{1-1/p}_{p,p}(S_2)]^4; \tag{18.109}$$

here $\widetilde{f}^{(0)} = \widetilde{f}^{(0)}_l\, l + \widetilde{f}^{(0)}_m\, m + \widetilde{f}_n\, n$ and $\widetilde{f}_n$ is given by (18.101).

It can be easily seen that (see (18.101) and (18.103))

$$\widetilde{f}|_{S_1} = \widetilde{f}^{(0)}|_{S_1} = \widetilde{f}^{(1)} \ \text{ on } \ S_1, \tag{18.110}$$

$$\widetilde{f}\cdot n|_{S_2} = \widetilde{f}^{(0)}\cdot n|_{S_2} = \widetilde{f}_n = \widetilde{f}^{(2)}_n \ \text{ on } \ S_2. \tag{18.111}$$

Now we are able to reduce the interface conditions (18.95)-(18.97) to the following equivalent equations in terms of the above introduced functions:

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \ \text{ on } \ S, \tag{18.112}$$

$$[U^{(1)}]^+_4 - [U^{(2)}]^-_4 = f_4 \ \text{ on } \ S, \tag{18.113}$$

$$[U^{(1)}]^+_k - [U^{(2)}]^-_k = f_k, \ \ k=1,2,3, \ \text{ on } \ S_1, \tag{18.114}$$

$$[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}\cdot n \ \text{ on } \ S_2, \tag{18.115}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot l]^+ + [P^{(2)}(D,n)U^{(2)}\cdot l]^- = \widetilde{\Phi}^{(+)}_l + \widetilde{\Phi}^{(-)}_l \ \text{ on } S_2, \tag{18.116}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot m]^+ + [P^{(2)}(D,n)U^{(2)}\cdot m]^- = \widetilde{\Phi}^{(+)}_m + \widetilde{\Phi}^{(-)}_m \ \text{ on } S_2, \tag{18.117}$$

where F, $\widetilde{f}$, and f_k are given by (18.99), (18.107)-(18.109).

After this reformulation we look for the solution of the problem under consideration in the form (18.32)-(18.33), where now F and f^0 defined by (18.99) and (18.108) are the given vector functions on S, while the vector function φ given by (18.109) is unknown. We observe that the conditions (18.112), (18.113), (18.114), and (18.115) are satisfied automatically, since the repesentations (18.32)-(18.33) yield

$$\begin{aligned} &[U^{(1)}]^+ - [U^{(2)}]^- = f^0 + \varphi, \\ &[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F. \end{aligned} \tag{18.118}$$

It remains only to meet conditions (18.116) and (18.117) which lead to the following system of ΨDEs on S_2 for the unknown functions φ_l and φ_m (see Subsection 15.2, formulae (15.73), (15.74))

$$\begin{aligned} &[P^{(1)}(D,n)U^{(1)}\cdot l]^+ + [P^{(2)}(D,n)U^{(2)}\cdot l]^- = \\ &\quad = [B^{(1)}(D,n)U^{(1)}\cdot l^*]^+ + [B^{(2)}(D,n)U^{(2)}\cdot l^*]^- = \\ &\quad = [\Psi_1\,\Psi\,F - \Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(f^0+\varphi)]\cdot l^* + \\ &\quad +[\Psi_2\,\Phi_2^{-1}\,\Phi_1\,\Psi\,F - \Psi_2\,\Phi_2^{-1}\,(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)(f^0+\varphi)]\cdot l^* = \\ &\quad = \widetilde{\Phi}_l^{(+)} + \widetilde{\Phi}_l^{(-)}, \\ &[P^{(1)}(D,n)U^{(1)}\cdot m]^+ + [P^{(2)}(D,n)U^{(2)}\cdot m]^- = \\ &\quad = [B^{(1)}(D,n)U^{(1)}\cdot m^*]^+ + [B^{(2)}(D,n)U^{(2)}\cdot m^*]^- = \\ &\quad = [\Psi_1\,\Psi\,F - \Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(f^0+\varphi)]\cdot m^* + \\ &\quad +[\Psi_2\,\Phi_2^{-1}\,\Phi_1\,\Psi\,F - \Psi_2\,\Phi_2^{-1}\,(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)(f^0+\varphi)]\cdot m^* = \\ &\quad = \widetilde{\Phi}_m^{(+)} + \widetilde{\Phi}_m^{(-)}, \end{aligned}$$

where $l^* = (l_1,l_2,l_3,0)^\top$ and $m^* = (m_1,m_2,m_3,0)^\top$ are the 4-vectors introduced in Section 14 (see (14.48)).

With the help of (15.80) we arrive at the system of equations

$$\left.\begin{aligned} r_{S_2}\,\mathcal{K}_G\,(\varphi_l l^* + \varphi_m m^*)\,\cdot l^* &= q_l, \\ r_{S_2}\,\mathcal{K}_G\,(\varphi_l l^* + \varphi_m m^*)\,\cdot m^* &= q_m, \end{aligned}\right\}\ \text{on}\ S_2, \tag{18.119}$$

where the ΨDE $\mathcal{K}_G$ is defined by (15.86), and

$$\begin{aligned} q_l &= 2^{-1}(\Phi_l^{(+)} + \Phi_l^{(-)}) - r_{S_2}\,\{\Psi_1\,\Psi - 2^{-1}I]\,F+ \\ &\qquad +\mathcal{K}_G\,f^0\}\,\cdot l^* \in B_{p,p}^{-1/p}(S_2), \\ q_m &= 2^{-1}(\Phi_m^{(+)} + \Phi_m^{(-)}) - r_{S_2}\,\{\Psi_1\,\Psi - 2^{-1}I]\,F+ \\ &\qquad +\mathcal{K}_G\,f^0\}\,\cdot m^* \in B_{p,p}^{-1/p}(S_2). \end{aligned} \tag{18.120}$$

Now, taking into account the formula (15.85), we can rewrite the above system in the matrix form

$$r_{S_2}\,\mathcal{M}_G\,h = g \ \ \text{on} \ \ S_2, \tag{18.121}$$

where $g = (q_l,q_m)^\top \in [B_{p,p}^{-1/p}(S_2)]^2$ is the given vector on S_2, and $h = (\varphi_l,\varphi_m)^\top \in [\widetilde{B}_{p,p}^{1-1/p}(S_2)]^2$ is the unknown vector. Due to Lemma 15.9

the principal homogeneous symbol matrix of the ΨDO $\mathcal{M}_G$ is positive definite. Therefore, by quite the same arguments as in the previous subsections and invoking Theorem 9.12 and Lemma 16.1, one can prove the following propositions.

Lemma 18.13. *The operators*

$$r_{S_2}\mathcal{M}_G \;:\; [\widetilde{B}^{s+1}_{p,q}(S_2)]^2 \to [B^{s}_{p,q}(S_2)]^2, \tag{18.122}$$
$$:\; [\widetilde{H}^{s+1}_{p}(S_2)]^2 \to [H^{s}_{p}(S_2)]^2, \tag{18.123}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Theorem 18.14. *Let* $4/3 < p < 4$ *and let the conditions* (18.98), (18.100), *and* (18.102) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C}-\mathcal{G})_\omega$ *is uniquely solvable in the class of vector functions* $(W^1_p(\Omega^1)\,,\,W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ *(with the parameters* r *and* ω *as in* (15.3)*) and the solution is representable by formulae* (18.32)–(18.33)*, where* F*,* f^0*, and* φ *are given by* (18.99), (18.108) *and* (18.109)*, respectively, and* $(\varphi_l,\varphi_m)^\top$ *is the unique solution of the ΨDE* (18.121).

Theorem 18.15. *Let the conditions* (18.98), (18.100), (18.102), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)},U^{(2)}) \in (W^1_p(\Omega^1)\,,\,W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C}-\mathcal{G})_\omega$.

In addition to (18.98), (18.100), (18.102),

i) *if*

$$\begin{aligned}
&f_4 \in B^{s+1}_{t,t}(S),\;\; F_4 \in B^{s}_{t,t}(S),\\
&\widetilde{\Phi}^{(\pm)}_l,\, \widetilde{\Phi}^{(\pm)}_m,\, \widetilde{F}^{(2)}_n \in B^{s}_{t,t}(S_2),\;\; \widetilde{f}^{(2)}_n \in B^{s+1}_{t,t}(S_2),\\
&\widetilde{f}^{(1)} \in [B^{s+1}_{t,t}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B^{s}_{t,t}(S_1)]^3,\\
&F \in [B^{s}_{t,t}(S)]^4,\;\; \widetilde{f}_n \in B^{s+1}_{t,t}(S),
\end{aligned} \tag{18.124}$$

then there holds the inclusion (18.22)*;*

ii) *if*

$$\begin{aligned}
&f_4 \in B^{s+1}_{t,q}(S),\;\; F_4 \in B^{s}_{t,q}(S),\\
&\widetilde{\Phi}^{(\pm)}_l,\, \widetilde{\Phi}^{(\pm)}_m,\, \widetilde{F}^{(2)}_n \in B^{s}_{t,q}(S_2),\;\; \widetilde{f}^{(2)}_n \in B^{s+1}_{t,q}(S_2),\\
&\widetilde{f}^{(1)} \in [B^{s+1}_{t,q}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B^{s}_{t,q}(S_1)]^3,\\
&F \in [B^{s}_{t,q}(S)]^4,\;\; \widetilde{f}_n \in B^{s+1}_{t,q}(S),
\end{aligned} \tag{18.125}$$

then there holds the inclusion (18.24)*;*

iii) *if*

$$\begin{aligned}
&f_4 \in \mathrm{C}^{\alpha}(S),\;\; F_4 \in B^{\alpha-1}_{\infty,\infty}(S),\\
&\widetilde{\Phi}^{(\pm)}_l,\, \widetilde{\Phi}^{(\pm)}_m,\, \widetilde{F}^{(2)}_n \in B^{\alpha-1}_{\infty,\infty}(S_2),\;\; \widetilde{f}^{(2)}_n \in \mathrm{C}^{\alpha}(S_2),\\
&\widetilde{f}^{(1)} \in [\mathrm{C}^{\alpha}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_1)]^3,\\
&F \in [B^{\alpha-1}_{\infty,\infty}(S)]^4,\;\; \widetilde{f}_n \in \mathrm{C}^{\alpha}(S),
\end{aligned} \tag{18.126}$$

for some $\alpha > 0$*, then there holds the inclusion* (18.26).

18.6. Problem $(\mathcal{C}-\mathcal{H})_\omega$. Again we start with the reformulation of the original interface conditions (see Subsection 7.2):

$$\begin{aligned}&[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4,\\ &[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4 \quad \text{on} \quad S,\end{aligned} \tag{18.127}$$

$$\begin{aligned}&[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}^{(1)},\\ &[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}^{(1)} \quad \text{on} \quad S_1,\end{aligned} \tag{18.128}$$

$$\left.\begin{aligned}&[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}_n^{(2)},\\ &[P^{(1)}(D,n)U^{(1)}\cdot n]^+ - [P^{(2)}(D,n)U^{(2)}\cdot n]^- = \widetilde{F}_n^{(2)},\\ &[u^{(1)}\cdot l]^+ = \widetilde{\varphi}_l^{(+)}, \quad [u^{(1)}\cdot m]^+ = \widetilde{\varphi}_m^{(+)},\\ &[u^{(2)}\cdot l]^- = \widetilde{\varphi}_l^{(-)}, \quad [u^{(2)}\cdot m]^- = \widetilde{\varphi}_m^{(-)},\end{aligned}\right\} \text{ on } S_2, \tag{18.129}$$

where

$$\begin{aligned}&\widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^\top \in [B_{p,p}^{1-1/p}(S_1)]^3, \;\; f_4 \in B_{p,p}^{1-1/p}(S),\\ &\widetilde{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [B_{p,p}^{-1/p}(S_1)]^3, \;\; F_4 \in B_{p,p}^{-1/p}(S),\\ &\widetilde{\varphi}_l^{(\pm)},\, \widetilde{\varphi}_m^{(\pm)},\, \widetilde{f}_n^{(2)} \in B_{p,p}^{1-1/p}(S_2), \;\; \widetilde{F}_n^{(2)} \in B_{p,p}^{-1/p}(S_2).\end{aligned} \tag{18.130}$$

The vector function

$$f := \begin{cases} (\widetilde{f}^{(1)}, f_4)^\top & \text{on} \quad S_1,\\ \Big([\widetilde{\varphi}_l^{(+)} - \widetilde{\varphi}_l^{(-)}]l + [\widetilde{\varphi}_m^{(+)} - \widetilde{\varphi}_m^{(-)}]m + \widetilde{f}_n^{(2)} n, f_4\Big)^\top & \text{on} \quad S_2,\end{cases} \tag{18.131}$$

represents the difference $[U^{(1)}]^+ - [U^{(2)}]^-$ on the interface S, and, therefore, we require the natural compatibility condition (cf. (7.28))

$$f = (f_1, \cdots, f_4)^\top \in [B_{p,p}^{1-1/p}(S)]^4. \tag{18.132}$$

Moreover, the function

$$\widetilde{F}_n := \begin{cases} \widetilde{F}^{(1)}\cdot n & \text{on} \quad S_1,\\ \widetilde{F}_n^{(2)} & \text{on} \quad S_2,\end{cases} \tag{18.133}$$

corresponds to the difference $[P^{(1)}(D,n)U^{(1)}\cdot n]^+ - [P^{(2)}(D,n)U^{(2)}\cdot n]^-$ on S, and, we again assume the natural compatibility condition

$$\widetilde{F}_n \in B_{p,p}^{-1/p}(S). \tag{18.134}$$

Next, let us represent the boundary vector function $\widetilde{F}^{(1)}$ in the form

$$\widetilde{F}^{(1)} = \widetilde{F}_l^{(1)}\, l + \widetilde{F}_m^{(1)}\, m + \widetilde{F}_n^{(1)}\, n \;\; \text{on} \;\; S_1, \tag{18.135}$$

where

$$\widetilde{F}_l^{(1)} = \widetilde{F}^{(1)}\cdot l, \;\; \widetilde{F}_m^{(1)} = \widetilde{F}^{(1)}\cdot m, \;\; \widetilde{F}_n^{(1)} = \widetilde{F}^{(1)}\cdot n. \tag{18.136}$$

Denote by $\widetilde{F}_l^{(0)}$ and $\widetilde{F}_m^{(0)}$ some fixed extensions of the functions $\widetilde{F}_l^{(1)}$ and $\widetilde{F}_m^{(1)}$ from S_1 onto S_2 preserving the functional space. Arbitrary extensions then can be represented as

$$\widetilde{F}_l = \widetilde{F}_l^{(0)} + \varphi_l, \;\; \widetilde{F}_m = \widetilde{F}_m^{(0)} + \varphi_m, \tag{18.137}$$

where

$$\begin{aligned} &\widetilde{F}_l,\, \widetilde{F}_l^{(0)},\, \widetilde{F}_m,\, \widetilde{F}_m^{(0)} \in B_{p,p}^{-1/p}(S), \quad \varphi_l,\, \varphi_m \in \widetilde{B}_{p,p}^{-1/p}(S_2), \\ &\widetilde{F}_l|_{S_1} = \widetilde{F}_l^{(0)}|_{S_1} = \widetilde{F}_l^{(1)}, \quad \widetilde{F}_m|_{S_1} = \widetilde{F}_m^{(0)}|_{S_1} = \widetilde{F}_m^{(1)}. \end{aligned} \tag{18.138}$$

Obviously, here φ_l and φ_m are arbitrary functions from $\widetilde{B}_{p,p}^{-1/p}(S_2)$.

Further, we set

$$F = (F_1, \cdots, F_4)^\top := F^0 + \varphi \in [B_{p,p}^{-1/p}(S)]^4, \;\; \widetilde{F} = (F_1, F_2, F_3)^\top, \tag{18.139}$$

where F_4 is the same function as above, while

$$F^0 := (\widetilde{F}_l^{(0)}\, l + \widetilde{F}_m^{(0)}\, m + \widetilde{F}_n\, n,\, F_4)^\top \in [B_{p,p}^{-1/p}(S)]^4 \tag{18.140}$$

with

$$\varphi = \varphi_l\, l^* + \varphi_m\, m^* = (\varphi_l\, l + \varphi_m\, m,\, 0)^\top \in [\widetilde{B}_{p,p}^{-1/p}(S_2)]^4. \tag{18.141}$$

Moreover, $\widetilde{F}^{(0)} = \widetilde{F}_l^{(0)}\, l + \widetilde{F}_m^{(0)}\, m + \widetilde{F}_n\, n$, the function $\widetilde{F}_n$ is given by (18.133), and the 4-vectors l^*, m^*, and n^* are defined by (14.48).

We note that (see (18.135))

$$\begin{aligned} &\widetilde{F}|_{S_1} = \widetilde{F}^{(0)}|_{S_1} = \widetilde{F}^{(1)} \;\text{ on }\; S_1, \\ &\widetilde{F} \cdot n|_{S_2} = \widetilde{F}^{(0)} \cdot n|_{S_2} = \widetilde{F}_n = \widetilde{F}_n^{(2)} \;\text{ on }\; S_2. \end{aligned} \tag{18.142}$$

Now we can easily see that the original interface conditions (18.127)-(18.129) are equivalent to the equations:

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \;\text{ on }\; S, \tag{18.143}$$

$$[B^{(1)}(D,n)U^{(1)}]_4^+ - [B^{(2)}(D,n)U^{(2)}]_4^- = F_4 \;\text{ on }\; S, \tag{18.144}$$

$$[B^{(1)}(D,n)U^{(1)}]_k^+ - [B^{(2)}(D,n)U^{(2)}]_k^- = F_k \text{ on } S_1, \; k = 1,2,3, \tag{18.145}$$

$$[B^{(1)}(D,n)U^{(1)} \cdot n^*]^+ - [B^{(2)}(D,n)U^{(2)} \cdot n^*]^- = F \cdot n^* \text{ on } S_2, \tag{18.146}$$

$$\left.\begin{aligned} &[U^{(1)} \cdot l^*]^+ + [U^{(2)} \cdot l^*]^- = \widetilde{\varphi}_l^{(+)} + \widetilde{\varphi}_l^{(-)}, \\ &[U^{(1)} \cdot m^*]^+ + [U^{(2)} \cdot m^*]^- = \widetilde{\varphi}_m^{(+)} + \widetilde{\varphi}_m^{(-)}, \end{aligned}\right\} \text{ on } S_2, \tag{18.147}$$

where f and F are given by (18.131) and (18.139), respectively.

Let us look for the solution of the reformulated problem in the form (18.9)-(18.10), where now f, F^0, and φ are defined by (18.131), (18.140), and (18.141). These representation formulae imply

$$[U^{(1)}]^+ - [U^{(2)}]^- = f, \;\; [B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^0 + \varphi,$$

which show that the conditions (18.143)-(18.146) are satisfied automatically.

The remaining conditions (18.147) yield the following system of ΨDEs on S_2 for the unknown scalar functions φ_l and φ_m (see (18.11))

$$\left.\begin{aligned} &r_{S_2}\, \Phi_1\, \Psi\, \varphi \,\cdot\, l^* \,=\, q_l, \\ &r_{S_2}\, \Phi_1\, \Psi\, \varphi \,\cdot\, m^* \,=\, q_m, \end{aligned}\right\} \text{ on } S_2, \tag{18.148}$$

where

$$\begin{aligned} q_l &= 2^{-1}(\varphi_l^{(+)} + \varphi_l^{(-)}) - \\ &\quad - r_{S_2}\{\Phi_1\,\Psi\,F^0 - [\Phi_1\,\Psi\,\Psi_2\Phi_2^{-1} + 2^{-1}I]\,f\}\,\cdot l^*, \\ q_m &= 2^{-1}(\varphi_m^{(+)} + \varphi_m^{(-)}) - \\ &\quad - r_{S_2}\{\Phi_1\,\Psi\,F^0 - [\Phi_1\,\Psi\,\Psi_2\Phi_2^{-1} + 2^{-1}I]\,f\}\,\cdot m^*. \end{aligned} \tag{18.149}$$

In accordance with the formula (15.104) this system can be written also as

$$r_{S_2}\,\mathcal{M}_H\,h = g \quad \text{on} \quad S_2, \tag{18.150}$$

where $g = (q_l, q_m)^\top \in [B_{p,p}^{1-1/p}(S_2)]^2$ is the given vector on S_2, and $h = (\varphi_l, \varphi_m)^\top \in [\widetilde{B}_{p,p}^{-1/p}(S_2)]^2$ is the unknown vector.

By virtue of Lemma 15.14 the principal homogeneous symbol matrix of the ΨDO $\mathcal{M}_H$ is positive definite which together with Theorem 9.12 and Lemma 16.1 implies the following existence and regularity results.

Lemma 18.16. *The operators*

$$r_{S_2}\,\mathcal{M}_H \;:\; [\widetilde{B}_{p,q}^{s}(S_2)]^2 \to [B_{p,q}^{s+1}(S_2)]^2, \tag{18.151}$$

$$:\; [\widetilde{H}_{p}^{s}(S_2)]^2 \to [H_{p}^{s+1}(S_2)]^2, \tag{18.152}$$

are bounded for any $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Theorem 18.17. *Let* $4/3 < p < 4$ *and let the conditions* (18.130), (18.132), *and* (18.134) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C} - \mathcal{H})_\omega$ *is uniquely solvable in the class of vector functions* $(W_p^1(\Omega^1)\,,\,W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ *(with the parameters* r *and* ω *as in* (15.3)*) and the solution is representable by formulae* (18.9)–(18.10)*, where* f, F^0, *and* φ *are given by* (18.131), (18.140), *and* (18.141), *respectively, and* $(\varphi_l, \varphi_m)^\top$ *is the unique solution of the* ΨDE (18.150).

Theorem 18.18. *Let the conditions* (18.130), (18.132), (18.134), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) \in (W_p^1(\Omega^1)\,,\,W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C} - \mathcal{H})_\omega$.

In addition to (18.130), (18.132), (18.134),

i) *if*

$$\begin{aligned} & f_4 \in B_{t,t}^{s+1}(S),\;\; F_4 \in B_{t,t}^{s}(S), \\ & \widetilde{\varphi}_l^{(\pm)},\, \widetilde{\varphi}_m^{(\pm)},\, \widetilde{f}_n^{(2)} \in B_{t,t}^{s+1}(S_2),\;\; \widetilde{F}_n^{(2)} \in B_{t,t}^{s}(S_2), \\ & \widetilde{f}^{(1)} \in [B_{t,t}^{s+1}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B_{t,t}^{s}(S_1)]^3, \\ & f \in [B_{t,t}^{s+1}(S)]^4,\;\; \widetilde{F}_n \in B_{t,t}^{s}(S), \end{aligned} \tag{18.153}$$

then there holds the inclusion (18.22);

ii) *if*

$$\begin{aligned}
&f_4 \in B^{s+1}_{t,q}(S),\;\; F_4 \in B^{s}_{t,q}(S),\\
&\widetilde{\varphi}^{(\pm)}_l,\, \widetilde{\varphi}^{(\pm)}_m,\, \widetilde{f}^{(2)}_n \in B^{s+1}_{t,q}(S_2),\;\; \widetilde{F}^{(2)}_n \in B^{s}_{t,q}(S_2),\\
&\widetilde{f}^{(1)} \in [B^{s+1}_{t,q}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B^{s}_{t,q}(S_1)]^3,\\
&f \in [B^{s+1}_{t,q}(S)]^4,\;\; \widetilde{F}_n \in B^{s}_{t,q}(S),
\end{aligned} \tag{18.154}$$

then there holds the inclusion (18.24);

iii) *if*

$$\begin{aligned}
&f_4 \in \mathrm{C}^{\alpha}(S),\;\; F_4 \in B^{\alpha-1}_{\infty,\infty}(S),\\
&\widetilde{\varphi}^{(\pm)}_l,\, \widetilde{\varphi}^{(\pm)}_m,\, \widetilde{f}^{(2)}_n \in \mathrm{C}^{\alpha}(S_2),\;\; \widetilde{F}^{(2)}_n \in B^{\alpha-1}_{\infty,\infty}(S_2),\\
&\widetilde{f}^{(1)} \in [\mathrm{C}^{\alpha}(S_1)]^3,\;\; \widetilde{F}^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_1)]^3,\\
&f \in [\mathrm{C}^{\alpha}(S)]^4,\;\; \widetilde{F}_n \in B^{\alpha-1}_{\infty,\infty}(S),
\end{aligned} \tag{18.155}$$

for some $\alpha > 0$*, then there holds the inclusion* (18.26).

19. Mixed Interface Problems of Pseudo-Oscillations

The mixed interface problems for the system of pseudo-oscillation equations are investigated by the approach developed in the previous section. In this case we have to apply the "explicit" representation formulae (14.24)–(14.25), obtained for the solution of the basic interface problem $(\mathcal{C})_\tau$, to reduce the mixed interface problems to the corresponding ΨDEs. For illustration of the method in this section we consider only the problems $(\mathcal{C}-\mathcal{DD})_\tau$ and $(\mathcal{C}-\mathcal{NN})_\tau$. The other mixed problems of pseudo-oscillations can be studied quite analogously.

19.1. Problem $(\mathcal{C}-\mathcal{DD})_\tau$**.** Let S, S_1, and S_2, be the same as in Section 18. The original formulation of the problem $(\mathcal{C}-\mathcal{DD})_\tau$ is the following (see Section 7): Find the pair of vector-functions $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1),\, W^1_p(\Omega^2))$ satisfying the differential equations

$$A^{(\mu)}(D,\tau)\,U^{(\mu)} \,=\, 0 \;\text{ in }\; \Omega^{(\mu)},\;\; \mu = 1,2, \tag{19.1}$$

and the mixed interface conditions on S

$$\begin{aligned}
&[U^{(1)}]^+ - [U^{(2)}]^- = f^{(1)},\\
&[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^{(1)} \;\;\text{on}\;\; S_1,
\end{aligned} \tag{19.2}$$

$$[U^{(1)}]^+ = \varphi^{(+)},\;\; [U^{(2)}]^- = \varphi^{(-)} \;\;\text{on}\;\; S_2; \tag{19.3}$$

moreover, $U^{(2)}$ satisfies the decay condition (1.30) at infinity.

Here $p > 1$ and

$$\begin{aligned}
&f^{(1)} = (f^{(1)}_1, \cdots, f^{(1)}_4)^\top \in B^{1-1/p}_{p,p}(S_1),\\
&F^{(1)} = (F^{(1)}_1, \cdots, F^{(1)}_4)^\top \in B^{-1/p}_{p,p}(S_1),
\end{aligned} \tag{19.4}$$

$$\varphi^{(\pm)} = (\varphi^{(\pm)}_1, \cdots, \varphi^{(\pm)}_4)^\top \in B^{1-1/p}_{p,p}(S_2). \tag{19.5}$$

Further, we assume that the vector function

$$f := \begin{cases} f^{(1)} & \text{on} \quad S_1, \\ \varphi^{(+)} - \varphi^{(-)} & \text{on} \quad S_2, \end{cases} \tag{19.6}$$

meets the necessary compatibility condition

$$f \in B^{1-1/p}_{p,p}(S). \tag{19.7}$$

Next, denote by $F^0 \in B^{-1/p}_{p,p}(S)$ some fixed extension of the vector function $F^{(1)}$ from the submanifold S_1 onto the whole surface S (i.e., $F^0|_{S_1} = F^{(1)}$ on S_1).

Evidently, an arbitrary extension (preserving the functional space) can be then represented as

$$F = F^0 + \varphi \in B^{-1/p}_{p,p}(S), \tag{19.8}$$

where $\varphi = (\varphi_1, \cdots, \varphi_4)^\top \in \widetilde{B}^{-1/p}_{p,p}(S_2)$ is an arbitrary function supported on $\overline{S}_2$.

Now we can reformulate the interface conditions (19.2)-(19.3) in the following equivalent form:

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \quad \text{on} \quad S, \tag{19.9}$$

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \quad \text{on} \quad S_1, \tag{19.10}$$

$$[U^{(1)}]^+ + [U^{(2)}]^- = \varphi^{(+)} + \varphi^{(-)} \quad \text{on} \quad S_2, \tag{19.11}$$

where $B^{(\mu)}(D,n)$ is defined again by (1.25), and f and F are given by (19.6) and (19.8), respectively.

Let us now look for the solution $(U^{(1)}, U^{(2)})$ to the problem $(\mathcal{C} - \mathcal{DD})_\tau$ as follows (cf. (14.24)–(14.25))

$$U^{(1)}(x) = V^{(1)}_\tau \Big((\mathcal{H}^{(1)}_\tau)^{-1} \mathcal{N}_\tau^{-1} [(F^0 + \varphi) + \mathcal{N}_{2,\tau} f] \Big)(x), \ \ x \in \Omega^1, \tag{19.12}$$

$$U^{(2)}(x) = V^{(2)}_\tau \Big((\mathcal{H}^{(2)}_\tau)^{-1} \mathcal{N}_\tau^{-1} [(F^0 + \varphi) - \mathcal{N}_{1,\tau} f] \Big)(x), \ \ x \in \Omega^2, \tag{19.13}$$

where $\varphi \in \widetilde{B}^{-1/p}_{p,p}(S_2)$ is the unknown vector function, $W^{(\mu)}_\tau$ and $V^{(\mu)}_\tau$ are the double and single layer potentials of pseudo-oscillations (see (11.1)–(11.2)), the boundary operators $\mathcal{H}^{(\mu)}_\tau$, $\mathcal{N}_\tau$, $\mathcal{N}_{1,\tau}$, and $\mathcal{N}_{2,\tau}$ are the same as in Section 14 (see (14.12)). Note that here and in what follows we keep all notations of Sections 11 and 14.

One can easily check that the interface conditions (19.9) and (19.10) are satisfied automatically, since (19.12) and (19.13) together with (14.12) imply

$$\begin{aligned} &[U^{(1)}]^+ - [U^{(2)}]^- = f, \\ &[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^0 + \varphi \quad \text{on} \quad S. \end{aligned} \tag{19.14}$$

It remains only to fulfil the condition (19.11) which yield the ΨDE for the unknown vector function φ

$$r_{S_2} \mathcal{N}_\tau^{-1} \varphi = q \quad \text{on} \quad S_2, \tag{19.15}$$

where r_{S_2} is again the restriction operator on S_2, the right-hand side vector q is given by

$$q = 2^{-1}(\varphi^{(+)} + \varphi^{(-)}) - r_{S_2}[\mathcal{N}_\tau^{-1} F^0 + 2^{-1}\mathcal{N}_\tau^{-1}(\mathcal{N}_{2,\tau} - \mathcal{N}_{1,\tau})f] \in B_{p,p}^{1-1/p}(S_2).$$

The operator $r_{S_2}\mathcal{N}_\tau^{-1}$ possesses the following properties.

Lemma 19.1. *The operators*

$$r_{S_2}\mathcal{N}_\tau^{-1} \;\;:\;\; [\widetilde{B}_{p,q}^s(S_2)]^4 \to [B_{p,q}^{s+1}(S_2)]^4, \tag{19.16}$$

$$: \;\; [\widetilde{H}_p^s(S_2)]^4 \to [H_p^{s+1}(S_2)]^4, \tag{19.17}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

Proof. Due to Theorem 11.3 and Lemma 14.2 we conclude that the mappings (19.16)-(19.17) are bounded and that their Fredholm indices equal zero, since the principal homogeneous symbol matrix of the operator $\mathcal{N}_\tau^{-1}$ is positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in I\!R^2 \setminus \{0\}$. It remains to prove that the corresponding null-spaces are trivial, i.e., we have to show that the homogeneous equation

$$r_{S_2}\mathcal{N}_\tau^{-1}\varphi = 0 \quad \text{on} \quad S_2 \tag{19.18}$$

has only the trivial solution in the spaces $\widetilde{B}_{p,q}^s(S_2)$ and $\widetilde{H}_p^s(S_2)$ with s and p satisfying the inequalities (16.15). We again consider the particular case $s = -1/2$ and $p = q = 2$ for which the condition (16.15) is fulfilled. Further, let $\varphi \in \widetilde{B}_{2,2}^{-1/2}(S_2) = \widetilde{H}_2^{-1/2}(S_2)$ be some solution to the equation (19.18), and construct the potentials:

$$U^{(1)}(x) = V_\tau^{(1)}\Big((\mathcal{H}_\tau^{(1)})^{-1}\mathcal{N}_\tau^{-1}\varphi\Big)(x), \;\; x \in \Omega^1, \tag{19.19}$$

$$U^{(2)}(x) = V_\tau^{(2)}\Big((\mathcal{H}_\tau^{(2)})^{-1}\mathcal{N}_\tau^{-1}\varphi\Big)(x), \;\; x \in \Omega^2. \tag{19.20}$$

Theorem 11.3 implies that the pair $(U^{(1)}, U^{(2)})$ represents a solution to the homogeneous problem $(\mathcal{C} - \mathcal{DD})_\tau$ in the space $(W_2^1(\Omega^1),\, W_2^1(\Omega^2))$. By Theorem 8.6 we then conclude $U^{(\mu)} = 0$ in Ω^μ, $\mu = 1, 2$, whence $[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = \varphi = 0$ follows. Therefore, the above homogeneous equation has no nontrivial solutions in the space $\widetilde{B}_{2,2}^{-1/2}(S_2)$. Now Lemma 16.1 completes the proof. □

Theorem 19.2. *Let* $4/3 < p < 4$ *and let the conditions* (19.4), (19.5), *and* (19.7) *be fulfilled. Then the problem* $(\mathcal{C} - \mathcal{DD})_\tau$ *is uniquely solvable in the class* $(W_p^1(\Omega^1),\, W_p^1(\Omega^2))$ *and the solution is representable in the form* (19.12)–(19.13), *where* φ *is the unique solution of the* ΨDE (19.15).

Proof. First we note that the condition (16.15) with $s = -1/p$ implies the inequality $4/3 < p < 4$. Next, Lemma 19.1, with $s = -1/p$ and $4/3 < p < 4$, shows that the ΨDE (19.15) is uniquely solvable. This together with the representation formulae (19.12)-(19.13) yields the solvability of the nonhomogeneous problem $(\mathcal{C} - \mathcal{DD})_\tau$ in the space indicated in the theorem.

It remains to prove the uniqueness of solution for $4/3 < p < 4$. Let $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1)\,,\, W^1_p(\Omega^2))$ be some solution of the homogeneous problem $(\mathcal{C} - \mathcal{DD})_\tau$. Clearly, then $[U^{(1)}]^+,\, [U^{(2)}]^- \in B^{1-1/p}_{p,p}(S)$ and $[B^{(1)}(D,n)U^{(1)}]^+,\, [B^{(2)}(D,n)U^{(2)}]^- \in B^{-1/p}_{p,p}(S)$. In addition, $f := [U^{(1)}]^+ - [U^{(2)}]^- = 0$ on S and $F := [B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = 0$ on S_1. Therefore, $F \in \widetilde{B}^{-1/p}_{p,p}(S_2)$. Due to Theorem 14.6, such solution is uniquely representable by formulae (14.24)–(14.25) which in the case in question read as

$$U^{(\mu)}(x) = V^{(\mu)}_\tau \left((\mathcal{H}^{(\mu)}_\tau)^{-1}\, \mathcal{N}^{-1}_\tau\, F \right)(x), \quad x \in \Omega^\mu, \quad \mu = 1,2, \tag{19.21}$$

with $F \in \widetilde{B}^{-1/p}_{p,p}(S_2)$.

The homogeneous versions of the conditions (19.2)-(19.3) (i.e., (19.9)-(19.11)) then shows that F has to satisfy the equation

$$r_{S_2}\, \mathcal{N}^{-1}_\tau\, F = 0 \quad \text{on} \quad S_2,$$

from which $F = 0$ on S_2 follows for arbitrary $p \in (4/3, 4)$ due to Lemma 19.1. Therefore, $U^{(\mu)} = 0$ in Ω^μ $(\mu = 1,2)$ in view of (19.21). This completes the proof. □

The next theorem deals with the smoothness of solutions to the mixed interface problem $(\mathcal{C} - \mathcal{DD})_\tau$.

Theorem 19.3. *Let the conditions* (19.4), (19.5), (19.7), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1)\,,\, W^1_{p,}(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C} - \mathcal{DD})_\tau$.

In addition to (19.4), (19.5), (19.7),

i) *if conditions* (18.21) *are satisfied, then*

$$(U^{(1)}, U^{(2)}) \in (H^{s+1+1/t}_t(\Omega^1)\,,\, H^{s+1+1/t}_t(\Omega^2));$$

ii) *if conditions* (18.23) *are satisfied, then*

$$(U^{(1)}, U^{(2)}) \in (B^{s+1+1/t}_{t,q}(\Omega^1)\,,\, B^{s+1+1/t}_{t,q}(\Omega^2));$$

iii) *if conditions* (18.25) *are satisfied for some* $\alpha > 0$, *then*

$$(U^{(1)}, U^{(2)}) \in (\mathrm{C}^\nu(\overline{\Omega^1})\,,\, \mathrm{C}^\nu(\overline{\Omega^2}))$$

with any $\nu \in (0, \alpha_0)$, $\alpha_0 := \min\{\alpha, 1/2\}$.

Proof. It is verbatim the proof of Theorem 16.5. □

19.2. Problem $(\mathcal{C} - \mathcal{NN})_\tau$. The original interface conditions for the problem $(\mathcal{C} - \mathcal{NN})_\tau$ read as

$$\begin{aligned} &[U^{(1)}]^+ - [U^{(2)}]^- = f^{(1)}, \\ &[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^{(1)} \quad \text{on} \quad S_1, \end{aligned} \tag{19.22}$$

$$[B^{(1)}(D,n)U^{(1)}]^+ = \Phi^{(+)}, \quad [B^{(2)}(D,n)U^{(2)}]^- = \Phi^{(-)} \quad \text{on} \quad S_2, \tag{19.23}$$

where

$$f^{(1)} \in B^{1-1/p}_{p,p}(S_1), \;\; F^{(1)} \in B^{-1/p}_{p,p}(S_1),$$
$$\Phi^{(\pm)} = (\Phi^{(\pm)}_1, \cdots, \Phi^{(\pm)}_4)^\top \in B^{-1/p}_{p,p}(S_2). \tag{19.24}$$

We require that the vector function

$$F := \begin{cases} F^{(1)} & \text{on} \quad S_1, \\ \Phi^{(+)} - \Phi^{(-)} & \text{on} \quad S_2. \end{cases} \tag{19.25}$$

satisfies the necessary compatibility condition

$$F \in B^{-1/p}_{p,p}(S). \tag{19.26}$$

Denote by $f^0 \in B^{1-1/p}_{p,p}(S)$ some fixed extension of the vector function $f^{(1)}$ from the submanifold S_1 onto the whole surface S. Then an arbitrary extension preserving the functional space is represented by formula

$$f = f^0 + \varphi \in B^{1-1/p}_{p,p}(S), \tag{19.27}$$

where $\varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$.

Next, we again reduce the above original interface conditions (19.22)-(19.23) to the equivalent equations:

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \;\; \text{on} \;\; S, \tag{19.28}$$
$$[U^{(1)}]^+ - [U^{(2)}]^- = f \;\; \text{on} \;\; S_1, \tag{19.29}$$
$$[B^{(1)}(D,n)U^{(1)}]^+ + [B^{(2)}(D,n)U^{(2)}]^- = \Phi^{(+)} + \Phi^{(-)} \;\; \text{on} \;\; S_2, \tag{19.30}$$

where F and f are given by (19.25) and (19.27), respectively.

Further, we look for the solution $(U^{(1)}, U^{(2)})$ to the problem $(\mathcal{C} - \mathcal{N}\mathcal{N})_\tau$ in the form (cf. (14.24)–(14.25))

$$U^{(1)}(x) = V^{(1)}_\tau \Big((\mathcal{H}^{(1)}_\tau)^{-1} \, \mathcal{N}^{-1}_\tau \, [F + \mathcal{N}_{2,\tau} \, (f^0 + \varphi)] \Big)(x), \;\; x \in \Omega^1, \tag{19.31}$$
$$U^{(2)}(x) = V^{(2)}_\tau \Big((\mathcal{H}^{(2)}_\tau)^{-1} \, \mathcal{N}^{-1}_\tau \, [F - \mathcal{N}_{1,\tau} \, (f^0 + \varphi)] \Big)(x), \;\; x \in \Omega^2, \tag{19.32}$$

where f^0 and F are the given vector functions on S and $\varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$ is the unknown vector function.

The conditions (19.28) and (19.29) are then satisfied automatically, while the condition (19.30) leads to the ΨDE for the unknown vector φ

$$r_{S_2} \, [\mathcal{N}_{1,\tau} \, \mathcal{N}^{-1}_\tau \, \mathcal{N}_{2,\tau} \, \varphi] \; = \; q \;\; \text{on} \;\; S_2, \tag{19.33}$$

where the right-hand side vector $q \in B^{-1/p}_{p,p}(S_2)$ reads as

$$q = 2^{-1}(\Phi^{(+)} + \Phi^{(-)}) \; + \; r_{S_2} \, [2^{-1}(\mathcal{N}_{2,\tau} - \mathcal{N}_{1,\tau})\mathcal{N}^{-1}_\tau \, F -$$
$$- \, \mathcal{N}_{1,\tau} \, \mathcal{N}^{-1}_\tau \, \mathcal{N}_{2,\tau} \, f^0]. \tag{19.34}$$

In the proof of Lemma 14.8 it has been shown that the principal homogeneous symbol matrix of the ΨDO $\mathcal{N}_{1,\tau} \, \mathcal{N}^{-1}_\tau \, \mathcal{N}_{2,\tau}$ is positive definite. Therefore, by the arguments employed above one can prove the following assertion (see the proof of Lemma 19.1).

Lemma 19.4. *The operators*

$$r_{S_2}\,\mathcal{N}_{1,\tau}\,\mathcal{N}_{\tau}^{-1}\,\mathcal{N}_{2,\tau} \quad : \quad [\widetilde{B}^{s+1}_{p,q}(S_2)]^4 \to [B^s_{p,q}(S_2)]^4, \tag{19.35}$$

$$: \quad [\widetilde{H}^{s+1}_{p}(S_2)]^4 \to [H^s_{p}(S_2)]^4, \tag{19.36}$$

are bounded for any $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$.

These operators are invertible if the condition (16.15) *holds.*

This lemma implies the existence and regularity results quite in the same way as in the previous subsection.

Theorem 19.5. *Let* $4/3 < p < 4$ *and let the conditions* (19.24) *and* (19.26) *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{C}-\mathcal{N}\mathcal{N})_\tau$ *is uniquely solvable in the class of vector functions* $(W^1_p(\Omega^1)\,,\,W^1_p(\Omega^2))$ *and the solution is representable in the form* (19.31)–(19.32)*, where* φ *is the unique solution of the* ΨDE (19.33).

Theorem 19.6. *Let the conditions* (19.24), (19.26), *and* (16.23) *be fulfilled, and let the pair* $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1)\,,\,W^1_p(\Omega^2))$ *be the unique solution to the problem* $(\mathcal{C}-\mathcal{N}\mathcal{N})_\tau$.

In addition to (19.24), (19.26),

i) *if conditions* (18.39) *hold, then*

$$(U^{(1)}, U^{(2)}) \in (H^{s+1+1/t}_t(\Omega^1)\,,\,H^{s+1+1/t}_t(\Omega^2));$$

ii) *if conditions* (18.40) *hold, then*

$$(U^{(1)}, U^{(2)}) \in (B^{s+1+1/t}_{t,q}(\Omega^1)\,,\,B^{s+1+1/t}_{t,q}(\Omega^2));$$

iii) *if conditions* (18.41) *hold for some* $\alpha > 0$*, then*

$$(U^{(1)}, U^{(2)}) \in (\mathrm{C}^\nu(\overline{\Omega^1})\,,\,\mathrm{C}^\nu(\overline{\Omega^2}))$$

with any $\nu \in (0, \alpha_0)$, $\alpha_0 := \min\{\alpha, 1/2\}$.

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