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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE GOURSAT PROBLEM FOR SYSTEMS OF FUNCTIONAL PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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In the rectangle $D_{ab} = [0, a] \times [0, b]$ let us consider the system of functional partial differential equations of hyperbolic type

$$\frac{\partial u_1(x,y)}{\partial x} = f_1(u_1, u_2)(x, y), \quad \frac{\partial u_2(x,y)}{\partial y} = f_2(u_1, u_2)(x, y) \tag{1}$$

with the initial conditions

$$u_1(0, y) = 0 \quad \text{for} \quad 0 \le y \le b, \quad u_2(x, 0) = 0 \quad \text{for} \quad 0 \le x \le a, \tag{2}$$

where $f_i : C(D_{ab}; R^n) \times C(D_{ab}; R^n) \to C(D_{ab}; R^n)$ $(i = 1, 2)$ are continuous Volterra operators.

A particular case of the system (1) is, for example, the system of integro-differential equations

$$\begin{aligned}
\frac{\partial u_1(x,y)}{\partial x} &= g_1\Big(x, y, u_1(\alpha_{11}(x), \beta_{11}(y)), u_2(\alpha_{12}(x), \beta_{12}(y)),\\
&\qquad \int\limits_0^{\beta_{13}(y)} G_1(x, t) u_1(\alpha_{13}(x), t)dt\Big),\\
\frac{\partial u_2(x,y)}{\partial y} &= g_2\Big(x, y, u_1(\alpha_{21}(x), \beta_{21}(y)), u_2(\alpha_{22}(x), \beta_{22}(y)),\\
&\qquad \int\limits_0^{\alpha_{23}(x)} G_2(s, y) u_2(s, \beta_{23}(y))ds\Big),
\end{aligned} \tag{1'}$$

where $g_i : D_{ab} \times R^n \times R^n \times R^n \to R^n$ and $G_i : D_{ab} \to R^{n\times n}$ $(i = 1, 2)$ are continuous vector and matrix functions, respectively, while $\alpha_{ik} : [0, a] \to [0, a]$ and $\beta_{ik} : [0, b] \to [0, b]$ $(i = 1, 2;\ k = 1, 2, 3)$ are continuous functions such that

$$\alpha_{ik}(x) \le x \ \text{ for } \ 0 \le x \le a, \ \ \beta_{ik}(y) \le y \ \text{ for } \ 0 \le y \le b \ \ (i = 1, 2;\ \ k = 1, 2, 3). \tag{3}$$

Along with $(1')$, (2) consider also the Goursat problem for the second order hyperbolic system with retarded arguments, i.e., the problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = g\big(x, y, u(\alpha_1(x), \beta_1(y)), p(\alpha_2(x), \beta_2(y)), q(\alpha_3(x), \beta_3(y))\big), \tag{4}$$

$$u(0, y) = 0, \quad \text{for} \quad 0 \le y \le b, \quad u(x, 0) = 0 \quad \text{for} \quad 0 \le x \le a, \tag{5}$$

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where

$$p(x,y)=\frac{\partial u(x,y)}{\partial x},\quad q(x,y)=\frac{\partial u(x,y)}{\partial y}, \tag{6}$$

$g : D_{ab}\times R^n\times R^n\times R^n\to R^n$ is a continuous vector function, while $\alpha_k : [0,a]\to[0,a]$ and $\beta_k : [0,b]\to[0,b]$ $(k=1,2,3)$ are continuous functions such that

$$\alpha_k(x)\le x \ \text{ for } \ 0\le x\le a,\quad \beta_k(y)\le y \ \text{ for } \ 0\le y\le b \ \ (k=1,2,3). \tag{7}$$

Let the problem (4), (5) have a solution u. Suppose

$$u_1(x,y)=\frac{\partial u(x,y)}{\partial y},\quad u_2(x,y)=\frac{\partial u(x,y)}{\partial x}.$$

Then

$$u(x,y)=\int\limits_0^y u_1(x,t)dt,\quad u(x,y)=\int\limits_0^x u_2(s,y)ds.$$

Therefore the vector function (u_1,u_2) is the solution of the problem $(1')$, (2), where

$$\begin{gathered} g_i(x,y,z_1,z_2,z_3)\equiv g(x,y,z_3,z_2,z_1),\quad G_i(x,y)\equiv E\ \ (i=1,2),\\ \alpha_{ik}(x)\equiv\alpha_{4-k}(x),\quad \beta_{ik}(x)\equiv\beta_{4-k}(x)\quad (i=1,2;\ \ k=1,2,3)\end{gathered} \tag{8}$$

and E is the unit $n\times n$ matrix.

The inverse assumption is obvious: if the identities (8) are fulfilled and the problem $(1')$, (2) has a solution (u_1,u_2), then the vector function u given by the equality

$$u(x,y)=\int\limits_0^y u_1(x,t)dt$$

is the solution of the problem (4), (5). Thus the problem (4), (5) is equivalent to the problem $(1')$, (2) for the case, where g_i, G_i, α_{ik}, and β_{ik} $(i=1,2;\ k=1,2,3)$ are given by the equalities (8).

In the cases, where $\alpha_k(x)\equiv x$, $\beta_k(y)\equiv y$ $(k=1,2,3)$, $\alpha_{ik}(x)\equiv x$, $\beta_{ik}(y)\equiv y$ $(i=1,2;\ \ k=1,2,3)$, the problems of the existence and uniqueness of a solution of the problems (4), (5), and $(1')$, (2) were investigated by many authors (see, e.g., [1–9] and references therein). However, the problem (1), (2) as well as the problems $(1')$, (2) and (4), (5) are studied insufficiently in the general case. The existence and uniqueness theorems formulated below concern with this case.

We shall use the following notation and definitions.

R^n is the space of n-dimensional real vectors in which under the norm $\|z\|$ of an arbitrary vector z is meant a sum of absolute values of this vector.

$$R^n_r=\{z\in R^n : \|z\|\le r\}.$$

$R^{n\times n}$ is the space of real $n\times n$ matrices in which under the norm $\|Z\|$ of an arbitrary matrix Z is meant maximal value among absolute values of components of this matrix.

$C(D_{ab};R^n)$ is the space of n-dimensional continuous vector functions $z : D_{ab}\to R^n$ with the norm

$$\|z\|_{a,b}=\max\Big\{\|z(x,y)\| :\ \ 0\le x\le a,\ \ 0\le y\le b\Big\}.$$

If $z\in C(D_{ab};R^n)$ and $(x,y)\in D_{ab}$, then

$$\|z\|_{x,y}=\max\Big\{\|z(s,t)\| :\ \ 0\le s\le x,\ \ 0\le t\le y\Big\}.$$

$C^{(1,0)}(D_{ab};R^n)$ is the space of continuous vector functions $z : D_{ab} \to R^n$ having the continuous partial derivative in the first argument.

$$C^{(1,0)}_{r,r_0}(D_{ab};R^n) =$$
$$= \left\{ z \in C_r(D_{ab};R^n) : \ \|z(a,y)\| \le r, \ \left\| \frac{\partial z(x,y)}{\partial x} \right\| \le r_0 \ \text{for} \ (x,y) \in D_{ab} \right\}.$$

$C^{(0,1)}(D_{ab};R^n)$ is the space of continuous vector functions $z : D_{ab} \to R^n$ having the continuous partial derivative in the second argument.

$$C^{(0,1)}_{r,r_0}(D_{ab};R^n) =$$
$$= \left\{ z \in C^{(0,1)}(D_{ab};R^n) : \ \|z(a,y)\| \le r, \ \left\| \frac{\partial z(x,y)}{\partial y} \right\| \le r_0 \ \text{for} \ (x,y) \in D_{ab} \right\}.$$

$\nu_1(z)(\cdot,y)$ and $\nu_2(z)(\cdot,x)$ are moduli of continuity of the vector function $(x,y) \to z(x,y)$ in the first and second arguments, i.e.,

$$\nu_1(z)(\delta,y) = \max\left\{ \|z(x,y) - z(\overline{x},y)\| : \ 0 \le x \le \overline{x} \le a, \ \overline{x} - x \le \delta \right\},$$
$$\nu_2(z)(\delta,x) = \max\left\{ \|z(x,y) - z(x,\overline{y})\| : \ 0 \le y \le \overline{y} \le b, \ \overline{y} - y \le \delta \right\}.$$

Definition 1. We say that $f : C(D_{ab};R^n) \times C(D_{ab};R^n) \to C(D_{ab};R^n)$ is a Volterra operator if for any z_i and $\overline{z}_i \in C(D_{ab};R^n)$ $(i=1,2)$ and $(x,y) \in D_{ab}$ from the equalities

$$z_i(s,t) = \overline{z}_i(s,t) \quad \text{for} \quad 0 \le s \le x, \quad 0 \le t \le y \quad (i=1,2)$$

there follows the equality $f(z_1,z_2)(x,y) = f(\overline{z}_1,\overline{z}_2)(x,y)$.

Definition 2. We say that the system (1) is evolutional if $f_i : C(D_{ab};R^n) \times C(D_{ab};R^n) \to C(D_{ab};R^n)$ $(i=1,2)$ are Volterra operators.

For example, the system $(1')$ is evolutional if the functions α_{ik} and β_{ik} $(i=1,2;$ $k=1,2,3)$ satisfy the inequalities (3).

Throughout the remainder it will be assumed that $f_i : C(D_{ab};R^n) \times C(D_{ab};R^n) \to C(D_{ab};R^n)$ $(i=1,2)$ are the continuous Volterra operators, i.e. the system (1) is evolutional.

By a solution of (1) we understand a vector function $(u_1,u_2) \in C^{(1,0)}(D_{ab};R^n) \times C^{(0,1)}(D_{ab};R^n)$ satisfying (1) on D_{ab}.

Theorem 1. *Let there exist positive numbers r, r_0, and continuous nondecreasing functions $\varepsilon : [0,+\infty[\to [0,+\infty[$, $\varphi : [0,+\infty[\to]0,+\infty[$, and $\omega_i : [0,+\infty[\to [0,+\infty[$ $(i=1,2)$ such that $\varepsilon(0)=0$,*

$$\int_0^\tau \frac{ds}{\varphi(s)} > 2(a+b), \quad r_0 = \varphi(r), \tag{9}$$

$$\omega_i(s) > 0 \ \textit{for} \ s > 0, \quad \int_0^1 \frac{ds}{\omega_i(s)} = +\infty \ \ (i=1,2), \tag{10}$$

and for any $z_1 \in C^{(1,0)}_{r,r_0}(D_{ab};R^n)$ and $z_2 \in C^{(0,1)}_{r,r_0}(D_{ab};R^n)$ the conditions

$$\Big\| f_i(z_1,z_2)(x,y) \Big\| \le \varphi\Big(\|z_1\|_{x,y} + \|z_2\|_{x,y} \Big) \quad (i=1,2), \tag{11}$$
$$\Big\| f_1(z_1,z_2)(x,\overline{y}) - f_1(z_1,z_2)(x,y) \Big\| \le \varepsilon(|y-\overline{y}|) + \omega_1\Big[\nu_2(z_1)(|y-\overline{y}|,x) \Big],$$
$$\Big\| f_2(z_1,z_2)(\overline{x},y) - f_2(z_1,z_2)(x,y) \Big\| \le \varepsilon(|x-\overline{x}|) + \omega_2\Big[\nu_1(z_2)(|x-\overline{x}|,y) \Big]$$

are fulfilled on D_{ab}. *Then the problem* (1), (2) *has at least one solution.*

Theorem 2. *Let there exist positive numbers* r, r_0, *and continuous nondecreasing functions* $\varphi : [0,+\infty[\to]0,+\infty[$ *and* $\omega : [0,+\infty[\to [0,+\infty[$ *such that for any* z_1 *and* $\overline{z}_1 \in C_r^{(1,0)}(D_{ab}; R^n)$ *and* z_2 *and* $\overline{z}_2 \in C_r^{(0,1)}(D_{ab}; R^n)$ *along with* (11) *the condition*

$$\Big\| f_i(z_1,z_2)(x,y) - f_i(\overline{z}_1,\overline{z}_2)(x,y)\Big\| \le \omega\Big(\|z_1-\overline{z}_1\|_{x,y} + \|z_2-\overline{z}_2\|_{x,y}\Big) \quad (i=1,2)$$

holds on D_{ab}. *Moreover, let*

$$\omega(0)=0, \quad \omega(s)>0 \ \text{ for } \ s>0, \quad \int_0^1 \frac{ds}{\omega(s)} = +\infty, \tag{12}$$

and let the condition (9) *hold. Then the problem* (1), (2) *has at most one solution.*

From Theorems 1 and 2 we obtain the following propositions on the solvability and unique solvability of the problems $(1')$, (2) and (4), (5).

Corollary 1. *Let there exist positive numbers* l, r, *and continuous nondecreasing functions* $\varphi : [0,+\infty[\to]0,+\infty[$ *and* $\omega_i : [0,+\infty[\to [0,+\infty[$ $(i=1,2)$ *such that*

$$\int_0^{\beta_{13}(y)} \|G_1(x,t)\|\,dt \le l, \qquad \int_0^{\alpha_{23}(x)} \|G_2(s,y)\|\,ds \le l \quad \text{for} \quad (x,y)\in D_{ab} \quad (i=1,2) \tag{13}$$

and the conditions

$$\begin{gathered}
\Big\|g_i(x,y,z_1,z_2,z_3)\Big\| \le \varphi\Big(\|z_1\|+\|z_2\|+\|z_3\|\Big) \quad (i=1,2), \\
\Big\|g_1(x,y,z_1,z_2,z_3) - g_1(x,y,\overline{z}_1,z_2,z_3)\Big\| \le \omega_1(\|z_1-\overline{z}_1\|), \\
\Big\|g_1(x,y,z_1,z_2,z_3) - g_1(x,y,z_1,\overline{z}_2,z_3)\Big\|\| \le \omega_2(\|z_2-\overline{z}_2\|)
\end{gathered} \tag{14}$$

hold on $D_{ab}\times R_r^n\times R_r^n\times R_r^n$. *Moreover, let*

$$\int_0^r \frac{ds}{\varphi(s)} > 2(1+l)(a+b) \tag{15}$$

and the functions α_{ik}, β_{ik}, ω_i $(i=1,2;\ k=1,2,3)$ *satisfy the conditions* (3) *and* (10). *Then the problem* $(1')$, (2) *has at least one solution.*

Corollary 2. *Let there exist positive numbers* l, r, *and continuous nondecreasing functions* $\varphi : [0,+\infty[\to]0,+\infty[$ *and* $\omega : [0,+\infty[\to [0,+\infty[$ *such that the matrix functions* G_i $(i=1,2)$ *satisfy the inequalities* (13), *and along with* (14) *the condition*

$$\begin{gathered}
\Big\|g_i(x,y,z_1,z_2,z_3) - g_i(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3)\Big\| \le \\
\le \omega\Big(\|z_1-\overline{z}_1\|+\|z_2-\overline{z}_2\|+\|z_3-\overline{z}_3\|\Big) \quad (i=1,2)
\end{gathered}$$

holds on $D_{ab}\times R_r^n\times R_r^n\times R_r^n$. *Moreover, let the functions* α_{ik}, β_{ik} $(i=1,2;\ k=1,2,3)$, ω *and* φ *satisfy the conditions* (3), (12) *and* (15). *Then the problem* $(1')$, (2) *has one and only one solution.*

Corollary 3. *Let there exist positive numbers l, r, and continuous nondecreasing functions* $\varphi : [0,+\infty[\to]0,+\infty[$ *and* $\omega_i : [0,+\infty[\to [0,+\infty$ $(i = 1,2)$ *such that*

$$\alpha_1(x) \le l \ \textit{for} \ 0 \le x \le a, \quad \beta_1(y) \le l \ \textit{for} \ 0 \le y \le b, \tag{16}$$

and the conditions

$$\begin{gathered} \Big\| g(x,y,z_1,z_2,z_3)\Big\| \le \varphi\Big(\|z_1\| + \|z_2\| + \|z_3\|\Big), \\ \Big\| g(x,y,z_1,z_2,z_3) - g(x,y,z_1,\overline{z}_2,z_3)\Big\| \le \omega_1\Big(\|z_2 - \overline{z}_2\|\Big), \\ \Big\| g(x,y,z_1,z_2,z_3) - g(x,y,z_1,z_2,\overline{z}_3)\Big\| \le \omega_2\Big(\|z_3 - \overline{z}_3\|\Big) \end{gathered} \tag{17}$$

hold on $D_{ab} \times R_r^n \times R_r^n \times R_r^n$. *Moreover, let the functions* α_k, β_k $(k = 1,2,3)$, ω_i $(i = 1,2)$ *and* φ *satisfy the conditions* (7), (10), *and* (15). *Then the problem* (4), (5) *has at least one solution.*

In the case, where $\alpha_k(x) \equiv x$ and $\beta_k(y) \equiv y$ $(k = 1,2,3)$, the results of Hartman–Wintner [1] and Alexiewicz–Orlicz [3] concerning the solvability of the Goursat problem follow from Corollary 3.

Corollary 4. *Let there exist positive numbers l, r, and continuous nondecreasing functions* $\varphi : [0,+\infty[\to]0,+\infty[$ *and* $\omega : [0,+\infty[\to [0,+\infty[$ *such that the inequalities* (7) *and* (16) *are fulfilled and along with* (17) *the condition*

$$\Big\| g(x,y,z_1,z_2,z_3) - g(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3)\Big\| \le \omega\Big(\|z_1 - \overline{z}_1\| + \|z_2 - \overline{z}_2\| + \|z_3 - \overline{z}_3\|\Big)$$

holds on $D_{ab} \times R^n \times R^n \times R^n$. *Moreover, let the functions* ω *and* φ *satisfy the conditions* (12) *and* (15). *Then the problem* (4), (5) *has one and only one solution.*

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