Mem. Differential Equations Math. Phys. 12(1997), 22–32

T. Burchuladze

ON THREE-DIMENSIONAL DYNAMICAL PROBLEMS OF THE GENERALIZED THEORY OF ELASTOTHERMODIFFUSION

Abstract. For Green–Lindsay's and Lord–Shulman's models, three–dimensional boundary value and contact dynamical problems of the mathematical theory of elasticity are considered. By the Riesz–Fisher–Kupradze method (the discrete singularity method), approximate solutions are effectively constructed.

რეზიუმე. ნაშრომში გრინ-ლინდსეისა და ლორდ-შულმანის მოდელებისათვის გამოკვლეულია დრაკადობის მათემატიკური თეორიის სამგანზომილებიანი სასაზღვრო და საკონტაქტო დინამიური ამოცანები. რის-ფიშერ-კუპრაძის მეთოდის გამოყენებით (დისკრეტული განსაკუთრებულობის მეთოდი) ეფექტურადაა აგებული მიახლოებითი ამონახსნები.

In the present paper, we investigate different three-dimensional boundary value and contact dynamical problems for new models of the mathematical theory of elasticity with conjugate fields. Intensively developing for the last years new branches require the construction of a general theory of solvability, the elaboration of analytic methods for solving complicated problems dealing with the interaction of fields of different nature. One of the possible approaches allowing us to solve these problems is the well-elaborated method of the potential theory and the theory of singular integral equations. We present here a complete mathematical analysis of these problems as well as give an effective algorithm for approximate construction of solutions of boundary-value and contact problems. The results of our investigations in this area can be found in [1, 2, 3] and in [4–14].

We consider a three-dimensional isotropic elastic medium in which thermodiffusion takes place. Deformation is described by the displacement vector $v(x,t) = (v_1, v_2, v_3)^T = \|v_k\|_{3\times 1}$ (one-column matrix), the variation of temperature $v_4(x,t)$ and the "chemical potential" of the medium $v_5(x,t)$; $x = (x_1, x_2, x_3)$ are the points of the Euclidean space $\mathbb{R}^3$, $t \geq 0$ is time, and the sign "T" stands for transposition.

1991 *Mathematics Subject Classification.* 73B30, 73C25.

Key words and phrases. Three–dimensional contact dynamical problems, conjugate fields, Green tensors, fundamantal solutions, approximate solutions.

The objects of our investigations are the following systems of partial differential equations of the generalized theory of elastothermodiffusion [1–3].

I. Green–Lindsay's model:

$$\begin{cases} \mu\Delta v + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,v - \rho\,\dfrac{\partial^2 v}{\partial t^2} - \sum\limits_{l=1}^{2}\gamma_l\Big(1+\tau^1\,\dfrac{\partial}{\partial t}\Big)\,\mathrm{grad}\,v_{3+l} = 0,\\ \delta_1\Delta v_4 - a_1\Big(1+\tau^0\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_4}{\partial t} - \gamma_1\,\dfrac{\partial}{\partial t}\,\mathrm{div}\,v - a_{12}\Big(1+\tau^0\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_5}{\partial t} = 0,\\ \delta_2\Delta v_5 - a_2\Big(1+\tau^0\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_5}{\partial t} - \gamma_2\,\dfrac{\partial}{\partial t}\,\mathrm{div}\,v - a_{12}\Big(1+\tau^0\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_4}{\partial t} = 0, \end{cases} \tag{1}$$

where the elastic, thermal, diffusion and relaxation constants satisfy the natural restrictions [3]

$$\mu > 0,\quad 3\lambda+2\mu > 0,\quad \rho > 0,\quad \delta_k > 0,\quad a_k > 0\quad (k=1,2),\quad a_1a_2 - a_{12}^2 > 0,$$
$$\tau_1 \ge \tau_0 \ge 0 \quad (\tau_1 = \tau_0 = 0\ -\ \text{ is the classical case}).$$

II. Lord–Shulman's model:

$$\begin{cases} \mu\Delta v + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,v - \rho\,\dfrac{\partial^2 v}{\partial t^2} - \sum\limits_{l=1}^{2}\gamma_l\,\mathrm{grad}\,v_{3+l} = 0,\\ \delta_1\Delta v_4 - a_1\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_4}{\partial t} - \gamma_1\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial}{\partial t}\,\mathrm{div}\,v -\\ \qquad -a_{12}\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_5}{\partial t} = 0,\\ \delta_2\Delta v_5 - a_2\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_5}{\partial t} - \gamma_2\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial}{\partial t}\,\mathrm{div}\,v -\\ \qquad -a_{12}\Big(1+\tau_t\,\dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_4}{\partial t} = 0; \end{cases} \tag{2}$$

here $\tau_t > 0$ is a relaxation constant.

The non-stationary systems (1) and (2) will be written below in the vector-matrix form

$$L\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)V(x,t) = 0, \tag{3}$$

where $V(x,t) = (v, v_4, v_5)^T = \|v_k\|_{5\times1}$, $LV = \|(LV)_k\|_{5\times1}$.

We consider two (possible) cases when unknown vector $V(x,t)$ depends on the time t:

(a) $v_k(x,t) = \mathrm{Re}[e^{-ipt}u_k(x,p)]$, steady (stationary) oscillations with the frequency $p > 0$;

(b) $v_k(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\zeta t}u_k(x,\zeta)\,d\zeta$, $\zeta = \sigma + iq$, $\sigma > 0$ is the representation given by Laplace–Mellin's integral (general dynamical case), $k = \overline{1,5}$.

It can be easily seen that in both cases the dynamical system (3) (and hence the systems (1) and (2)) is reduced with respect to the vector $U(x,\omega) =$

$\|u_k\|_{5\times 1}$ to the form

$$L\Big(\frac{\partial}{\partial x}, -i\omega\Big)U = 0, \tag{4}$$

where $\omega = p > 0$ in the case (a) and $\omega = i\zeta$ in the case (b). $L(\frac{\partial}{\partial x}, -i\omega)$ is an elliptic matrix differential operator. Denote by $\Phi(x; -i\omega)$ a matrix of fundamental solutions of this operator. It can be constructed explicitly in terms of elementary functions and has the form [3, 13] $\Phi(x; -i\omega) = \|\Phi_{jk}\|_{5\times 5} = \|\overset{1}{\Phi}, \overset{2}{\Phi}, \dots, \overset{5}{\Phi}\|_{5\times 5}$, where $\overset{k}{\Phi} = (\Phi_{1k}, \Phi_{2k}, \dots, \Phi_{5k})^T$, $k = \overline{1,5}$, are column-vectors; $\Phi(x; -i\omega) = \sum_{k=1}^{4} \Omega^k(\frac{\partial}{\partial x})\frac{e^{i\lambda_k|x|}}{|x|}$, where $\Omega^k(\frac{\partial}{\partial x})$ are explicitly specified matrix differential operators and $\lambda_k(\omega)$ are the so-called characteristic constants expressed explicitly in terms of the coefficients of the differential operator under consideration. Behavior of the matrix of fundamental solutions Φ depends on the properties $\lambda_k(\omega)$; all the necessary properties are established.

We have the following relation: $\Phi^T(-x; -i\omega) = \widetilde{\Phi}(x; -i\omega)$, where $\widetilde{\Phi}(x; -i\omega)$ is the matrix of fundamental solutions of the associate (conjugate) operator $\widetilde{L}(\frac{\partial}{\partial x}, -i\omega) \equiv L^T(-\frac{\partial}{\partial x}, -i\omega)$.

For the systems (3) and (4), the basic initial–boundary and boundary value problems for finite and infinite (unbounded) domains are investigated; appropriate theorems for the uniqueness and existence of solutions are proved; integral formulas convenient for numerical realizations are constructed. Along with general theoretical problems, great attention is given to the approximate and efficient construction of solutions [3,4,5,7,8,13, 14].

Principal boundary differential operators of this theory are of the form

$$\begin{aligned}
P_{(k)}\Big(\frac{\partial}{\partial x}, n\Big)U(x) = \Big(T\Big(\frac{\partial}{\partial x}, n\Big)u(x) - n(x)\sum_{l=1}^{2}\gamma_l(1-\tau^1 i\omega)u_{3+l},\\
-(\delta_{1k}+\delta_{2k})u_4 + (\delta_{3k}+\delta_{0k})\delta_1\frac{\partial v_4}{\partial n},\\
-(\delta_{1k}+\delta_{3k})u_5 + (\delta_{2k}+\delta_{0k})\delta_2\frac{\partial v_5}{\partial n}\Big)^T,\\
Q_{(k)}\Big(\frac{\partial}{\partial x}, n\Big)U(x) = \Big(u, (\delta_{1k}+\delta_{2k})\delta_1\frac{\partial v_4}{\partial n} + (\delta_{3k}+\delta_{0k})v_4,\\
+(\delta_{1k}+\delta_{3k})\delta_2\frac{\partial v_5}{\partial n} + (\delta_{2k}+\delta_{0k})v_5\Big)^T, \quad k = \overline{0,3},
\end{aligned}$$

where $T(\frac{\partial}{\partial x}, n) = \|\mu\delta_{jk}\frac{\partial}{\partial n} + \lambda n_j\frac{\partial}{\partial x_k} + \mu n_k\frac{\partial}{\partial x_j}\|_{3\times 3}$ is a matrix differential stress operator of the classical theory of elasticity [1], $n(x) = (n_1, n_2, n_3)$ is the unit vector and δ_{jk} is the Kronecker symbol. The corresponding associate operators are denoted by $\widetilde{P}_{(k)}$, $\widetilde{Q}_{(k)}$. Moreover, $\widetilde{Q}_{(k)} = Q_{(k)}$, $k = \overline{0,3}$.

Let ${}_1D \subset \mathbb{R}^3$ be a finite domain bounded by a surface $S \in \mathcal{L}_2(\alpha)$, $\alpha > 0$ [1], and ${}_2D = \mathbb{R}^3 \setminus {}_1\overline{D}$ be an infinite domain. We consider the boundary value problems for the system (4):

Problem $P^j_{(q)}(\omega)$**.** In the domain ${}_jD$ $(j = 1, 2)$, find a regular vector $U = (u, u_4, u_5)^T \in C^1({}_j\overline{D}) \cap C^2({}_jD)$ satisfying

$$\forall x \in {}_jD: \ L\Big(\frac{\partial}{\partial x}, -i\omega\Big)U = 0,$$
$$\forall z \in S: \ \Big(P_{(q)}\Big(\frac{\partial}{\partial z}, n\Big)U\Big)^j = F^{(1)}(z),$$

where

$$\Big(P_{(q)}\Big(\frac{\partial}{\partial x}, n\Big)U(z)\Big)^j = \lim_{{}_jD \ni x \to z \in S} P_{(q)}\Big(\frac{\partial}{\partial x}, n\Big)U(x).$$

Problem $Q^j_{(q)}(\omega)$**.** In the domain ${}_jD$ $(j = 1, 2)$ find a regular vector $U = (u, u_4, u_5)^T \in C^1({}_j\overline{D}) \cap C^2({}_jD)$ satisfying

$$\forall x \in {}_jD: \ L\Big(\frac{\partial}{\partial x}, -i\omega\Big)U = 0,$$
$$\forall z \in S: \ \Big(Q_{(q)}\Big(\frac{\partial}{\partial z}, n\Big)U\Big)^j = F^{(2)}(z),$$

where

$$q = \overline{0, 3}, \quad Q_{(q)}F^{(1)} \in C^{0,\alpha}(S), \quad P_{(q)}F^{(2)} \in C^{0,al}(S).$$

(In case $j = 2$, the solution U satisfies certain decrease conditions at infinity [3]).

General theory of solvability of the above-mentioned problems is constructed [3–14]. (Corresponding theorems for the existence and uniqueness of the solution are proved; the problems of smoothness are considered and the estimates with respect to the parameter are given).

Here we begin with the actual construction of solutions. Consider, for example, Problem $P^1_{(q)}(i\zeta)$. We denote Green's tensor of this problem by $G_{P_{(q)}}(x, y; i\zeta, {}_1D)$. We have [1, 3]:

$$G_{P_{(q)}}(x, y; i\zeta, {}_1D) = \Phi(x - y; \zeta) - g_{P_{(q)}}(x, y; i\zeta, {}_1D),$$

where $g_{P_{(q)}}$ is a regular component. The representation

$$\forall x^0 \in {}_1D: U(x^0) = \int_S \Big[Q_{(q)}\Big(\frac{\partial}{\partial y}, n\Big) G^T_{P_{(q)}}(x^0, y; i\zeta, {}_1D)\Big]^T F^{(1)}(y)\, d_yS \quad (5)$$

is valid [1, 3].

Thus we have [1, 3] $G^T_{P_{(q)}}(x^0,x) = \widetilde{G}_{P_{(q)}}(x,x^0)$, where $\widetilde{G}_{P_{(q)}}(x,x^0) = \widetilde{\Phi}(x-x^0) - \widetilde{g}_{P_{(q)}}(x,x^0)$ is Green's tensor of the associate problem $\widetilde{P}^1_{(q)}(i\zeta)$ (with x^0 as a pole):

$$\forall x \in D_1: \ \widetilde{L}\Big(\frac{\partial}{\partial x},\zeta\Big)\widetilde{U} = 0, \quad \forall y \in S: \ \big(\widetilde{P}_{(q)}\widetilde{U}\big)^1 = 0.$$

Consequently, the representation (5) implies

$$\forall x^0 \in {}_1D: \ U(x^0) = \int_S \Big[\widetilde{Q}_{(q)}\Big(\frac{\partial}{\partial y},n\Big)\Phi^T(x^0-y)\Big]^T F^{(1)}(y)\,d_yS - \\ - \int_S \Big[\widetilde{Q}_{(q)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_{P_{(q)}}(y,x^0)\Big]^T F^{(1)}(y)\,d_yS. \tag{6}$$

It turns out that we can find a value $\widetilde{Q}_{(q)}\widetilde{g}_{P_{(q)}}(y,x^0)|_S$, $x^0 \in {}_1D$ without solving the problem.

Let ${}_2\widetilde{S} \in \mathcal{L}_2(\alpha)$ be an arbitrary closed surface covering ${}_1D$ and let $\{{}_2x^k\}_{k=1}^{\infty} \subset {}_2\widetilde{S}$ be an everywhere dense countable set of points. Let $\overset{s}{\widetilde{g}}_{P_{(q)}}$, $s=\overline{1,5}$, be the s-th vertical vector $\widetilde{g}_{P_{(q)}} = \|\overset{1}{\widetilde{g}}_{P_{(q)}},\dots,\overset{5}{\widetilde{g}}_{P_{(q)}}\|_{5\times 5}$.

By the formula of general integral representation of the vector $\overset{s}{\widetilde{g}}_{P_{(q)}}$, we have [1, 3]

$$\int_S \Gamma^T(y-{}_2x^k,\zeta)\widetilde{\psi}^s(y,x^0)\,d_yS = \Theta({}_2x^k,x^0), \tag{7}$$

where

$$\Gamma(y-x,\zeta) = P_{(q)}\Big(\frac{\partial}{\partial y},n\Big)\Phi(y-x;\zeta),$$
$$\widetilde{\psi}^s(y,x^0) = \widetilde{Q}_{(q)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}(y,x^0), \quad x^0 \in {}_1D,$$
$$\Theta({}_2x^k,x^0) = \int_S \big[Q_{(q)}\Phi(y-{}_2x^k,\zeta)\big]^T\widetilde{P}_{(q)}\overset{s}{\widetilde{\Phi}}(y-x^0)\,d_yS.$$

The vector equality (7) can be rewritten in terms of components as

$$\int_S [\overset{k}{\psi}(y)]^T\widetilde{\psi}^s(y,x^0)\,d_yS = \Theta_{l_k}\big({}_2x^{[\frac{k+4}{5}]},x^0\big), \tag{8}$$

where

$$\overset{k}{\psi}(y) = P_{(q)}\Big(\frac{\partial}{\partial y},n\Big)\overset{l_k}{\Phi}\big(y-{}_2x^{[\frac{k+4}{5}]},\zeta\big), \quad l_k = k-5\Big[\frac{k-1}{5}\Big], \quad k=\overline{1,\infty}.$$

($[k]$ is the integral part of the number k). The following theorem is valid.

Theorem 1. *A system of vectors* $\{\overset{k}{\psi}(y)\}_{k=1}^{\infty}$ *is linearly independent and complete in a vector (five-dimensional) Hilbert space* $L_2(S)$ *(i.e., this system is a basis in this space).*

For the proof of the theorem see [1, 3].

Determine now the coefficients α_k^s, $k=\overline{1,N}$, from the condition

$$\min_{\alpha_k^s}\Big\|\widetilde{\psi}^s(z)-\sum_{k=1}^N\alpha_k^s\overline{\overset{k}{\psi}}(z)\Big\|_{L_2(S)}.$$

(here the sign – stands for "complex-conjugate").

By Theorem 1 and the equality (8), we obtain a uniquely solvable algebraic system with respect to α_k^s:

$$\sum_{k=1}^N\alpha_k^s(\overline{\overset{k}{\psi}},\overline{\overset{j}{\psi}})=(\widetilde{\psi}^s,\overline{\overset{j}{\psi}}),\quad j=\overline{1,N}, \tag{9}$$

where for the scalar product we adopt the notation

$$(\overset{k}{\psi},\overline{\psi})=\int_S[\overset{k}{\psi}]^T\psi\,dS=(\psi,\overline{\overset{k}{\psi}}).$$

According to the property of the Hilbert space $L_2(S)$, we have

$$\lim_{N\to\infty}\Big\|\widetilde{\psi}^s(z)-\sum_{k=1}^N\alpha_k^s\overline{\overset{k}{\psi}}(z)\Big\|_{L_2(S)}=0. \tag{10}$$

Introduce the notation:

$$\begin{aligned}
{}_N\widetilde{\psi}^s(z)&=\sum_{k=1}^N\alpha_k^s\overline{\overset{k}{\psi}}(z)\equiv{}_N\big(\widetilde{Q}_{(q)},\overset{s}{\widetilde{g}}(z,x^0)\big)^T,\quad s=\overline{1,5},\\
{}_N\Big(\widetilde{Q}_{(q)}\Big(\frac{\partial}{\partial z}\Big)\widetilde{g}(z,x^0)\Big)&\equiv\Big\|{}_N(\widetilde{Q}_{(q)}\overset{1}{\widetilde{g}}),\dots,{}_N(\widetilde{Q}_{(q)}\overset{5}{\widetilde{g}})\Big\|_{5\times5}\equiv\\
&\equiv\big\|{}_N\widetilde{\psi}^1(z),\dots,{}_N\widetilde{\psi}^5(z)\big\|_{5\times5}\equiv\\
&\equiv\Big\|\sum_{k=1}^N\alpha_k^1\overline{\overset{k}{\psi}}(z),\dots,\sum_{k=1}^N\alpha_k^5\overline{\overset{k}{\psi}}(z)\Big\|_{5\times5},\\
{}_NU(x^0)&=\int_S\Big[\widetilde{Q}_{(q)}\Big(\frac{\partial}{\partial y}\Big)\Phi^T(x^0-y)\Big]^TF^{(1)}(y)\,d_yS-\\
&-\int_S\big\|{}_N\widetilde{\psi}^1(y,x^0),\dots,{}_N\widetilde{\psi}^5(y,x^0)\big\|_{5\times5}^TF^{(1)}(y)\,d_yS.
\end{aligned} \tag{11}$$

Thus due to (6), (8), (10) and (11), we finally have $\forall x^0\in{}_1D\cup{}_2D$ and for an arbitrary natural N

$$\big|U(x^0)-{}_NU(x^0)\big|\le$$

$$\leq \sum_{s=1}^{5} \big\| \widetilde{\psi}^s(y,x^0) - {}_N\widetilde{\psi}^s(y,x^0) \big\|_{L_2(S)} \cdot \|F^{(1)}(y)\|_{L_2(S)}. \tag{12}$$

Consider now the so-called basic contact problem for an inhomogeneous medium ${}_1D \cup {}_2D$ [1]: ${}_1D$ and ${}_2D$ are filled with different homogeneous isotropic elastic materials. Denote the elastothermodiffusion constants for the domains ${}_jD$, $j = 1,2$, by ${}_j\lambda$, ${}_j\mu$, ${}_j\rho$, $\dots$, and the differential operators by ${}_jL$, ${}_jP_{(q)}$, and so on.

Problem $A^c(i\zeta)$. Define in ${}_1D \cup {}_2D$ a regular vector

$$U(x,\zeta) = (u,u_4,u_5)T \in C^1({}_1\overline{D} \cup {}_2\overline{D}) \cap C^2({}_1D \cup {}_2D),$$
$$\zeta \in \Pi_{\sigma_0^*} \equiv \big\{\zeta: \ \ \mathrm{Re}\,\zeta > \sigma_0^*\big\},$$

($\sigma_0^* > 0$ is a given constant) satisfying

$$\forall x \in {}_jD: \ {}_jL\Big(\frac{\partial}{\partial x},\zeta\Big)U(x,\zeta) = 0, \ \ j = 1,2,$$
$$\forall y \in S: \ \Big({}_1Q_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)U(y,\zeta)\Big)^1 - \Big({}_2Q_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)U(y,\zeta)\Big)^2 = f(y,\zeta),$$
$$\Big({}_1P_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)U(x,\zeta)\Big)^1 - \Big({}_2P_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)U(x,\zeta)\Big)^2 = F(y,\zeta),$$

($k_0 = \overline{0,3}$ is a fixed number) and the asymptotic conditions at infinity ($|x| \to \infty$)

$$U(x,\zeta) = O(|x|^{-1}), \quad \frac{\partial}{\partial x_k}U(x,\zeta) = O(|x|^{-2}), \ \ k = \overline{1,3}.$$

Let

$$G_c(x,x^0;i\zeta) = {}_j\Phi(x-x^0,\zeta) - {}_jg_c(x,x^0;i\zeta), \quad x \in {}_jD, \ \ x^0 \in \mathbb{R}^3\backslash S,$$

where $j = 1,2$ is Green's tensor of the contact problem $A^c(i\zeta)$, x^0 is a pole and ${}_jg_c$ is a regular component. Denote $g_c(x,x^0;i\zeta) = {}_jg_c(x,x^0;i\zeta)$, $x \in {}_jD$, $j = 1,2$, $g_c = \|\overset{1}{g}_c,\dots,\overset{5}{g}_c\|_{5\times 5}$, where $\overset{s}{g}_c$ is a column-vector, $s = \overline{1,5}$. The existence of this tensor and its basic properties are established as usual [1, 3]. By means of the latter we can write the formula of general representation of the solution of Problem $A^c(i\zeta)$. We can easily see that the following formula is valid:

$$\forall x^0 \in {}_1D \cup {}_2D: U(x,\zeta) = \int_S \Big\{\big[{}_1\widetilde{Q}_{(k_0)}\widetilde{G}_c(y,x^0)\big]^T\Big\}^1 F(y)\,d_yS -$$
$$- \int_S \Big\{\big[{}_1\widetilde{P}_{(k_0)}\widetilde{G}_c(y,x^0)\big]^T\Big\}^1 f(y)\,d_yS, \tag{13}$$

where

$$\big[\widetilde{G}_c(y,x^0)\big]^T = G_c(x^0,y),$$

$\widetilde{G}_c(x,x^0;i\zeta)$ is Green's tensor of the conjugate Problem $\widetilde{A}^c(i\zeta)$.

Having written (13) in an expanded form, we obtain

$$\forall x^0 \in {}_jD: \; U(x^0,\zeta) = \int_S \left[{}_1\widetilde{Q}_{(k_0)}{}_j\Phi^T(x^0-y)\right]^T F(y)\,d_yS -$$
$$- \int_S \left\{\left[{}_1\widetilde{Q}_{(k_0)}\widetilde{g}_c(y,x^0)\right]^T\right\}^1 F(y)\,d_yS -$$
$$- \int_S \left[{}_1\widetilde{P}_{(k_0)}{}_j\Phi^T(x^0-y)\right]^T f(y)\,d_yS +$$
$$+ \int_S \left\{\left[{}_1\widetilde{P}_{(k_0)}\widetilde{g}_c(y,x^0)\right]^T\right\}^1 f(y)\,d_yS \quad (j=1,2). \tag{14}$$

Taking into account (14), we make an important conclusion: in order to construct (explicitly or approximately) the solution of Problem $A^c(i\zeta)$, it is sufficient to know on the surface S the following quantities:

$$\left[{}_1\widetilde{Q}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c(y,x^0)\right]^1 \quad \text{and} \quad \left[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c(y,x^0)\right]^1, \;\; y\in S, \;\; x^0 \overline{\in} S.$$

With regard for the definition of $\overset{s}{\widetilde{g}}_c(x,x^0)$, $s=\overline{1,5}$, the formula of general integral representation of solution of boundary value problems of this theory gives

$$\forall x \in {}_{j+1}D: \; \int_S \left[{}_jP_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi(y-x)\right]^T \left[{}_1\widetilde{Q}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\right]^1 d_yS -$$
$$- \int_S \left[{}_jQ_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi(y-x)\right]^T \left[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\right]^1 d_yS =$$
$$= {}_j\Theta(x), \tag{15}$$

where $j=1,2$, ${}_3D \equiv {}_1D$, ${}_1\Theta \equiv 0$, ${}_2\Theta$ is a well–determined vector which is expressed by ${}_1P_{(k_0)1}\Phi - {}_2P_{k_0)2}\Phi$ and ${}_1Q_{(k_0)1}\Phi - {}_2Q_{k_0)2}\Phi$.

Introduce the notation:

$$\widetilde{\psi}^s(y,\zeta) = \|\widetilde{\psi}_k^s\|_{10\times 1} =$$
$$= \left(\left[{}_1\widetilde{Q}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\right]^1, \left[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\right]^1\right)^T$$

– an unknown vector,

$${}_j\Psi(y-x,\zeta) =$$
$$= \left\| \boxed{\left[{}_jP_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi(y-x)\right]^T}_{5\times 5}, \;\; \boxed{-\left[{}_jQ_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi(y-x)\right]^T}_{5\times 5} \right\|_{5\times 10} \tag{16}$$
$$(j=1,2).$$

Hence (15) takes the form

$$\forall x \in {}_{j+1}D: \; \int_S {}_j\Psi(y-x;\zeta)\widetilde{\psi}^s(y,\zeta)\,d_yS = {}_j\Theta(x), \;\; j=1,2. \tag{17}$$

Let ${}_1\widetilde{S}$ be a closed surface placed strictly in ${}_1D$ and $\{{}_jx^k\}_{k=1}^{\infty} \subset {}_j\widetilde{S}$, $j = 1, 2$, be an everywhere dense countable set of points. By (10) we have

$$\begin{aligned} &\int_S {}_1\Psi(y - {}_2x^k, \zeta)\widetilde{\psi}^s(y, \zeta)\, d_yS = {}_1\Theta({}_2x^k), \\ &\int_S {}_2\Psi(y - {}_1x^k, \zeta)\widetilde{\psi}^s(y, \zeta)\, d_yS = {}_2\Theta({}_1x^k). \end{aligned} \tag{18}$$

Denote by ${}_j\Psi^1, \dots, {}_j\Psi^5$ the columns of the matrix ${}_j\Psi^T$. Then the following theorem holds.

Theorem 2. *The countable set of vectors*

$$\big\{{}_1\Psi^l(y - {}_2x^k)\big\}_{k=1,\, l=1}^{\infty,\, 5} \cup \big\{{}_2\Psi^l(y - {}_1x^k)\big\}_{k=1,\, l=1}^{\infty,\, 5} \tag{19}$$

is linearly independent and complete in a vector (*ten-dimensional*) *Hilbert space* $L_2(S)$.

(For the proof see [1, 3]).

Renumerate (19) as follows:

$$\overset{k}{\psi}(y) = {}_{a_k}\Psi^{l_k}(y - {}_{b_k}x^{q_k}, \zeta), \quad k = \overline{1, \infty}, \tag{20}$$

where

$$\begin{gathered} a_k = k - 2\Big[\frac{k-1}{2}\Big], \quad b_k = 2\Big[\frac{k+1}{2}\Big] - k + 1, \\ q_k = \left[\frac{[\frac{k+1}{2}] + 4}{5}\right], \quad l_k = \Big[\frac{k+1}{2}\Big] - 5\left[\frac{[\frac{k+1}{2}] - 1}{5}\right]. \end{gathered}$$

According to (18), the scalar product

$$(\overset{k}{\psi}, \overline{\widetilde{\psi}}^s) = \int_S [\overset{k}{\psi}]^T \widetilde{\psi}^s\, dS = \int_S [\widetilde{\psi}^s]^T \overset{k}{\psi}\, dS = (\widetilde{\psi}^s, \overline{\overset{k}{\psi}}) = {}_{a_k}\Theta_{l_k}({}_{b_k}x^{q_k}), \quad k = \overline{1, \infty} \tag{21}$$

is known. Determine now the coefficients β_k^s, $k = \overline{1, N}$, $s = \overline{1, 5}$ from the condition

$$\min_{\beta_k^s} \Big\| \widetilde{\psi}^s(z) - \sum_{k=1}^{N} \beta_k^s \overline{\overset{k}{\psi}}(z) \Big\|_{L_2(S)}.$$

Repeating word by word the above-said, we obtain the following approximate values:

$$
\begin{aligned}
{}_N\widetilde{\psi}^s(y) &= \sum_{k=1}^{N} \beta_k^s \overset{\overline{k}}{\psi}(y),\\
{}_N\Big({}_1Q_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\Big)^1 &\equiv \big({}_N\widetilde{\psi}_1^s, {}_N\widetilde{\psi}_2^s, \dots, {}_N\widetilde{\psi}_5^s\big)^T \equiv\\
&\equiv \sum_{k=1}^{N} \beta_k^s \big(\overset{\overline{k}}{\psi}_1, \overset{\overline{k}}{\psi}_2, \dots, \overset{\overline{k}}{\psi}_5\big)^T, \qquad (22)\\
{}_N\Big(\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\overset{s}{\widetilde{g}}_c(y,x^0)\Big)^1 &\equiv \big({}_N\widetilde{\psi}_6^s, {}_N\widetilde{\psi}_7^s, \dots, {}_N\widetilde{\psi}_{10}^s\big)^T \equiv\\
&\equiv \sum_{k=1}^{N} \beta_k^s \big(\overset{\overline{k}}{\psi}_6, \overset{\overline{k}}{\psi}_7, \dots, \overset{\overline{k}}{\psi}_{10}\big)^T.
\end{aligned}
$$

Substituting the above values in (14), we construct the vector

$$
\begin{aligned}
\forall x^0 \in {}_jD:\ {}_NU(x^0,\zeta) = &\int_S \Big[{}_1Q_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi^T(x^0-y)\Big]^T F(y)\,d_yS -\\
&- \int_S \Big({}_N\Big[{}_1\widetilde{Q}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c(y,x^0)\Big]^T\Big)^1 F(y)\,d_yS -\\
&- \int_S \Big[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big){}_j\Phi^T(x^0-y)\Big]^T f(y)\,d_yS +\\
&+ \int_S \Big({}_N\Big[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c(y,x^0)\Big]^T\Big)^1 f(y)\,d_yS \qquad (23)\\
&(j=1,2).
\end{aligned}
$$

Denote

$$
\begin{aligned}
\Big(\Big[{}_1\widetilde{Q}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c\Big]^T\Big)^1 &= \Psi_Q(y,x^0) = \|\Psi_{Q^{ik}}\|_{5\times 5},\\
\Big(\Big[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c\Big]^T\Big)^1 &= \Psi_P(y,x^0) = \|\Psi_{P^{ik}}\|_{5\times 5},
\end{aligned}
$$

$$
\Big({}_N\Big[{}_1Q_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c\Big]^T\Big)^1 = {}_N\Psi_Q(y,x^0), \quad \Big({}_N\Big[{}_1\widetilde{P}_{(k_0)}\Big(\frac{\partial}{\partial y}\Big)\widetilde{g}_c\Big]^T\Big)^1 = {}_N\Psi_P.
$$

From (14) and (23) we finally find that

$$
\begin{aligned}
\forall x^0 \in {}_1D \cup {}_2D:\ \ \big|U(x^0,\zeta) - {}_N(x^0,\zeta)\big| \le \big\|\Psi_Q(x^0) -\\
-{}_N\Psi_Q(x^0)\big\|_{L_2(S)} \cdot \|F\|_{L_2(S)} + \big\|\Psi_P(x^0) - {}_N\Psi_P(x^0)\big\|_{L_2(S)} \cdot \|f\|_{L_2(S)}.
\end{aligned}
$$

It should be noted that this method can be extended to some other more complicated problems.

References

1. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Translated from Russian) *North-Holland Publishing Company; Amsterdam-New York-Oxford*, 1979; *Russian original: Nauka, Moscow*, 1976.

2. V. D. Kupradze and T. V. Burchuladze, Dynamical problems of the theory of elasticity and thermoelasticity. (Russian) *Current problems in mathematics, v.* 7, 163–294; *Itogi Nauki i Tekhniki Akad. Nauk SSSR, Moscow*, 1975; *English translation: J. Soviet Math.* 7(1977), No. 3.

3. T. V. Burchuladze and T. G. Gegelia, The development of the method of potential in the elasticity theory. (Russian) *Metsniereba, Tbilisi*, 1985.

4. T. V. Burchuladze, Green formulas in the ganeralized theory of elastothermodiffusion in deformed solid elastic bodies. (Russian) *Trudy Tbilis. Math. Inst. Razmadze* 90(1988), 25–33.

5. T. V. Burchuladze, On Dynamic three–dimensional contact problems of generalized theory of elastothermodiffution. *Bulletin of the Georgian Acad. Sci.* 150(1994), No. 32, 223–228.

6. T. V. Burchuladze, Dynamic problems of the non–classical theory of thermoelasicity. (Russian) *Zeitschrift für Analysis und ihre Anwendungen* (*ZAA*)*, Band* 3(5)(1984), 441–455.

7. T. V. Burchuladze, To the theory of dynamic mathematical problems of generalized elastothermodiffusion. *Proc. of A. Razmadze Math. Inst., Tbilisi* 100(1992), 10–38.

8. T. V. Burchuladze, To the theory of mathematical problems of generalized elastothermodiffusion. *Bull. Acad. Sci. Georgian SSR* 153(1996), No. 22, 201–205.

9. T. V. Burchuladze, On nonstationary three–dimensional contact problems of the generalized theory of elastothermodiffusion. *Facta Universitatis, series mechanics, automatic control and robotics* 2(1966), No. 6, 33–37; *Published by the University of Nis, Serbia, Yugoslavia.*

10. T. V. Burchuladze, Non–stationary problems of generalized elastothermodiffusion for inhomogeneous media. *Georgian Math. J.* 1(1994), No. 6, 587–596.

11. T. V. Burchuladze, Nonstationary initial boundary value contact problems of generalized elastothermodiffusion. *Georgian Math. J.* 4(1997), No. 1, 1–18.

12. T. V. Burchuladze, Three–dimensional boundary value problems of elastothermodiffusion with mixed boundary conditions. *Georgian Math. J.* 4(1997), No. 3, 243–258 (*jointly with Yu. A. Bezhuashvili*).

13. T. V. Burchuladze, Mathematical theory of boundary value problems of thermodiffusion in deformable solid elastic bodies.*Trudy Tbilis. Math. Inst. Razmadze* 65(1980), 5–23.

14. T. V. Burchuladze, Approximate solutions of boundary value problems of thermodiffusion of solid deformable elastic bodies. *Trudy Tbilis. Math. Inst. Razmadze* 71(1982), 19–30.

(Received 25.07.1997)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia