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ADVANCES IN UAS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. This paper develops the method of Lyapunov functions on product spaces for functional differential equations and discusses stability properties in terms of two measures. It presents an up-to-date information regarding uniform asymptotic stability.

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1. Introduction

Let $\mathcal{C} = C[[-\tau, 0], R^N]$ with the norm $|\phi|_0 = \max\limits_{-\tau \le s \le 0} |\phi(s)|$. Consider

$$x'(t) = f(t, x_t), x_{t_0} = \phi_0 \in \mathcal{C}, t_0 \ge 0, \tag{1.1}$$

where $f \in C[R_+ \times \mathcal{C}, R^N]$ and for $x \in C[[t_0 - \tau, \infty), R^n]$, $x_t \in \mathcal{C}$ implies $x_t(s) = x(t+s)$, $-\tau \le s \le 0$. Assume, for convenience, the existence and uniqueness of solutions.

In extending stability theory of Lyapunov to delay equations or Volterra integro-differential equations, there have been two approaches:

(i) using Lyapunov functions; (ii) using Lyapunov functionals.

Krasovskǐ [4] introduced the method of Lyapunov functionals because, in this setup, converse theorems can be proved. A result corresponding to Lyapunov's Second Theorem for ODE is as follows:

Theorem 1.1. *Assume that there exists a Lyapunov functional satisfying*

(i) $b(|\phi|_0) \le V(t, \phi) \le a(|\phi|_0)$,

(ii) $D^+(V(t, \phi)) \le -c(|\phi|_0)$, $\quad a, b, c \in \mathcal{K}$.

Then we have UAS of the trivial solution of (1.1).

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Krasovskǐ realized that to verify the existence of

$$D^+V(t,\phi) \equiv \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\phi)) - V(t,\phi)],$$

(where $x(t,\phi)$ is any solution of (1.1) with the initial function ϕ at time t) would be very difficult in the investigation of actual problems since no simple formula exists for $D^+V(t,\phi)$ comparable to the formula for ODE. Also, one would rarely be able to construct a functional satisfying the conditions of Theorem 1.1. He, therefore, suggested the following change.

Theorem 1.2. *Assume that there exists a Lyapunov functional satisfying*

(i) $b(|\phi(0)| \leq V(t,\phi) \leq a(|\phi|_0)$;
(ii) $D^+V(t,\phi) \leq -c(|\phi(0)|), a, b, c, \in \mathcal{K}$;
(iii) *for any* $\alpha > 0$, $\exists L > 0$ *such that* $|f(t,\phi)| \leq L$ *whenever* $|\phi|_0 \leq \alpha$.

Then UAS follows.

This result corresponds to Marachkov's result in ODE. Krasovskǐ also indicated that to construct functionals, it is convenient to employ the L_2 norm in $\mathcal{C}$ and proved the following result.

Theorem 1.3. *Assume*

(i) $b(|\phi(0)|) \leq V(t,\phi) \leq a_0(\phi(0)|) + a_1(|\phi|_2)$,
(ii) $D^+V(t,\phi) \leq -c(|\phi(0)|), a_0, a_1, b, c \in \mathcal{K}$.

Then UAS results.

Moreover, Krasovskǐ used Lyapunov functions stressing the importance of what is now known as Razumikhin's method.

When we examine the Lyapunov functionals constructed for the examples that have been discussed in the literature, we find that the investigators, inadvertently, employ a combination of a functional and a function in such a way that the corresponding derivative can be estimated suitably without demanding the knowledge of solutions and minimal classes of functions. This observation leads to the development of the method of Lyapunov functions on product spaces [6]. Also studying the stability properties in terms of two measures unifies several known concepts of stability. See [7].

2. Main Results

There are several papers devoted to the problem of weakening the condition (iii) of Theorem 1.2. See, for references, Hatvani [4]. Using an annulus argument, Hatvani [4] has proved very general results which include known results in this direction. A simplified corollary of Hatvani's results is the following theorem which gives the idea.

Theorem 2.1. *Assume that there exists a Lyapunov functional* $V(t,\phi)$ *satisfying* (i) *of Theorem* 1.2. *Suppose further that there are locally integrable functions* $\eta, M : R_+ \to R_+$ *such that* $V'(t,\phi) \leq -\eta(t)c(|\phi(0)|)$, $c \in \mathcal{K}$ *and*

$$V'(t,\phi) \leq -[\phi^T(0)Df(t,\phi)]_+ + M(t)d(|\phi)_0),$$

where $d \in \mathcal{K}$, D is a symmetric positively definite matrix and $[a]_+ = \max[a, 0]$. *If* $\int_s^t \eta(\sigma)d\sigma \geq w(\int_s^t M(\sigma)d\sigma)$, *for all* $s \leq t$, *where* $w \in \mathcal{K}$, *then we have UAS.*

Another direction of proving UAS is the following simple result of Hering [3]. See Bo Zhang [9], Lakshmikantham [5], and Lakshmikantham and Vatsala [8] for extensions.

Theorem 2.2. *Assume that there exists a Lyapunov functional* $V(t, \phi)$ *satisfying*

(i) $b(|\phi(0)|) \leq V(t, \phi) \leq a_0(|\phi(0)|) + a_1(|\phi|_0)$;

(ii) $V'(t, \phi) \leq -\eta(t)c(|\phi(0)|), a_0, a_1, b, c \in \mathcal{K}$ *and* $\eta \geq 0$ *is continuous on* $R_+ \to R_+$ *such that* $\int_t^{t+L} \eta(\sigma)d\sigma \geq M$ *for any* $M > 0 \exists L > 0$;

(iii) $b(r) > a_1(r)$ *for* $r \in (0, r_0)$ *for some* $r_0 > 0$.

Then UAS is valid.

We will next offer another direction of extension which unifies the methods of variation of parameters and Lyapunov functions. This approach was introduced for ODE in [6]. This unification will be called variational Lyapunov method.

Consider two different systems

$$y' = F(t, y), y(t_0) = x_0, \tag{2.1}$$

$$z' = G(t, z_t), z_{t_0} = \phi_0, \tag{2.2}$$

where $F \in C[R_+ \times R^n, R^n]$, $G \in C[R_+ \times \mathcal{C}, R^n]$, $\mathcal{C} = C[[-\tau, 0], R^n]$ with $\phi_0(0) = x_0$. Assume that the solutions of (2.1) and (2.2), $y(t, t_0, x_0)$ and $z(t_0, \phi_0)(t)$, respectively, exist, are unique and continuously depend on initial data for $t \geq t_0$. Also, suppose that $|y(t, t_0, x_0)|$ is locally Lipschitzian in x_0. We define for any $V \in \mathcal{C}[R_+ \times R^n \times \mathcal{C}, R_+]$ and for $t_0 \leq s \leq t$,

$$\begin{aligned} D_-V(t, s, x, \phi) \equiv \liminf_{h \to 0^-} \frac{1}{h}[&V(s + h, y(t, s + h, x + hF(s, x)), \\ &z_t(s + h, x_{s+h})) - V(s, y(t, s, x), z_t(s, \phi))]. \end{aligned}$$

Then we can prove the following general comparison result.

Theorem 2.3. *Assume that* $V \in C[R_+ \times R^n \times \mathcal{C}, R_+]$, $V(t, x, \phi)$ *is locally Lipschitzian in* x *and for* $t_0 \leq s \leq t$,

$$D_-V(t, s, x, \phi) \leq g(t, s, V(s, y(t, s, x), z_t(s, \phi))),$$

where $g \in C[R_+^3, R]$. *Let* $r(t, s, t_0, u_0)$ *be the maximal solution of* $u'(s) = g(t, s, u)$, $u(t_0) = u_0$, $t_0 \leq s \leq t$ *existing for* $s \geq t_0$ *for each* $t \in R_+$. *Then whenever* $u_0 = V(t_0, y(t, t_0, \phi_0(0))$, $z_t(t_0, \phi_0)$ *we have for* $t \geq t_0$,

$$V(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0)) \leq r(t, t, t_0, V(t_0, y(t, t_0, \phi_0(0)), z_t(t_0, \phi_0))).$$

The special cases of (2.1) and (2.2) yield several variations of Theorem 2.3. For example, if $F \equiv G \equiv 0$, we get the known comparison result in terms of the Lyapunov function on product spaces (See [9]). Using this comparison result, one can prove the following extensions of Lyapunov theorems.

Theorem 2.4. *Assume that*

(i) $b(|\phi(0)|) \le V(t, \phi(0), \phi) \le a_0(|\phi(0)|) + a_1(|\phi|_0)$;

(ii) $D_-V(t, s, x, \phi) \le 0$;

(iii) *the trivial solutions of* (2.1) *and* (2.2) *are US.*

Then the trivial solution of (1.1) *is US.*

To prove a UAS result for (1.1) in this set up, we need the concept of strict US of (2.1) which we define below.

Definition. The trivial solution of (2.1) is strictly US, if, given $\epsilon_1 > \epsilon_2 > 0$ and $t_0 \in R_+$, there exist $\delta_1, \delta_2 > 0$ such that $\epsilon_2 < \delta_2 < \delta_1 < \epsilon_1$ with $\delta_2 < |x_0| < \delta_1 \Rightarrow \epsilon_2 < |y(t, t_0, x_0)| < \epsilon_1$ for $t \ge t_0$.

Theorem 2.5. *Assume that* (i) *of Theorem* 2.4 *holds. Suppose further that*

(ii*) $D_-V(t, s, x, \phi) \le -c(|y(t, s, x)|)$, $c \in \mathcal{K}$;

(iii) *the trivial solution of* (2.1) *is strictly US and that of* (2.2) *is US;*

(iv) $a_1(u) < b(u)$ *for* $0 < u < r_0$ *for some* $r_0 > 0$.

Then (1.1) *is UAS.*

One can extend the ideas of Theorem 2.1 in this framework when (2.1) satisfies strict US with suitable modifications. The advantage of variational Lyapunov method is that the good behavior of perturbation terms can be exploited successfully compared to the usual perturbation theory where one can only preserve at best the properties of the unperturbed systems.

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