Mem. Differential Equations Math. Phys. 12(1997), 99–105

G. Kharatishvili and T. Tadumadze

DEVELOPMENT OF THE THEORY OF OPTIMAL SYSTEMS WITH DELAYED ARGUMENTS IN GEORGIA

Abstract. Main results obtained by Georgian mathematicians in the theory of optimal control are presented. These results deal with necessary conditions of optimality, existence, continuous dependence on regular perturbations of the minimum of the functional.

რეზიუმე. გადმოცემულია დაგვიანებულ არგუმენტიანი ოპტიმალური მართვის თეორიაში ქართველ მათემატიკოსთა მიერ მიღებული ძირითადი შედეგები. სახელდობრ, მოყვანილია ოპტიმალობის აუცილებელი პირობები, არსებობის თეორემა, თეორემა ფუნქციონალის მინიმუმის რეგულარულ შეშფოთებებზე უწყვეტად დამოკიდებულების შესახებ.

In 1961, an analogue of Pontryagin's maximum principle [1] was proved for control systems with one constant delay in the phase coordinates [2]. This results became the basis for the development of mathematical theory of optimal systems with delayed arguments in many countries, including Georgia.

Initially, the optimal problem had the form

$$\dot{x}(t) = f(x(t), x(t-\tau), u(t)),\ t \in [t_0, t_1],\quad u(\cdot) \in \Omega,\ \tau > 0,$$
$$x(t) = \varphi_0(t), \quad t \in [t_0-\tau, t_0], \quad x(t_1) = x_1,$$
$$t_1 - t_0 \to \min,$$

with f, f_x, f_y continuous on $R^n \times R^n \times U$, $U \subset R^r$, Ω the set of piecwise continuous functions $u : [t_0, t_1] \to U$ with a finite number of discontinuities of the first kind at which $u(t) = u(t-)$, $\varphi_0 : [t_0-\tau, t_0] \to R^n$ a fixed continuous function and t_0, x_0 fixed points.

Theorem 1 (Maximum principle). *Let $\widetilde{u}(t)$, $t \in [t_0, \widetilde{t}_1]$, be an optimal control and $\widetilde{x}(t)$ be the corresponding optimal trajectory. Then there ex-*

1991 *Mathematics Subject Classification.* 49K25.

Key words and phrases. Optimal systems with delays, necessary conditions, existence theorem, regular perturbations, computational algorithm of the optimal control.

ists a non-zero solution $\psi(t)$, $t \in [t_0, \widetilde{t}_1]$ *of the system*

$$\dot{\psi}(t) = - H_x(\psi(t), \widetilde{x}(t), \widetilde{x}(t-\tau), \widetilde{u}(t)) - \\ - H_y(\psi(t+\tau), \widetilde{x}(t+\tau), \widetilde{x}(t), \widetilde{u}(t+\tau)), \\ t \in [t_0, \widetilde{t}_1], \psi(t) = 0, \quad t > \widetilde{t}_1,$$

such that:

1^0. *For each* $t \in [t_0, \widetilde{t}_1]$, *the maximum condition is fulfilled:*

$$H(\psi(t), \widetilde{x}(t), \widetilde{x}(t-\tau), \widetilde{u}(t)) = M(\psi(t), \widetilde{x}(t), \widetilde{x}(t-\tau));$$

2^0. *At the moment* $\hat{t}_1$, *the inequality*

$$M(\psi(\widetilde{t}_1), \widetilde{x}(\widetilde{t}_1), \widetilde{x}(\widetilde{t}_1 - \tau)) \geq 0,$$

is fulfilled, where

$$H = \langle \psi, f \rangle, M(\psi, x, y) = \sup_{u \in U} H(\psi, x, y, u).$$

Then Theorem 1 was generalized for systems of the form [3]

$$\dot{x}(t) = f(t, x(t-\tau_1), \dots, x(t-\tau_s), u(t)), \quad \tau_s > \dots > \tau_1 = 0.$$

Further for control systems

$$\dot{x}(t) = f(t, x(t-\tau_1), \dots, x(t-\tau_s), u(t-m_1 h), \dots, u(t-m_\nu h)),$$

where $m_1 = 0$, m_i, $i = 2, \dots, \nu$, are natural numbers and $h > 0$, necessary conditions were proved for the optimality of a control and an initial function in the form of an integral (pointwise) maximum principle and conditions of transversality [4,5].

After that, analogous results were received for optimal systems containing: variable delays both in phase coordinates and in the controls [6-8]; distributed delays in the controls [9]; neutral type equations [10]; equations with variable structure [11-13]; functional-differential equations [14]; equations with mixed restrictions [15]; hyperbolic partial differential equations [16,17].

Problems of optimal control for the most part of the enumerated systems were investigated within the framework of the general theory of extremal problems which was elabarated in [18-21].

Now let's formulate necessary conditions for the optimal problem with single delays both in the phase coordinates and in the controls:

$$\dot{x}(t) = f(t, x(t), x(\tau(t)), u(t), u(\theta(t))), \tag{1}$$

$$t \in [t_0, t_1] \subset J = [a, b], u(\cdot) \in \Omega_1,$$

$$x(t) = \varphi(t), t \in [\tau(t_0), t_0), \quad x(t_0) = x_0, \quad \varphi(\cdot) \in \Phi, \tag{2}$$

$$q^i(t_0, t_1, x_0, x(t_1)) = 0, \quad i = 1, \dots, l, \tag{3}$$

$$q^0(t_0, t_1, x_0, x(t_1)) \to \min,$$

where the function $f : J \times O^2 \times G^2 \to R^n$ is continuous and continuously differentiable with respect to $(x, y) \in O^2$ $O \subset R^n$ and $G \subset R^r$ are open sets; $\tau : R^1 \to R^1$ $\theta : R^1 \to R^1$ are absolutly continuous functions satisfying $\tau(t) < t$, $\dot{\tau}(t) > 0$, $\theta(t) < t$, $\dot{\theta}(t) > 0$; $\Omega_1 = \Omega$ $([\theta(a), b], U)$ is the set of all measurable functions $u : [\theta(a), b] \to U$ such that the conditions $cl(u([\theta(a), b]))$ is a compact lying in G; $\Phi = \Phi([\tau(a), b], N)$ is the set of all piecewise continuous functions $\varphi : [\tau(a), b] \to N$ with a finite number of points of discontinuity; $N \subset O$ is a convex bounded set; $q^i : J^2 \times O^2 \to R^1$, $i = 0, \ldots, l$, are functions continuously differentiable with respect to all arguments.

Definition 1. An element $z = (t_0, t_1, x_0, \varphi(\cdot), x(\cdot), u(\cdot))$ is said to be admissible, if $(t_0, t_1, x_0, \varphi(\cdot), u(\cdot)) \in J^2 \times O \times \Phi \times \Omega_1$, $x(t) \in O$, $t \in [t_0, t_1]$, is absolutely continuous on $[t_0, t_1]$ and satisfies (2), (3), and the pair $(x(\cdot), u(\cdot))$ satisfies (1) almost everywhere on $[t_0, t_1]$.

The set of admissible elements will be denoted by Δ.

Definition 2. An element $\widetilde{z} \in \Delta$ is called optimal if for an arbitrary element $z \in \Delta$, is fulfilled the inequality

$$q^0(\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{x}(t_1)) \le q^0(t_0, t_1, x_0, x(t_1)).$$

The problem of optimal control consists in finding an optimal element.

Theorem 2. *Let* $\widetilde{z} = (\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{\varphi}(\cdot), \widetilde{x}(\cdot), \widetilde{u}(\cdot))$ *be an optimal element,* $\tau(\widetilde{t}_1) > \widetilde{t}_0, \widetilde{t}_i \in (a, b)$, $i = 0, 1$; *let the function* $(\widetilde{u}(t), \widetilde{u}(\theta(t)))$ *be continuous at the points* $\widetilde{t}_0$, $\gamma_0 = \gamma(\widetilde{t}_0)$ *and the function* $(\dot{\gamma}(t), \widetilde{\varphi}(\tau(t)), \widetilde{\varphi}(t))$ *be continuous at the point* $\widetilde{t}_0$. *Then there exist a non-zero vector* $\pi = (\pi_0, \ldots, \pi_l)$, $\pi_0 \le 0$, *and a solution* $\psi(t)$ *of the equation*

$$\dot{\psi}(t) = -\psi(t)\widetilde{f}_x[t] - \psi(\gamma(t))\widetilde{f}_y[\gamma(t)]\dot{\gamma}(t), \quad t \in [\widetilde{t}_0, t_1], \quad \psi(t) = 0, \quad t > \widetilde{t}_1,$$

such that the following conditions are fulfilled:

3^0. *The integral maximum principle*

$$\int\limits_{\tau(\widetilde{t}_0)}^{\widetilde{t}_0} \psi(\gamma(t))\widetilde{f}_y[\gamma(t)]\dot{\gamma}(t)\widetilde{\varphi}(t)dt \ge$$

$$\ge \int\limits_{\tau(\widetilde{t}_0)}^{\widetilde{t}_0} \psi(\gamma(t))\widetilde{f}_y[\gamma(t)]\dot{\gamma}(t)\varphi(t)dt \quad \forall \varphi(\cdot) \in \Phi,$$

$$\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1} \psi(t)\widetilde{f}[t]dt \ge \int\limits_{\widetilde{t}_0}^{\widetilde{t}_1} \psi(t)f(t, \widetilde{x}(t), \widetilde{x}(\tau(t)), u(t), u(\theta(t)))dt, \quad \forall u(\cdot) \in \Omega_1;$$

4^0. *The transversality conditions*

$$\pi\widetilde{Q}_{t_0} - \psi(\widetilde{t}_0)\widetilde{f}[\widetilde{t}_0] + \psi(\gamma_0)[f(\gamma_0, \widetilde{x}(\gamma_0), \widetilde{\varphi}(\widetilde{t}_0), \widetilde{u}(\gamma_0), \widetilde{u}(\theta(\gamma_0))) -$$
$$-f(\gamma_0, \widetilde{x}(\gamma_0), \widetilde{x}_0, \widetilde{u}(\gamma_0), \widetilde{u}(\theta(\gamma_0)))]\dot{\gamma}(\widetilde{t}_0),$$
$$\pi\widetilde{Q}_{t_1} = -\psi(\widetilde{t}_1)\widetilde{f}[\widetilde{t}_1], \quad \pi\widetilde{Q}_{x_0} = -\psi(\widetilde{t}_0), \quad \pi\widetilde{Q}_{x_1} = \psi(\widetilde{t}_1).$$

Here

$$\widetilde{f}[t] = f(t, \widetilde{x}(t), \widetilde{x}(\tau(t)), \widetilde{u}(t), \widetilde{u}(\theta(t))),$$
$$\widetilde{f}_x[t] = f_x(t, \widetilde{x}(t), \widetilde{x}(\tau(t)), \widetilde{u}(t), \widetilde{u}(\theta(t)));$$

the tilde over Q denotes that the corresponding gradient is calculated at the point $(\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}_0, \widetilde{x}(\widetilde{t}_1))$; $\gamma(t)$ is the function inverse to $\tau(t)$. If the rank of the matrix

$$(\widetilde{Q}_{t_0}, \widetilde{Q}_{t_1}, \widetilde{Q}_{x_0}, \widetilde{Q}_{x_1})$$

is equal to $1 + l$, then $\Psi(t) \not\equiv 0$.

The researches [22-29] are dedicated to theorems of existence and regular perturbations in optimal systems with delays.

Now let's formulate for a simplified case basic theorems obtained in this direction.

Consider the optimal problem

$$\dot{x}(t) = f(t, x(t), x(\tau(t)), u(t), u(\theta(t))), \tag{4}$$
$$t \in J_0 = [t_0, t_1], \quad u(\cdot) \in \Omega([\theta(t_0), t_1], U),$$
$$x(t) = \varphi_0(t), \quad t \in [\tau(t_0), t_0), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{5}$$
$$I(z) = \int_{t_0}^{t_1} f^0(t, x(t), x(\tau(t)), u(t), u(\theta(t)))dt \to \min, \quad z \in \Delta_1, \tag{6}$$

where U is a compact set; $\varphi_0(\cdot) \in \Phi([\tau(t_0), t_0], O)$; $F = (f^0, f) : J_0 \times O^2 \times U^2 \to R^{1+n}$ is a Carathéodory function satisfying the following conditions: for each compact set $K \subset O$, there exist $m_K(\cdot)$ and $L_K(\cdot) \in L_1(J_0, R^1_+)$, $R^1_+ = [0, \infty)$ such that

$$|F(t, x, y, u, v)| \leq m_K(t) \ \ \forall(t, x, y, u, v) \in J \times K^2 \times U^2,$$
$$|F(t, x', y', u, v) - F(t, x'', y'', u, v)| \leq L_K(t)(|x' - x''| + |y' - y''|),$$
$$\forall(t, x', x'', y', y'', u, v) \in J \times K^4 \times U^2;$$

Δ_1 is the set of admissible elements $z = (x(\cdot), u(\cdot))$.

Now we introduce the set P. To this aim, divide the interval $[\theta_0, t_1]$ into subintervals $[s_i, s_{i+1}]$, $i = -1, 0, \dots, m$, $s_0 = t_0$, $s_{m+1} = t_1$, $s_i = \theta(s_{i+1})$,

$i = -1, 0, \dots, m-1$, $\theta(t_1) \le s_m < t_1$, and put

$$P(t, x_0, y_0, \dots, x_m, y_m) = \{q = (q_0, \dots, q_m);$$
$$q_i = \dot{\varrho}^i(t) F(\dot{\varrho}^i(t), x_i, y_i, u_i, u_{i-1}), \quad i = 0, \dots, m,$$
$$u_i \in U, \quad i = -1, \dots m\}, \quad t \in [s_0, s_1],$$

where $\varrho(t)$ is the function inverse to $\theta(t)$ and $\varrho^i(t) = \varrho(\varrho^{i-1}(t))$. We assume that $\varrho^0(t) = t$ and $\varrho(t) = t_1$ if $t \ge \theta(t_1)$

Theorem 3. *For the problem* (4)–(6), *an optimal element* $\widetilde{z} = (\widetilde{x}(\cdot), \widetilde{u}(\cdot))$ *exists if the following conditions are satisfied:*

5^0.$\Delta_1 \ne \oslash$;

6^0. *There exists a compact* $K_0 \subset O$ *such that* $x(t) \in K_0$, $t \in J_0$, $\forall z = (x(\cdot), u(\cdot)) \in \Delta_1$;

7^0. *The set* $P(t, x_0, y_0, \dots, x_m, y_m)$ *is convex for each fixed* $t \in [s_0, s_1]$, $(x_i, y_i) \in O^2$, $i = 0, \dots, m$.

Theorem 4. *Let the assumptions of Theorem* 3 *hold. Then for every* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon) > 0$ *such that for every* $(\delta x_0, \delta\varphi(\cdot), \delta F(\cdot))$ *satisfying*

$$|\delta x_0| + \sup_{t \in J_0} |\delta\varphi(t)| + \max_{t', t'' \in J_0\ x, y, \in K_1} \left| \int_{t'}^{t''} \delta F(t, x, y) dt \right| \le \epsilon,$$
$$\delta F = (\delta f_0, \delta f),$$

the perturbed problem

$$\dot{x}(t) = f(t, x(t), x(\tau(t)), u(t), u(\theta(t))) + \delta f(t, x(t), x(\tau(t))),$$
$$x(t) = \varphi_0(t) + \delta\varphi(t), \quad t \in [\tau(t_0), t_0),$$
$$x(t_0) = x_0 + \delta x_0, \quad |x_1 - x(t_1)| \le \epsilon,$$

$$I(z, \epsilon) = \int_{t_0}^{t_1} [f^0(t, x(t), x(\tau(t)), u(t), u(\theta(t))) + \delta f^0(t, x(t), x(\tau(t)))] dt \to \min,$$

has a solution $\widetilde{z}_\epsilon$ *and*

$$\lim_{\epsilon \to 0} I(\widetilde{z}_\epsilon, \epsilon) = I(\widetilde{z}).$$

Here $\varphi_0(\cdot) + \delta\varphi(\cdot) \in \Phi([\tau(t_0), t_0], O)$, $x_0 + \delta x_0 \in O$, the Caretheódory functions $\delta F : J_0 \times O^2 \to R^{1+n}$ satisfy the conditions

$$|\delta F(t, x, y)| \le m_{\delta F}(t) \ \ \forall (t, x, y) \in J_0 \times K_1^2,$$
$$|\delta F(t, x', y') - \delta F(t, x'', y'')| \le L_{\delta F}(t)(|x' - x''| + |y' - y''|)$$
$$\forall (t, x', y', x'', y'') \in J_0 \times K_1^4,$$
$$\int_{J_0} (m_{\delta F}(t) + L_{\delta F}(t)) dt \le const,$$

$K_1 \subset O$ is a compact set containing a neighborhood of K_0, $m_{\delta F}(\cdot), L_{\delta F}(\cdot) \in L_1(J_0, R^1_+)$.

In conclusion note that computational algoritms of the optimal control elabarated in [30,31] are extended for the linear system [6]

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)u(t) + \\ + D(t)u(t-\theta) + f(t), \quad \theta > 0.$$

References

1. L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze, and E. F. Mishchenko, The mathematical theory of optimal processes. (Russian) *Nauka, Moscow,* 1961.

2. G. L. Kharatishvili, The maximum principle in the theory of optimal processes with delay. (Russian) *Dokl. Akad. Nauk SSSR* 136(1961), No. 1, 39–42.

3. G. L. Kharatishvili, Optimal processes with delays. (Russian) *Metsniereba, Tbilisi,* 1966.

4. G. L. Kharatishvili, A maximum principle in extremal problems with delays. *In: Mathematical theory of control, Academic press,* 1967, 26–34.

5. G. L. Kharatishvili, A maximum principle in extremal problems with delays. (Russian) *Trudy Tbiliss. Gos. Univ.* 128(1968), 149–156.

6. G. L. Kharatishvili, Z. A. Machaidze, N. I. Markozashvili, and T. A. Tadumadze, Abstract variational theory and its applications to optimal problems with delays. (Russian) *Metsniereba, Tbilisi,* 1973.

7. G. L. Kharatishvili and T. A. Tadumadze, Nonlinear optimal control systems with variable delays. (Russian) *Mat. Sb.* 107(149)(1978), No. 4(12), 613–633.

8. T. A. Tadumadze, The maximum principle and existence theorem in the optimal problems with delay and non-fixed initial function. *Seminar of I. Vekua Institute of Applied Mathematics. Reports.* 22(1993), 102–107.

9. G. L. Kharatishvili and T. A. Tadumadze, The maximum principle in optimal control problems with concentrated and distributed delays in controls. *Georgian Math. J.* 2(1995), No. 6, 577–591.

10. T. A. Tadumadze, The maximum principle for some controlled systems of neutral type. (Russian) *Seminar of I. Vekua Institute of Applied Mathematics, Abstracts of Reports,* 9(1974), 9–13.

11. A. I. Arsenashvili, An optimal control problem for variable structure systems with delay in phase coordinates. (Russian) *In: Theory and equipment of automatic control systems* XXI:1,(Russian) *Metsniereba, Tbilisi,* 1982, 11–18.

12. N. M. Avalishvili, Maximum principle for an optimal problem with variable structure and delays. (Russian) *In: Optimal problems in systems with variable structure, Tbilisi University Press, Tbilisi,* 1985, 48–71.

13. G. L. Kharatishvili and T. A. Tadumadze, Problem of optimal control for nonlinear systems with variable structure, delays and pecewise-continuous prehistory. *Mem. Differential. Equations Math. Phys.* 11(1997), 67–88.

14. N. I. Markozashvili, Necessary conditions of optimality for controlled systems with prehistory. (Russian) *In: Some questions of mathematical theory of optimal control, (Russian) Tbilisi University Press, Tbilisi,* 1975, 151–180.

15. Z. A. Tsintsadze, Optimal problem with delays and mixed restrictions. (Russian) *Bull. Acad. Sci. Georgian SSR* 92(1978), No. 3, 569–572.

16. T. G. Tsutsunava, Optimal processes with delay for hyperbolic equations. (Russian) *Trudy Probl. Labor. Avtom. i Vichisl. Tekhniki Gruzinskogo Politekhnicheskogo Instituta,* 1967.

17. M. T. Tsutsunava, On necessary condition of optimality for hyperbolic partial differential equations with delays. (Russian) *In: Theory and equipment of automatic control systems XIX:1, (Russian) Metsniereba, Tbilisi,* 1980, 33–38.

18. R. V. Gamkrelidze and G. L. Kharatishvili, Extremal problems in linear topological spaces, 1. *Math. Systems Theory* 1(1967), No. 3, 229–256.

19. R. V. Gamkrelidze and G. L. Kharatishvili, Extremal problems in linear topological spaces. (Russian) *Izv. Akad. Nauk. SSSR, Ser. Mat.* 33(1969), No. 4, 781–839.

20. R. V. Gamkrelidze, Necessary conditions of first kind and axomatics of extremal problems. (Russian). *Trudy Mat. Inst. Steklov* 62(1971), 781–839.

21. K. Sh. Tsiskaridze, Extremal problems in Banach spaces. (Russian) *In: Some questions of the mathematical theory of optimal control, (Russian) Tbilisi State University Press, Tbilisi,* 1975, 1–150.

22. T. A. Tadumadze, On the existence of solution in optimal problems with a deviating argument. (Russian) *Bull. Acad. Sci. Georgian SSR* 89(1978), No. 2, 313–316.

23. T. A. Tadumadze, On the existence of solution in optimal problems of neutral type. (Russian) *Bull. Acad. Sci. Georgian SSR* 97(1980), No. 1, 33–36.

24. T. A. Tadumadze, Existence theorems for solution of optimal problems with variable delays. *Control and Cybernetics* 10(1981), Nos. 3–4, 125–134.

25. T. A. Tadumadze, On the existence of solution in optimal problems described by nonlinear functional differential equations. (Russian) *Differentsial'nye Uravneniya* 20(1984), No. 4, 597–604.

26. T. A. Tadumadze, Some problems of the qualitative theory of optimal control. (Russian) *Tbilisi University Press, Tbilisi,* 1983.

27. G. L. Kharatishvili and T. A. Tadumadze, Regular perturbations in optimal control problems with variable delays and a free right end.(Russian) *Dokl. Akad. Nauk SSSR* 314(1990), No. 1, 151–156.

28. A. V. Tavadze, Some problems of qualitative theory of optimal control of neutral type. (Russian) *Trudy Inst. Prikl. Mat. I. Vekua* 27(1988), 107–136.

29. F. Dvalishvili, Some problems of qualititave theory of optimal control with distributed delay. (Russian) *Proc. I. Vekua Inst. Appl. Math.* 41(1991), 81–113.

30. T. G. Babunashvili, Synthesis of linear optimal systems. (Russian) *Dokl. Akad. Nauk SSSR* 155(1964), No. 2, 295–298.

31. T. G. Babunashvili, Z. A. Machaidze, G. L. Kharatishvili, and K. Sh. Tsiskaridze, Calcucation of linear time-optimal systems. (Russian) *Trudy Inst. Prikl. Mat.* 3(1972), 208–238.

(Received 25.06.1997)

Authors' addresses:

Guram Kharatishvili
Institute of Cybernetics
Georgian Academy of Sciences
Georgia

Tamaz Tadumadze
I. Vekua Institute of Applied Mathematics
Tbilisi State University
Georgia