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THE LAGRANGE PRINCIPLE OF TAKING RESTRICTIONS OFF AND ITS APPLICATION TO LINEAR OPTIMAL CONTROL PROBLEMS IN THE PRESENCE OF MIXED RESTRICTIONS AND DELAYS

Abstract. The Lagrange principle of taking restrictions off is proved for the problems of conditional minimization in infinite-dimensional spaces when the function to be minimized and the mapping specifying the restrictions of the problem satisfy certain convexity and continuity conditions.

On the basis of the obtained result, necessary conditions of optimality are derived in terms of an analogue of Pontryagin's maximum principle for some classes of linear problems of optimal control when mixed restrictions and delays take place.

რეზიუმე. დამტკიცებულია შეზღუდვების მოხსნის ლაგრანჟისეული პრინციპი პირობითი მინიმიზაციის ამოცანებისათვის უსასრულო განზომილებიან სივრცეებში იმ შემთხვევაში, როცა მინიმიზებადი ფუნქცია და ასახვა, რომელიც შეზღუდვებს განსაზღვრავს, აკმაყოფილებენ უწყვეტობის და ამოზნექილობის გარკვეულ პირობებს. მიღებული შედეგის საფუძველზე ოპტიმალური მართვის შერეულშეზღუდვებიანი და დაგვიანებულარგუმენტიანი ამოცანების ზოგიერთი კლასებისათვის მიღებულია ოპტიმალურობის აუცილებელი პირობები პონტრიაგინის მაქსიმუმის პრინციპის ანალოგის ფორმით.

Introduction. Lagrange's idea of studying the problem of conditional minimization by means of the corresponding auxiliary problem of absolute minimization is realized for wide classes of extremal problems in terms of the rule of Lagrange's multipliers. According to this rule, the necessary conditions of minimum in the initial problem coincide with those of free minimum of the Lagrange's function in the auxiliary problem. There however exist extremal problems in which Lagrange's idea is realized in a more strengthened form. Namely, it can be claimed that the solution of the initial problem of conditional minimization is a solution of the corresponding problem of absolute minimization. The proof of this assertion (i.e., the Lagrange's principle of taking restrictions off) for a wide class of extremal problems serves, in particular, as a basis for obtaining necessary conditions of optimality in optimal control problems.

Theorem 1 (Lagrange's principle of taking the restrictions off) proven below enables us to obtain necessary conditions both for solutions of problems of particular importance referred in [1] and [2] and for solutions of extremal problems distinguished by their specific character, for example, for solutions of the linear optimal problem when mixed restrictions and delays take place. Note that in the latter case we succeeded in obtaining necessary conditions of optimality in a form which is completely analogous to that of Pontryagin's maximum principle cited in the fundamental work [3] for the problem free from mixed restrictions.

In what follows, the use will be made of the notation accepted in [4].

1. Statement of the Problem. Proof of the Main Theorem. Consider an extremal problem in the form

$$f_0(w) \to \inf \tag{1}$$

$$F(w) = \mathbf{0}, \tag{2}$$

$$f_i(w) \le 0, \quad i = \overline{1, s}, \tag{3}$$

$$w \in W. \tag{4}$$

Here $f_i : W \to R$, $i = \overline{0, s}$, $F : W \to Y$ are given mappings, W is a topological space, R is the set of all real numbers, Y is a locally convex Hausdorff (separable) linear topological space, $\mathbf{0}$ is the zero element in Y.

The Lagrange function for the problem (1)–(4) has the form

$$\mathcal{L}(w, \lambda, y^*) = \sum_{i=0}^{s} \lambda_i f_i(w) + \langle y^*, F(w) \rangle,$$

where $\lambda = \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_s \end{pmatrix}$, $\lambda_i \in R$, $i = \overline{0, s}$, $y^* \in Y^*$, Y^* is the space conjugate to Y.

Theorem 1. (*The Lagrange principle of taking restrictions off*). *Let for the problem* (1)–(4) *the following assumptions be fulfilled:*

a) *for* $\forall \ w_1 \in W, \ w_2 \in W \ and \ \alpha \in [0,1] \ \exists w \in W \ such \ that*

$$F(w) = \alpha F(w_1) + (1-\alpha)F(w_2),$$

$$f_i(w) \leq \alpha f_i(w_1) + (1-\alpha)f_i(w_2), \quad i = \overline{0,s};$$

b) *the functions* f_i, $i = \overline{0,s}$ *are continuous on* W *and for any neighborhood* $U(w)$ *of the point* $w \in \ker F$ *the set* $F(U(w))$ *contains a neighborhood of zero of the space* Y.

Then *for any solution* $\widehat{w}$ *of the problem* (1)–(4), *there exist numbers* $\lambda_i \geq 0$, $i = \overline{0,s}$ *and an element* y^* *of the conjugate space* Y^* *such that the conditions*

$$(\lambda_0, \ldots, \lambda_s, y^*) \neq (0, \ldots, 0, \mathbf{0}), \tag{5}$$

$$\lambda_i f_i(\widehat{w}) = 0, \quad i = \overline{1,s} \tag{6}$$

$$\mathcal{L}(\widehat{w}, \lambda, y^*) = \min_{w \in W} \mathcal{L}(w, \lambda, y^*) \tag{7}$$

are fulfilled.

Proof. In the space $R^{s+1} \times Y$, let us consider the set

$$C := \big\{ (\mu_0, \ldots, \mu_s, y) \in R^{s+1} \times Y \, | \, \exists w \in W, \quad \text{such that}$$

$$\mu_0 > f_0(w) - f_0(\widehat{w}), \mu_i \geq f_i(w), i = \overline{1,s}, y = F(w) \big\}.$$

The set C possesses the following properties:

Property 1. $\mathrm{int}\, C \neq \varnothing$.

Indeed, consider an element $(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0}) \in R^{s+1} \times Y$, where $\widehat{\mu}_i$, $i = \overline{0,s}$, are fixed positive numbers. Obviously $(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0}) \in C$ since for the element $w = \widehat{w}$ the conditions $\widehat{\mu}_0 > f_0(\widehat{w}) - f_0(\widehat{w})$, $\widehat{\mu}_i > f_i(\widehat{w})$, $i = \overline{1,s}$, $\mathbf{0} = F(\widehat{w})$ are valid. Let us now show that $(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0})$ is an inner point of the set C. Because the functions f_i, $i = \overline{0,s}$, are continuous, there exists a neighborhood $W_0 \subset W$ of the element $\widehat{w}$ such that

$$\widehat{\mu}_0 > f_0(w) - f_0(\widehat{w}) + \frac{\varepsilon}{2}, \quad \widehat{\mu}_i > f_i(w) + \frac{\varepsilon}{2}, \quad i = \overline{1,s}, \quad \forall w \in W_0,$$

where ε is defined from the condition $\varepsilon = \min\limits_{i=\overline{0,s}}\{\widehat{\mu}_i\}$. But then for every point $(\mu_0, \ldots, \mu_s, y)$ satisfying $|\mu_i - \widehat{\mu}_i| < \frac{\varepsilon}{2}$, $i = \overline{0,s}$, and $y \in F(W_0)$, $\exists w \in W_0$ such that

$$\mu_0 > f_0(w) - f_0(\widehat{w}), \quad \mu_i > f_i(w), \quad i = \overline{1,s}, \quad y = F(w). \tag{8}$$

Taking now into account the assumption b) concerning the mapping F, we can see that $F(w_0)$ contains a neighborhood of zero the space Y. Denote it by $F_0(w_0)$ and consider the neighborhood $V(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0}) = \big\{ \mu_i \, \big| \, |\mu_i - \widehat{\mu}_i| < \frac{\varepsilon}{2}, i = \overline{0,s} \big\} \times F_0(w_0)$ of the point $(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0}) \in C$. For $\forall (\mu_0, \ldots, \mu_s, y) \in V(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0}) \ \exists w \in W_0 \subset W$ such that the conditions

(8) are fulfilled. Thus the set C together with the point $(\widehat{\mu}_0, \ldots, \widehat{\mu}_3, \mathbf{0})$ contains also its neighborhood $V(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0})$, i.e., $(\widehat{\mu}_0, \ldots, \widehat{\mu}_s, \mathbf{0})$ is an inner point of the set C.

Property 2. The set C does not contain the zero element. Really, if $(0, \ldots, 0, \mathbf{0}) \in C$, then by definition of the set C, $\exists w \in W$ such that

$$0 > f_0(w) - f_0(\widehat{w}), \quad 0 \geq f_i(w), \quad i = \overline{1, s}, \quad \mathbf{0} = F(w). \tag{9}$$

But the condition (9) contradicts the fact that $\widehat{w}$ is a solution of the problem (1)–(4).

Property 3. The set C is convex.

Let $(\mu_0^k, \ldots, \mu_s^k, y^k) \in C$, $k = \overline{1, 2}$, be two arbitrary points from C. Then there exist $w^k \in W$, $k = \overline{1, 2}$, for which

$$\mu_0^k > f_0(w^k) - f_0(\widehat{w}), \quad \mu_i^k \geq f_i(w^k), \quad i = \overline{1, s}, \quad y^k = F(w^k).$$

But then for $\forall \alpha \in [0, 1]$ the following relations are valid:

$$\alpha \mu_0^1 + (1 - \alpha) \mu_0^2 > \alpha f_0(w^1) + (1 - \alpha) f_0(w^2) - f_0(\widehat{w}),$$
$$\alpha \mu_i^1 + (1 - \alpha) \mu_i^2 > \alpha f_i(w^1) + (1 - \alpha) f_i(w^2), \quad i = \overline{1, s},$$

and

$$\alpha y^1 + (1 - \alpha) y^2 = \alpha F(w^1) + (1 - \alpha) F(w^2).$$

Taking into account the assumption a), from the above relations we obtain that for $\forall \alpha \in [0, 1]$, $\exists w \in W$ such that

$$\alpha \mu_0^1 + (1 - \alpha) \mu_0^2 > f_0(w) - f_0(\widehat{w}),$$
$$\alpha \mu_i^1 + (1 - \alpha) \mu_i^2 > f_i(w), \quad i = \overline{1, s},$$
$$\alpha y^1 + (1 - \alpha) y^2 = F(w).$$

Hence, for any $(\mu_0^k, \ldots, \mu_s^k, y^k) \in C$, $k = \overline{1, 2}$, the point $(\alpha \mu_0^1 + (1 - \alpha) \mu_0^2, \ldots, \alpha \mu_s^1 + (1 - \alpha) \mu_s^2, \alpha y^1 + (1 - \alpha) y^2)$ also belongs to C, which implies that the set C is convex.

Taking into consideration the above stated properties of the set C, we conclude that the sets $A := \{\underbrace{0, \ldots, 0}_{s+1}, \mathbf{0}\}$ and C from $R^{s+1} \times Y$ are convex, do not intersect and, moreover, $\operatorname{int} C \neq \varnothing$. Then by the theorem on separability (see [4], p. 25), there exists on $R^{s+1} \times Y$ a non-zero continuous linear functional separating the sets A and C, i.e., there exists a non-trivial vector $(\lambda_0, \ldots, \lambda_s, y^*)$, where $\lambda_i \in R$, $i = \overline{0, s}$, and $y^* \in Y^*$ such that for $\forall (\mu_0, \ldots, \mu_s, y) \in C$ the condition

$$\sum_{i=0}^{s} \lambda_i \mu_i + \langle y^*, y \rangle \geq 0 \tag{10}$$

is fulfilled.

The inequality (10) implies that the conditions $\lambda_i \geq 0$, $i = \overline{0, s}$, are valid. Indeed, as far as $(\mu_0, \ldots, \mu_s, \mathbf{0}) \in C$ for $\mu_i = \varepsilon$ $(i = \overline{0, s}, i \neq i_0)$ and $\mu_{i_0} = 1$, where $\varepsilon > 0$ and i_0 $\forall$ is a fixed index, substitution of this vector into (10) yields $\lambda_{i_0} \geq -\varepsilon \sum_{i \neq i_0} \lambda_i$, whence, it follows that $\varepsilon > 0$ since $\lambda_{i_0} \geq 0$ is arbitrary.

To prove the condition (6), we note that if for some $i_0 \neq 0$ $f_{i_0}(\widehat{w}) = -\alpha < 0$, then for $\forall \varepsilon > 0$ the vector $(\mu_0, \ldots, \mu_s, \mathbf{0}) \in C$ if and only if $\mu_i = \varepsilon$ $(i = \overline{0, s}, i \neq i_0)$ and $\mu_{i_0} = -\alpha$. Substituting this vector into (10) and letting ε approach zero, we obtain that $-\lambda_{i_0} \alpha \geq 0$, whence $\lambda_{i_0} \leq 0$. Therefore $\lambda_{i_0} = 0$. Thus $\lambda_i = 0$ if $f_i(\widehat{w}) < 0$, $i = \overline{1, s}$, and hence $\lambda_i f_i(\widehat{w}) = 0$ for all $i = 1, \ldots, s$.

To complete the proof of the theorem it remains, for us to show that the conditions (7) are valid. Since the vector $(\mu_0, \ldots, \mu_s, y) = (f_0(w) - f_0(\widehat{w}) + \varepsilon, f_1(w), \ldots, f_s(w), F(w))$ belongs to the set C for all $\varepsilon > 0$, $w \in W$, owing to (10) we have $\sum_{i=0}^{s} \lambda_i f_i(w) + \langle y^*, F(w) \rangle \geq \lambda_0 f_0(\widehat{w}) - \lambda_0 \varepsilon$ whence, due to the fact that $\varepsilon > 0$ is arbitrary, we have

$$\sum_{i=0}^{s} \lambda_i f_i(w) + \langle y^*, F(w) \rangle \geq \lambda_0 f_0(\widehat{w}). \tag{11}$$

Taking now into account the above proven condition (6), we obtain from (11) that

$$\sum_{i=0}^{s} \lambda_i f_i(w) + \langle y^*, F(w) \rangle \geq \sum_{i=0}^{s} \lambda_i f_i(\widehat{w}) + \langle y^*, F(\widehat{w}) \rangle,$$

for $\forall w \in W$ which signifies the fulfillment of the conditions (7). $\blacksquare$

The above proven theorem is used as a basis for obtaining necessary conditions for the solution of special type smooth-convex problems of minimization (see [5]). Just to such type of smooth-convex problems belong the optimal problems which will be considered in the subsequent sections.

2.Necessary Conditions of Optimality for the Linear Problem of Optimal Control with Delays in Phase Coordinates in the Presence of Mixed Restrictions. Consider now the optimal problem

$$\int_{t_0}^{t_1} f^0(x(t), x(t - \tau), u(t)) dt \to \min \tag{12}$$

under the restrictions

$$\dot{x}(t) = f(x(t), x(t - \tau), u(t)), \tag{13}$$

$$g(x(t), x(t - \tau), u(t)) \leq 0, \tag{14}$$

$$x(t_1) = x_1, \tag{15}$$

$$x(t) = \widehat{\varphi}(t), \quad t_0 - \tau \leq t \leq t_0, \tag{16}$$

where τ is a fixed positive number, the scalar function f^0 and the vector functions $f = (f^1, \ldots, f^n)$, $g = (g^1, \ldots, g^m)$ are linear with respect to all their arguments, $x(t) \in W^n_{1,1}[t_0 - \tau, t_1]$, $u(t) \in L^r_1[t_0, t_1]$, and x_1 is a fixed point from R^n. Note that the conditions (13) and (14) are fulfilled almost everywhere on $[t_0, t_1]$ and the restrictions (14) satisfy the conditions of generality: for any (x, z, u) satisfying (14), the system of vectors $\text{grad}_u g^j(x, z, u)$, $j \in J(x, z, u)$, is linearly independent. Here by $J(x, z, u)$ we denote the set of such indices $j \in \{1, 2, \ldots, m\}$ for which $g^j(x, z, u) = 0$.

In the problem under consideration the initial function $\widehat{\varphi}(t) \in W^n_{1,1}[t_0 - \tau, t_0]$ is a priori fixed and the moments t_0 and t_1 are assumed to be known.

Theorem 2. *Let $(\widehat{x}(t), \widehat{u}(t))$ be a solution of the problem (12)–(16). Then there exist non-trivial multipliers $\psi_0 = \text{const} \geq 0$, $\psi(t) \in W^n_{1,1}[t_0, t_1]$ and $\mu(t) \in L^m_\infty[t_0, t_1]$ such that the following conditions are fulfilled: almost everywhere on $[t_0, t_1]$*

$$\mu_j(t) \geq 0, \quad j = \overline{1, m}, \tag{17}$$

$$\mu_j(t) g^j(\widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t)) = 0, \quad j = \overline{1, m}; \tag{18}$$

almost everywhere on $t_0 \leq t \leq t_1 - \tau$

$$\frac{d\psi}{dt} = \frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t))}{\partial x} +$$
$$+ \frac{\partial \mathcal{H}(\psi_0, \psi(t + \tau), \mu(t + \tau), \widehat{x}(t + \tau), \widehat{x}(t), \widehat{u}(t + \tau))}{\partial z}; \tag{19}$$

almost everywhere on $t_1 - \tau \leq t \leq t_1$

$$\frac{d\psi}{dt} = \frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t))}{\partial x}; \tag{20}$$

for almost all $t \in [t_0, t_1]$

$$H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t)) = \min H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau), u), \tag{21}$$
$$u \in \{u | g(\widehat{x}(t), \widehat{x}(t - \tau), u) \leq 0\},$$

where $\mathcal{H}(\psi_0, \psi(t), \mu(t), x(t), x(t - \tau), u(t)) = H(\psi_0, \psi(t), x(t), x(t - \tau), u(t)) + \sum_{j=1}^m \mu_j(t) g^j(x(t), x(t - \tau), u(t))$, $H(\psi_0, \psi(t), x(t), x(t - \tau), u(t)) = \psi_0 f^0(x(t), x(t - \tau), u(t)) - \sum_{i=1}^n \psi_i(t) f^i(x(t), x(t - \tau), u(t))$. Moreover, for almost all $t \in [t_0, t_1]$

$$\frac{\partial \mathcal{H}(\widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t))}{\partial u} = 0. \tag{22}$$

Proof. Introduce an auxiliary vector function $y(t) \in L^m_2[t_0, t_1]$ whose coordinates satisfy almost everywhere on $[t_0, t_1]$ the conditions

$$y^2_j(t) + g^j(x(t), x(t - \tau), u(t)) = 0, \tag{23}$$

and represent the problem (12)–(16) in the form of

Problem 1. Find the minimum (12) under the restrictions (13)–(15) in the case where $x(t) \in W_{1,1}^n[t_0, t_1]$, $y(t) \in L_2^m[t_0, t_1]$, $u(t) \in L_1^r[t_0, t_1]$ and the conditions

$$x(t) = \widehat{\varphi}(t), \quad t_0 - \tau \leq t \leq t_0, \tag{24}$$

$$\widehat{\varphi}(t_0) = x(t_1) - \int\limits_{t_0}^{t_1} \dot{x}(t) dt \tag{25}$$

are fulfilled.

Note that the conditions (24)–(25) make it possible to continue the absolutely continuous functions $x(t) \in W_{1,1}^n[t_1, t_1]$ on the whole interval $[t_0 - \tau, t_1]$ so that the continued function turns out to be from the class $W_{1,1}^n[t_0 - \tau, t_1]$. As is easily seen, Problem 1 is a particular case of the problem (1)–(4). Really,

$$W = W_{1,1}^n[t_0 - \tau, t_1] \times W_{1,1}^n[t_0 - \tau, t_0] \times L_1^r[t_0, t_1] \times L_2^m[t_0, t_1],$$
$$Y = W_{1,1}^n[t_0 - \tau, t_0] \times L_1^{n+m}[t_0, t_1],$$

$w = (x, \widehat{\varphi}, u, y)$, where $x = x(t) \in W_{1,1}^n[t_0 - \tau, t_1]$, $\widehat{\varphi} = \widehat{\varphi}(t) \in W_{1,1}^n[t_0 - \tau, t_0]$, $u = u(t) \in L_1^r[t_0, t_1]$, $y = y(t) \in L_2^m[t_0, t_1]$, $f_0(w) = \int_{t_0}^{t_1} f^0(x(t), x(t - \tau), u(t)) dt$,

$$F(w) = \begin{cases} x(t) - \widehat{\varphi}(t), & t_0 - \tau \leq t \leq t_0, \\ \dot{x}(t) - f(x(t), x(t - \tau), u(t)) & t_0 \leq t \leq t_1, \\ y_j^2(t) + g^j(x(t), x(t - \tau), u(t)), & j = \overline{1, m} \quad t_0 \leq t \leq t_1, \end{cases}$$

$$f_i(w) = x^i(t_1) - x_1^i, \quad i = \overline{1, n},$$

$$f_{n+i}(w) = \widehat{\varphi}^i(t_0) - x^i(t_1) + \int\limits_{t_0}^{t_1} x^i(t) dt, \quad i = \overline{1, n}.$$

The functions $f_i : W \to R$, $i = \overline{0, 2n}$ defined above and the mapping $F : W \to Y$ satisfy all conditions of Theorem 1. Indeed, W and Y are Banach spaces and hence they are locally convex and separable linear topological spaces; the condition a) and the first part of the conditions b) can be verified directly from the definitions, and the condition of generality ensures the fulfillment of the second part of the conditions b).

The Lagrange function for Problem 1 is of the form

$$\mathcal{L}(x(\cdot), \widehat{\varphi}(\cdot), u(\cdot), y(\cdot), \psi_0, \lambda_1, \ldots, \lambda_{2n}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)) =$$

$$= \psi_0 f^0(w) + \sum_{\lambda=1}^{2n} \lambda_i f^i(w) + \langle y^*, F(w) \rangle =$$

$$= \psi_0 \int\limits_{t_0}^{t_1} f^0(x(t), x(t-\tau), u(t))dt +$$

$$+(a_0|(x(t_0 - \tau) - \widehat{\varphi}(t_0 - \tau))) + \int\limits_{t_0-\tau}^{t_0} (\sigma(t)|(\dot{x}(t) - \widehat{\varphi}(t)))dt +$$

$$+ \int\limits_{t_0}^{t_1} \{(\psi(t)|(\dot{x}(t) - f(x(t), x(t-\tau), u(t)))) +$$

$$+(\mu(t)|(y^2(t) + g(x(t), x(t-\tau), u(t))))\}dt +$$

$$+(\lambda_{(1)}|(x(t_1) - x_1)) + (\lambda_{(2)}|(\widehat{\varphi}(t_0) - x(t_1) + \int\limits_{t_0}^{t_1} \dot{x}(t)dt), \qquad (26)$$

where $(a_0, \sigma(\cdot)) \in W_{1,1}^{n*}[t_0 - \tau, t_0]$, $a_0 \in R^n$, $\sigma(t) \in L_\infty^n[t_0 - \tau, t_0]$, $\lambda_{(1)} \in R^n$,

$\lambda_{(2)} \in R^n$, $\psi_0 \in R$, $\psi(t) \in L_\infty^n[t_0, t_1]$, $\mu(t) \in L_\infty^m[t_0, t_1]$, $y^2(t) = \begin{pmatrix} y_1^2(t) \\ \vdots \\ y_m^2(t) \end{pmatrix}$.

Let $(\widehat{x}(\cdot), \widehat{u}(\cdot), \widehat{y}(\cdot))$ be a solution of Problem 1. Then because of Theorem 1, there exist non-trivial Lagrange multipliers $\psi_0 \geq 0$, $\lambda_{(1)}$, $\lambda_{(2)}$, $(a_0, \sigma(\cdot))$, $\psi(\cdot)$, $\mu(\cdot)$ such that the condition

$$\mathcal{L}(\widehat{x}(\cdot), \widehat{\varphi}(\cdot), \widehat{u}(\cdot), \widehat{y}(\cdot), \psi_0, \lambda_{(1)}, \lambda_{(2)}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)) \leq$$
$$\leq \mathcal{L}(x(\cdot), \widehat{\varphi}(\cdot), u(\cdot), y(\cdot), \psi_0, \lambda_{(1)}, \lambda_{(2)}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)) \qquad (27)$$

is fulfilled for all $(x(\cdot), \widehat{\varphi}(\cdot), u(\cdot), y(\cdot)) \in W$. The inequality (27) implies

$$\mathcal{L}(\widehat{x}(\cdot), \widehat{\varphi}(\cdot), \widehat{u}(\cdot), \widehat{y}(\cdot), \psi_0, \lambda_{(1)}, \lambda_{(2)}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)) \leq$$
$$\leq \mathcal{L}(\widehat{x}(\cdot), \widehat{\varphi}(\cdot), u(\cdot), \widehat{y}(\cdot), \psi_0, \lambda_{(1)}, \lambda_{(2)}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)), \qquad (28)$$

for all $u(\cdot) \in L_1^r[t_0, t_1]$. Consequently, if we take

$$u(t) = \begin{cases} \widehat{u}(t), & t \notin [s, s+h] \\ v, & t \in [s, s+h], \end{cases}$$

where s is any fixed time moment from (t_0, t_1), v is any fixed element from R^r and $h > 0$ satisfies the condition $[s, s+h] \subset [t_0, t_1]$, then (28) results in

$$\int\limits_s^{s+h} \left\{ \psi_0 f^0(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t)) - (\psi(t)|f(\widehat{x}(t), \widehat{x}(t-\tau), u(t))) + \right.$$

$$\left. +(\mu(t)|g(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))) \right\}dt \leq \int\limits_s^{s+h} \left\{ \psi_0 f^0(\widehat{x}(t), \widehat{x}(t-\tau), v) - \right.$$

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$$-(\psi(t)|f(\widehat{x}(t), \widehat{x}(t-\tau), v)) + (\mu(t)|g(\widehat{x}(t), \widehat{x}(t-\tau), v))\big\} dt \qquad (29)$$

for $\forall v \in R^r$.

If we divide both sides of the inequality (29) by $h > 0$ and then pass to limit as $h \to 0$, we will obtain the condition of minimum at the point s for almost all $s \in [t_0, t_1]$:

$$\psi_0 f^0(\widehat{x}(s), \widehat{x}(s-\tau), \widehat{u}(s)) - (\psi(s)|f(\widehat{x}(s), \widehat{x}(s-\tau), \widehat{u}(s))) +$$
$$+(\mu(s)|g(\widehat{x}(s), \widehat{x}(s-\tau), \widehat{u}(s))) \le \psi_0 f^0(\widehat{x}(s), \widehat{x}(s-\tau), v) -$$
$$-(\psi(s)|f(\widehat{x}(s), \widehat{x}(s-\tau), v)) + (\mu(s)|g(\widehat{x}(s), \widehat{x}(s-\tau), v)), \qquad (30)$$

for $\forall v \in R^r$.

Since the Lagrange function (26) is Frechet differentiable on W, from (27) we get three groups of conditions:

$$\mathcal{L}_x(\widehat{w}(\cdot)) = 0, \quad \mathcal{L}_y(\widehat{w}(\cdot)) = 0, \quad \mathcal{L}_u(\widehat{w}(\cdot)) = 0. \qquad (31)$$

The first group of these conditions yields

$$\mathcal{L}_x(\widehat{w}(\cdot))(\xi(\cdot)) = \int\limits_{t_0}^{t_1} \Big\{ \Big(\psi_0 \frac{\partial f^0(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial x} |\xi(t) \Big) +$$

$$+\Big(\psi_0 \cdot \frac{\partial f^0(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial z} |\xi(t-\tau) \Big) \Big\} dt + (a_0|\xi(t_0 - \tau)) +$$

$$+\int\limits_{t_0 - \tau}^{t_0} (\sigma(t)|\dot{\xi}(t)) dt + \int\limits_{t_0}^{t_1} \Big\{ \Big(\psi(t)|\Big[\dot{\xi}(t) -$$

$$-\Big(\frac{\partial f(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial x} |\xi(t) \Big) -$$

$$-\Big(\frac{\partial f(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial z} |\xi(t-\tau) \Big) \Big] \Big) +$$

$$+\Big(\mu(t)|\Big[\Big(\frac{\partial g(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial x} |\xi(t) \Big) +$$

$$+\Big(\frac{\partial g(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial z} |\xi(t-\tau) \Big) \Big] \Big) \Big\} dt +$$

$$+([\lambda_{(1)} - \lambda_{(2)}]|\xi(t_1)) + \lambda_{(2)} \cdot \int\limits_{t_0}^{t_1} \dot{\xi}(t) dt = 0 \qquad (32)$$

for $\forall \xi(t) \in W_{1,1}^n[t_0 - \tau, t_1]$. We transform the left-hand side of (32) in an obvious manner:

$$\mathcal{L}_x(\widehat{w}(\cdot))(\xi(\cdot)) = \int\limits_{t_0}^{t_1} \Big(\Big[\psi_0 \frac{\partial f^0(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t))}{\partial x} -$$

$$-\left(\psi(t)|\frac{\partial f(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}\right)+\left(\mu(t)|\frac{\partial g(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}\right)\Big]|\xi(t)\right)dt+$$

$$+\int\limits_{t_0}^{t_1}\left(\Big[\psi_0\frac{\partial f^0(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial z}-\left(\psi(t)|\frac{\partial f(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial z}\right)+\right.$$

$$+\left(\mu(t)|\frac{\partial g(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial z}\right)\Big]|\xi(t-\tau)\right)dt+\int\limits_{t_0}^{t_1}\left(\Big[\psi(t)+\lambda_{(2)}\Big]|\dot{\xi}(t)\right)dt+$$

$$+\int\limits_{t_0-\tau}^{t_0}(\sigma(t)|\dot{\xi}(t))dt+(a_0|\xi(t_0-\tau))+$$

$$+([\lambda_{(1)}-\lambda_{(2)}]|\xi(t_1))=\int\limits_{t_0-\tau}^{t_0}\left(\Big[\psi_0\frac{\partial f^0(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}-\right.$$

$$-\left(\psi(t+\tau)|\frac{\partial f(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}\right)+$$

$$+\left(\mu(t+\tau)|\frac{\partial g(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}\right)\Big]|\xi(t)\right)dt+$$

$$+\int\limits_{t_0}^{t_1-\tau}\left(\Big[\psi_0\frac{\partial f^0(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}-\left(\psi(t)|\frac{\partial f(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}\right)+\right.$$

$$+\left(\mu(t)|\frac{\partial g(\widehat{x}(t),\widehat{x}(t-\tau),u(t))}{\partial x}\right)+\psi_0\frac{\partial f^0(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}-$$

$$-\left(\psi(t+\tau)|\frac{\partial f(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}\right)+$$

$$+\left(\mu(t+\tau)|\frac{\partial g(\widehat{x}(t+\tau),\widehat{x}(t),\widehat{u}(t+\tau))}{\partial z}\right)\Big]|\xi(t)\right)dt+$$

$$+\int\limits_{t_1-\tau}^{t_1}\left(\Big[\psi_0\frac{\partial f^0(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}-\left(\psi(t)|\frac{\partial f(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}\right)+\right.$$

$$+\left(\mu(t)|\frac{\partial g(\widehat{x}(t),\widehat{x}(t-\tau),\widehat{u}(t))}{\partial x}\right)\Big]|\xi(t)\right)dt+$$

$$+\int\limits_{t_0}^{t_1}([\psi(t)+\lambda_{(2)}]|\dot{\xi}(t))dt+\int\limits_{t_0-\tau}^{t_0}(\sigma(t)|\dot{\xi}(t))dt+$$

$$+(a_0|\xi(t_0-\tau))+\left([\lambda_{(1)}-\lambda_{(2)}]|\Big[\xi(t_0)+\int\limits_{t_0}^{t_1}\dot{\xi}(t)dt\Big]\right)=$$

$$= \int\limits_{t_0-\tau}^{t_0} \left(\frac{\partial \mathcal{H}[t+\tau]}{\partial z} | \xi(t) \right) dt + \int\limits_{t_0}^{t_1-\tau} \left(\left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t+\tau]}{\partial z} \right] | \xi(t) \right) dt +$$

$$+ \int\limits_{t_1-\tau}^{t_1} \left(\frac{\partial \mathcal{H}[t]}{\partial x} | \xi(t) \right) dt + \int\limits_{t_0}^{t_1} ([\psi(t) + \lambda_{(1)}] | \dot{\xi}(t)) dt + \int\limits_{t_0-\tau}^{t_0} (\sigma(t) | \dot{\xi}(t)) dt +$$

$$+ (a_0 | \xi(t_0 - \tau)) + ([\lambda_{(1)} - \lambda_{(2)}] | \xi(t_0)), \tag{33}$$

where $\mathcal{H}[t] = \psi_0 f^0[t] - (\psi(t) | f[t]) + (\mu(t) | g[t])$, and then we integrate by parts the second and the third summands of (33):

$$\int\limits_{t_0}^{t_1-\tau} \left(\left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t+\tau]}{\partial z} \right] | \xi(t) \right) dt =$$

$$= \left(\int\limits_{t_0}^{t_1-\tau} \left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t+\tau]}{\partial z} \right] dt | \xi(t_0) \right) +$$

$$+ \int\limits_{t_0}^{t_1-\tau} \left(\left\{ \int\limits_{t}^{t_1-\tau} \left[\frac{\mathcal{H}[s]}{\partial x} + \frac{\partial x \mathcal{H}[s+\tau]}{\partial z} \right] ds \right\} | \dot{\xi}(t) \right) dt \tag{34}$$

$$\int\limits_{t_1-\tau}^{t_1} \left(\frac{\partial \mathcal{H}[t]}{\partial x} | \xi(t) \right) dt =$$

$$= \left(\int\limits_{t_1-\tau}^{t_1} \frac{\partial \mathcal{H}[t]}{\partial x} dt | \xi(t_1 - \tau) \right) + \int\limits_{t_1-\tau}^{t_1} \left(\int\limits_{t}^{t_1} \frac{\partial \mathcal{H}[s]}{\partial x} ds | \dot{\xi}(t) \right) dt. \tag{35}$$

The condition (32) with regard for (33), (34) and (35) will take the form

$$\int\limits_{t_0-\tau}^{t_0} \left(\frac{\partial \mathcal{H}[t+\tau]}{\partial z} | \xi(t) \right) dt + \left(\xi(t_0) | \left\{ [\lambda_{(1)} - \lambda_{(2)}] + \right. \right.$$

$$\left. + \int\limits_{t_0}^{t_1-\tau} \left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t+\tau]}{\partial z} \right] \right\} \right) +$$

$$+ \int\limits_{t_0}^{t_1-\tau} \left(\dot{\xi}(t) | \left\{ \psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1-\tau} \left[\frac{\partial \mathcal{H}[s]}{\partial x} + \frac{\partial \mathcal{H}[s+\tau]}{\partial z} \right] ds \right\} \right) dt +$$

$$+ \int\limits_{t_1-\tau}^{t_1} \left(\dot{\xi}(t) | \left\{ \psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1} \frac{\partial \mathcal{H}[s]}{\partial x} ds \right\} \right) dt +$$

$$+\xi(t_1 - \tau)|\int\limits_{t_1-\tau}^{t_1} \frac{\partial \mathcal{H}[t]}{\partial x}dt + (a_0|\xi(t_0 - \tau)) + \int\limits_{t_0-\tau}^{t_0} (\sigma(t)|\dot{\xi}(t))dt = 0 \quad (36)$$

for $\forall \xi(t) \in W_{1,1}^n[t_0 - \tau, t_1]$

Since

$$\int\limits_{t_0-\tau}^{t_0} \left(\frac{\partial \mathcal{H}[t+\tau]}{\partial z}|\xi(t)\right)dt = \left(\int\limits_{t_0-\tau}^{t_0} \frac{\partial \mathcal{H}[t+\tau]}{\partial z}dt|\xi(t_0 - \tau)\right) +$$

$$+ \int\limits_{t_0-\tau}^{t_0} \left(\int\limits_{t}^{t_0} \frac{\partial \mathcal{H}[s+\tau]}{\partial z}ds|\dot{\xi}(t)\right)dt,$$

from (36) we have

$$\left(\xi(t_0 - \tau)|\left\{a_0 + \int\limits_{t_0-\tau}^{t_0} \frac{\partial \mathcal{H}[t+\tau]}{\partial z}dt\right\}\right) +$$

$$+ \int\limits_{t_0-\tau}^{t_0} \left(\left[\sigma(t) + \int\limits_{t}^{t_0} \frac{\partial \mathcal{H}[s+\tau]}{\partial z}ds\right]|\dot{\xi}(t)\right)dt +$$

$$+ \left(\xi(t_0)|\left\{\lambda_{(1)} - \lambda_{(2)} + \int\limits_{t_0}^{t_1-\tau} \left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t+\tau]}{\partial z}\right]dt\right\}\right) +$$

$$+ \int\limits_{t_0}^{t_1-\tau} \left(\dot{\xi}(t)|\left\{\psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1-\tau} \left[\frac{\partial \mathcal{H}[s]}{\partial x} + \frac{\partial \mathcal{H}[s+\tau]}{\partial z}\right]ds\right\}\right)dt +$$

$$+ \int\limits_{t_1-\tau}^{t_1} \left(\dot{\xi}(t)|\left\{\psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1} \frac{\partial \mathcal{H}[s]}{\partial x}ds\right\}\right)dt +$$

$$+ \left(\xi(t_1 - \tau)|\int\limits_{t_1-\tau}^{t_1} \frac{\partial \mathcal{H}[t]}{\partial x}dt\right) = 0 \quad (37)$$

for $\forall \xi(t) \in W_{1,1}^n[t_0 - \tau, t_0]$.

Since $\xi(t) \in W_{1,1}^n[t_0 - \tau, t_0]$ is taken arbitrarily, from (37) we obtain

$$a_0 + \int\limits_{t_0-\tau}^{t_0} \frac{\partial \mathcal{H}[t+\tau]}{\partial z}dt = 0 \quad (38)$$

$$\sigma(t) + \int\limits_{t}^{t_0} \frac{\partial \mathcal{H}[s+\tau]}{\partial z}dt = 0, \quad \text{for} \quad t \in [t_0 - \tau, t_0]; \quad (39)$$

$$\lambda_{(1)} - \lambda_{(2)} + \int\limits_{t_0}^{t_1 - \tau} \left[\frac{\partial \mathcal{H}[t]}{\partial x} + \frac{\partial \mathcal{H}[t + \tau]}{\partial z} \right] dt = 0, \tag{40}$$

$$\psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1 - \tau} \left[\frac{\partial \mathcal{H}[s]}{\partial x} + \frac{\partial \mathcal{H}[s + \tau]}{\partial z} \right] ds = 0, \tag{41}$$

$$\text{for} \quad t \in [t_0, t_1 - \tau];$$

$$\int\limits_{t_1 - \tau}^{t_1} \frac{\partial \mathcal{H}[t]}{\partial x} dt = 0, \tag{42}$$

$$\psi(t) + \lambda_{(1)} + \int\limits_{t}^{t_1} \frac{\partial \mathcal{H}[s]}{\partial x} ds = 0, \quad \text{for} \quad t \in [t_1 - \tau, t_1]. \tag{43}$$

It follows from (41) that the function $\psi(t)$ is absolutely continuous in the interval $[t_0, t_1 - \tau]$, and the condition

$$\psi(t_1 - \tau) = -\lambda_{(1)} \tag{44}$$

is fulfilled. It follows from (43) that the function $\psi(t)$ is absolutely continuous in the interval $[t_1 - \tau, t_1]$, and

$$\psi(t_1 - \tau) + \lambda_{(1)} + \int\limits_{t_1 - \tau}^{t_1} \frac{\partial \mathcal{H}[s]}{\partial x} ds = 0,$$

that is, taking into account (42), we get (44). Thus the function $\psi(t)$ is absolutely continuous in the interval $[t_0, t_1]$ and therefore almost everywhere on $[t_0, t_1]$ there exists a derivative $\frac{d\psi}{dt}$ which, as is seen from (41) and (43), satisfies the conjugate system of the equations (19) and (20).

From the second group of the conditions (31) we have

$$\mathcal{L}_y(\widehat{w}(\cdot))(\eta(\cdot)) = 2 \int\limits_{t_0}^{t_1} \big((\mu(t)|\widehat{y}(t))|\eta(t)\big) dt = 0, \tag{45}$$

for $\forall \eta(t) \in L_2^m[t_0, t_1]$ whence, because of the arbitrariness of $\eta(t) \in L_2^m[t_0, t_1]$, we get that almost everywhere on $[t_0, t_1]$ the condition

$$\mu_j(t)\widehat{y}_j(t) = 0, \quad j = \overline{1, m} \tag{46}$$

is fulfilled.

Since $\widehat{y}_j^2(t) + g^j(\widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t)) = 0, j = \overline{1, m}$, from (46) we immediately obtain (18).

Next, from (27) it follows

$$\mathcal{L}_{yy}(\widehat{w}(\cdot))(\eta(\cdot)) = 2 \int\limits_{t_0}^{t_1} \big(\mu(t)|\eta^2(t)\big)\, dt \geq 0$$

for $\forall \eta(t) \in L_2^m[t_0, t_1]$, whence $\mu_j(t) \geq 0$, $j = \overline{1, m}$, for almost all $t \in [t_0, t_1]$, i.e., we obtain the condition (17).

With regard for (30), from (17) and (18) we conclude that (21) is valid.

The third group of the conditions (31) results in

$$\mathcal{L}_u(\widehat{w}(\cdot))(v(\cdot)) = \int\limits_{t_0}^{t_1} \Big(\frac{\partial \mathcal{H}(\widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t))}{\partial u}|v(t)\Big) dt = 0.$$

for $\forall v(t) \in L_1^r[t_0, t_1]$, whence, owing to the arbitrariness of $v(t) \in L_1^r[t_0, t_1]$, we obtain the condition (22) for almost all $t \in [t_0, t_1]$.

To complete the proof of Theorem 2, it remains to prove the non-triviality of the vector $(\psi_0, \psi(\cdot))$. Suppose on the contrary that $(\psi_0, \psi(t)) \equiv (0, 0)$ on the interval $t \in [t_0, t_1]$. Then, taking into account the condition of generality, we obtain from (22) that $\mu_j(t) = 0$, $j = \overline{1, m}$, almost everywhere on $[t_0, t_1]$. But then (38) and (39) imply that $a_0 = 0$, $\sigma_i(t) = 0$, $i = \overline{1, n}$, on the interval $t \in [t_0 - \tau, t_0]$. It turns out that since by (40) and (41) for the trivial $\psi(t) \equiv 0$ the multipliers $\lambda_{(1)}$ and $\lambda_{(2)}$ are trivial, the whole system of the Lagrange multipliers $(\psi_0, \lambda_{(1)}, \lambda_{(2)}, (a_0, \sigma(\cdot)), \psi(\cdot), \mu(\cdot))$ is also trivial, which contradicts Theorem 1. ∎

3. Problems with incommensurable delays in phase coordinates and controls. Consider the problem

$$\int\limits_{t_0}^{t_1} f^0(x(t), x(t - \tau), u(t), u(t - \vartheta))dt \to \min \qquad (47)$$

under the restrictions

$$\dot{x}(t) = f(x(t), x(t - \tau), u(t), u(t - \vartheta)), \qquad (48)$$
$$g(x(t), x(t - \tau), u(t), u(t - \vartheta)) \leq 0, \quad t_0 \leq t \leq t_1 \qquad (49)$$
$$x(t_1) = x_1, \qquad (50)$$
$$x(t) = \widehat{\varphi}(t), \quad \text{for} \quad t_0 - \tau \leq t \leq t_0, \qquad (51)$$

where $\tau > 0$, $\vartheta > 0$ (note that the constant numbers τ and ϑ may be incommensurable; for example, τ may be rational and ϑ irrational), $u(t) \in L_1^r[t_0 - \vartheta, t_1]$, and all the remaining conditions formulated in the previous section for the optimal problem (12)–(16) with natural modifications connected with the type of the element $(x(t), x(t - \tau), u(t), u(t - \vartheta)) = (x, z, u, v)$, are satisfied.

In a way similar to that of the previous section, we represent the problem (47)–(51) as a particular case of the problem (1)–(4) and construct the appropriate Lagrange function $\mathcal{L}$. It follows from Theorem 1 that the conditions (7) are valid, whence

$$
\int\limits_{t_0}^{t_1} [\psi_0 f^0(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta)) -
$$
$$
-(\psi(t)|f(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{x}(t-\vartheta)))) +
$$
$$
+(\mu(t)|g(\widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))))]dt \leq
$$
$$
\leq \int\limits_{t_0}^{t_1} [\psi_0 f^0(\widehat{x}(t), \widehat{x}(t-\vartheta), u(t), u(t-\vartheta)) -
$$
$$
-(\psi(t)|f(\widehat{x}(t), \widehat{x}(t-\tau), u(t), u(t-\vartheta)))) +
$$
$$
+(\mu(t)|g(\widehat{x}(t), \widehat{x}(t-\tau), u(t), u(t-\vartheta)))]dt, \tag{52}
$$

for $\forall u(t) \in L_1^r[t_0 - \vartheta, t_1]$.

With the help of the function $\mathcal{H}$, the condition (52) can be written as

$$
\int\limits_{t_0}^{t_1} \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))dt \leq
$$
$$
\int\limits_{t_0}^{t_1} \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), u(t), u(t-\vartheta))dt \tag{53}
$$

for $\forall u(t) \in L_1^r[t_0 - \vartheta, t_1]$.

Let

$$
u(t) = \begin{cases} \widehat{u}(t), & t \notin [s, s+h] \\ \nu, & t \in [s, s+h], \end{cases} \tag{54}
$$

where s is any fixed moment from $(t_0 - \vartheta, t_0)$, ν is any fixed vector from R^r, and $h > 0$ satisfies the condition $[s, s+h] \subseteq [t_0 - \vartheta, t_0]$. Then from (53) it follows

$$
\int\limits_{s}^{s+h} \mathcal{H}(\psi_0, \psi(t+\vartheta), \mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau),
$$
$$
\widehat{u}(t+\vartheta), \widehat{u}(t))dt \leq \int\limits_{s}^{s+h} \mathcal{H}(\psi_0, \psi(t+\vartheta,
$$
$$
\mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \nu))dt, \tag{55}
$$

for $\forall \nu \in R^r$.

Dividing both parts of the inequality (55) by h and passing to limit as $h \to 0$, we obtain that for almost all $s \in [t_0 - \vartheta, t_0]$ the inequalities

$$
\mathcal{H}(\psi_0, \psi(s+\vartheta), \mu(s+\vartheta), \widehat{x}(s+\vartheta), \widehat{x}(s+\vartheta-\tau), \widehat{u}(s+\vartheta), \widehat{u}(s)) \leq
$$
$$
\leq \mathcal{H}(\psi_0, \psi(s+\vartheta), \mu(s+\vartheta), \widehat{x}(s+\vartheta), \widehat{x}(s+\vartheta-\tau), \widehat{u}(s+\vartheta), \nu), \quad (56)
$$

are valid for $\forall \nu \in R^r$.

If in (54) $s \in (t_0, t_0 + \vartheta)$ and h is a sufficiently small positive number, then from (53) we have

$$
\frac{1}{h} \int_s^{s+h} \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))dt \leq
$$
$$
\leq \frac{1}{h} \int_s^{s+h} \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \nu, \widehat{u}(t-\vartheta))dt \quad (57)
$$

for $\forall \nu \in R^r$.

Passing in (57) to limit as $h \to 0$, we obtain that for almost all $s \in [t_0, t_0 + \vartheta]$ the inequalities

$$
\mathcal{H}(\psi_0, \psi(s), \mu(s), \widehat{x}(s), \widehat{x}(s-\tau), \widehat{u}(s), \widehat{u}(s-\vartheta)) \leq
$$
$$
\leq \mathcal{H}(\psi_0, \psi(s), \mu(s), \widehat{x}(s), \widehat{x}(s-\tau), \nu, \widehat{u}(s-\vartheta)) \quad (58)
$$

are valid for $\forall \nu \in R^r$.

Thus, as is seen from (56) and (58), for almost all $t \in [t_0, t_0 + \vartheta]$ the relation

$$
\mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta)) +
$$
$$
+ \mathcal{H}(\psi_0, \psi(t+\vartheta), \mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \widehat{u}(t)) \leq
$$
$$
\leq \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \nu, \widehat{u}(t-\vartheta)) +
$$
$$
+ \mathcal{H}(\psi_0, \psi(t+\vartheta), \mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \nu), \quad (59)
$$

is fulfilled for $\forall \nu \in R^r$.

Owing to (54), in the interval $s \in (t_0 + \vartheta, t_0 + 2\vartheta)$ (53) again results in (56), while in the interval $s \in (t_0 + 2\vartheta, t_0 + 3\vartheta)$ we have (58), i.e., for almost all $t \in [t_0, t_0 + 3\vartheta]$ we get (59). Moreover, it is clear that if $s \in [t_1 - \vartheta, t_1]$, then instead of (56) there takes place (58). So, it becomes evident that owing to (54), from (53) it follows the validity of the conditions (59) and (58) for almost all $t \in [t_0, t_1 - \vartheta]$ and $t \in [t_1 - \vartheta, t_1]$, respectively.

It is evident that, the condition $\mathcal{L}_x(\widehat{w}(\cdot))(\xi(\cdot)) = 0$, $\mathcal{L}_y(\widehat{w}(\cdot))(\eta(\cdot)) = 0$, $\mathcal{L}_{yy}(\widehat{w}(\cdot))(\eta(\cdot)) \geq 0$ derived from (7) give the same conditions as in the

previous section, while the condition $\mathcal{L}_u(\widehat{w}(\cdot))(v(\cdot)) = 0$, derived also from (7) yields the condition

$$
\int\limits_{t_0}^{t_1} \left(\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial u} |v(t) \right) dt +
$$
$$
+ \int\limits_{t_0}^{t_1} \left(\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial v} |v(t-\vartheta) \right) dt = 0 \quad (60)
$$

for $\forall v(t) \in L_1^r[t_0 - \vartheta, t_1]$.

From (60) we get

$$
\int\limits_{t_0-\vartheta}^{t_1-\vartheta} \left(\left[\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial u} + \right.\right.
$$
$$
\left.\left. + \frac{\partial \mathcal{H}(\psi_0, \psi(t+\vartheta), \mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \widehat{u}(t))}{\partial v} \right] |v(t) \right) dt +
$$
$$
+ \int\limits_{t_1-\vartheta}^{t_1} \left(\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial u} |v(t) \right) dt = 0 \quad (61)
$$

for $\forall v(t) \in L_1^r[t_0 - \vartheta, t_1]$.

Due to the fact that $v(t) \in L_1^r[t_0 - \vartheta, t_1]$ is arbitrary, it follows from (61) that the condition

$$
\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(x-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial u} +
$$
$$
+ \frac{\partial \mathcal{H}(\psi_0, \psi(t+\vartheta), \mu(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \widehat{u}(t))}{\partial v} = 0 \quad (62)
$$

is fulfilled almost everywhere on $t \in [t_0, t_1 - \vartheta]$, and the condition

$$
\frac{\partial \mathcal{H}(\psi_0, \psi(t), \mu(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta))}{\partial u} = 0 \quad (63)
$$

is fulfilled for almost all $t \in [t_1 - \vartheta, t_1]$.

With regard for the conditions (62) and (63), we obtain as in the previous section that the vector $(\psi_0, \psi(\cdot))$ is non-trivial, and from the conditions (17), (18), (59) and (58) it follows the validity of the following assertions: for almost all $t \in [t_0, t_1 - \tau]$ the inequality

$$
H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t-\tau), \widehat{u}(t), \widehat{u}(t-\vartheta)) +
$$
$$
+ H(\psi_0, \psi(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \widehat{u}(t)) \le
$$
$$
\le H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t-\tau), \nu, \widehat{u}(t-\vartheta)) +
$$
$$
+ H(\psi_0, \psi(t+\vartheta), \widehat{x}(t+\vartheta), \widehat{x}(t+\vartheta-\tau), \widehat{u}(t+\vartheta), \nu), \quad (64)
$$

is fulfilled for $\forall \nu$ satisfying $g(\widehat{x}(t), \widehat{x}(t - \tau), \nu, \widehat{u}(t - \vartheta)) \leq 0$ and $g(\widehat{x}(t + \vartheta), \widehat{x}(t + \vartheta - \tau), \widehat{u}(t + \vartheta), \nu) \leq 0$; for almost all $t \in [t_1 - \vartheta, t_1]$ the inequality

$$H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau), \widehat{u}(t), \widehat{u}(t - \vartheta)) \leq$$
$$\leq H(\psi_0, \psi(t), x(t), \widehat{x}(t - \tau), \nu, \widehat{u}(t - \vartheta)), \tag{65}$$

is valid for $\forall \nu$ satisfying $g(\widehat{x}(t), \widehat{x}(t - \tau), \nu, \widehat{u}(t - \vartheta)) \leq 0$.

From the above obtained assertions we arrive to the following

Theorem 3. *Let* $(\widehat{x}(t), \widehat{u}(t))$ *be a solution of the problem* (47)–(51)*. Then there exist non-trivial Lagrange multipliers* $\psi_0 \geq 0$ *and* $\psi(t) \in W_{1,1}^n[t_0, t_1]$ *satisfying* (17)–(20), (62), (63) *as well as the "minimum" conditions* (64) *and* (65)*.*

It is clear that in the conditions (18)–(20) the functions g^j, $j = \overline{1, m}$, and $\mathcal{H}$ along with the other arguments depend also on the argument $\widehat{u}(t - \vartheta)$.

Suppose that we are given real positive numbers $\tau_1 < \tau_2 < \cdots < \tau_p$ and $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_q$, and it is required to find the minimum of the functional

$$I = \int\limits_{t_0}^{t_1} f^0(x(t), x(t - \tau_1), \ldots, x(t - \tau_p),$$
$$u(t), u(t - \vartheta_1), \ldots, u(t - \vartheta_q))dt \tag{66}$$

under the restrictions

$$\dot{x}(t) = f(x(t), x(t - \tau_1), \ldots, x(t - \tau_p),$$
$$u(t), u(t - \vartheta_1), \ldots, u(t - \vartheta_q)), \tag{67}$$
$$g(x(t), x(t - \tau_1), \ldots, x(t - \tau_p), u(t), \ldots, u(t - \vartheta_q)) \leq 0, \tag{68}$$
$$x(t_1) = x_1, \tag{69}$$
$$x(t) = \widehat{\varphi}(t), \quad \text{for} \quad t \in [t_0 - \tau_p, t_0], \tag{70}$$

where $u(t) \in L_1^r[t_0 - \vartheta_q, t_1]$, and all the remaining conditions formulated for the optimal problem (12)–(16) with natural modifications connected with the type of of the element

$$x(t), x(t - \tau_1), \ldots, x(t - \tau_p), u(t), u(t - \vartheta_1), \ldots, u(t - \vartheta_q) =$$
$$= (x, z_1, \ldots, z_p, u, \vartheta_1, \ldots, \vartheta_q)$$

are fulfilled.

After the representing of the problem (66)–(70) as a particular case of the problem (1)–(4), the Lagrange function takes the form

$$\mathcal{L}(x(\cdot), \widehat{\varphi}(\cdot), u(\cdot), y(\cdot), \psi_0, \lambda_{(1)}, \lambda_{(2)}, \psi(\cdot), (a_0, \sigma(\cdot)), \mu(\cdot)) =$$

$$= (a_0|[x(t_0 - \tau_p) - \widehat{\varphi}(t_0 - \tau_p)] + \int\limits_{t_0 - \tau_p}^{t_0} (\sigma(t)|[\dot{x}(t) - \widehat{\varphi}(t)])dt +$$

$$+ \int\limits_{t_0}^{t_1} \Big\{ \psi_0 f^0(x(t), \ldots, x(t - \tau_p), u(t), \ldots, u(t - \vartheta_q)) +$$

$$+ (\psi(t)|[\dot{x}(t) - f(x(t), \ldots, x(t - \tau_p), u(t), \ldots, u(t - \vartheta_q))])) +$$

$$+ (\mu(t)|[y^2(t) + g(x(t), \ldots, x(t - \tau_p), u(t), \ldots, u(t - \vartheta_q))]) \Big\} dt +$$

$$+ (\lambda_{(1)}|[x(t_1) - x_1] + (\lambda_{(2)}|[\widehat{\varphi}(t_0) - x(t_1) + \int\limits_{t_0}^{t_1} \dot{x}(t)dt]). \qquad (71)$$

Using Theorem 1, we can see that the condition (7) is valid for the function $\mathcal{L}$ from (71) which provides us with the condition (28) for any $u(\cdot) \in L_1^r[t_0 - \vartheta_q, t_1]$. From this condition, applying now the expression (54) subsequently to $s \in (t_0 - \vartheta_j, \vartheta_{j-1})$ $(j = q, q-1, \ldots, 2)$, $s \in (t_0 - \vartheta_1, t_0), \ldots, s \in (t_1 - \vartheta_1, t_1)$ and passing every time to limit as $h \to 0$, we obtain the conditions analogous to (59) and (58) which are valid for almost all $t \in [t_0, t_1 - \vartheta_1]$ and $t \in [t_1 - \vartheta_1, t_1]$, respectively. The conditions $\mathcal{L}_y(\widehat{w}(\cdot))(\eta(\cdot)) = 0$ and $\mathcal{L}_{yy}(\widehat{w}(\cdot))(\eta(\cdot)) = 0$ remain unchanged and result respectively in the conditions (17) and (18) (note that in the condition (18) the function g will take the form $g = g(x(t), x(t - \tau_1), \ldots, x(t - \tau_p), u(t), u(t - \vartheta_1), \ldots, u(t - \vartheta_q))$). Taking into consideration (17) and (18), from the analogues of (59) and (58) we respectively get

$$H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau_1), \ldots, \widehat{x}(t - \tau_p), \widehat{u}(t), \ldots, \widehat{u}(t - \vartheta_q)) +$$
$$+ H(\psi_0, \psi(t + \vartheta_1), \widehat{x}(t + \vartheta_1), \widehat{x}(t + \vartheta_1 - \tau_1), \ldots, \widehat{x}(t + \vartheta_1 - \tau_p),$$
$$\widehat{u}(t + \vartheta_1), u(t), \widehat{u}(t + \vartheta_1 - \vartheta_2), \ldots, \widehat{u}(t + \vartheta_1 - \vartheta_q)) + \cdots +$$
$$H(\psi_0, \psi(t + \vartheta_q), \widehat{x}(t + \vartheta_q), \widehat{x}(t + \vartheta_q - \tau_1), \ldots, \widehat{x}(t + \vartheta_q - \tau_p), u(t + \vartheta_q),$$
$$\ldots, \widehat{u}(t + \vartheta_q - \vartheta_{q-1}, \widehat{u}(t)) \le H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau_1), \ldots, \widehat{x}(t - \tau_p), u,$$
$$\widehat{u}(t - \vartheta_1), \ldots, \widehat{u}(t - \vartheta_q) + H(\psi_0, \psi(t + \vartheta_1), \widehat{x}(t + \vartheta_1), \widehat{x}(t + \vartheta_1 - \tau_1),$$
$$\ldots, \widehat{x}(t + \vartheta_1 - \tau_p), \widehat{u}(t + \vartheta_1), u, \widehat{u}(t + \vartheta_1 - \vartheta_2), \ldots, \widehat{u}(t + \vartheta_1 - \vartheta_q)) +$$
$$+ \cdots + H(\psi_0, \psi(t + \vartheta_q), \widehat{x}(t + \vartheta_q), \ldots,$$
$$\widehat{x}(t + \vartheta_q - \tau_p), \widehat{u}(t + \vartheta_q), \ldots, \widehat{u}(t + \vartheta_q - \vartheta_{q-1}), u)) \qquad (72)$$

for almost all $t \in [t_0, t_1 - \vartheta_1]$ and $\forall u \in \omega_0$;

$$H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau_1), \ldots, \widehat{x}(t - \tau_p), \widehat{u}(t), \ldots, \widehat{u}(t - \vartheta_q)) \le$$
$$\le H(\psi_0, \psi(t), \widehat{x}(t), \widehat{x}(t - \tau_1), \ldots, \widehat{x}(t - \tau_p), u, \widehat{u}(t - \vartheta_1), \ldots, \widehat{u}(t - \vartheta_q)) \quad (73)$$

for almost all $t \in [t_1 - \vartheta_1, t_1]$ and $\forall u \in \omega_1$.

Here ω_0 denotes the set of the elements u for which the inequalities

$$g(\widehat{x}(t), \ldots, \widehat{x}(t - \tau_p), u, \widehat{u}(t - \vartheta_1), \ldots, \widehat{u}(t - \vartheta_q)) \le 0,$$
$$g(\widehat{x}(t + \vartheta_1), \ldots, \widehat{x}(t + \vartheta_1 - \tau_p), \widehat{u}(t + \vartheta_1), u, \ldots, \widehat{u}(t + \vartheta_1 - \vartheta_q)) \le 0,$$

....

....

....

$$g(\widehat{x}(t+\vartheta_q),\ldots,\widehat{x}(t+\vartheta_q-\tau_p),\widehat{u}(t+\vartheta_q),\ldots,\widehat{u}(t+\vartheta_q-\vartheta_{q-1}),u)\leq 0$$

are fulfilled, and ω_1 denotes the set of elements u for which

$$g(\widehat{x}(t),\ldots,\widehat{x}(t-\tau_p),u,\widehat{u}(t-\vartheta_1),\ldots,\widehat{u}(t-\vartheta_q))\leq 0.$$

The condition $\mathcal{L}_x(\widehat{w}(\cdot))(\xi(\cdot))=0$ for the function $\mathcal{L}$ defined by the formula (71) will take the form

$$(a_0|\xi(t_0-\tau_p))+\int\limits_{t_0-\tau_p}^{t_0}(\sigma(t)|\dot{\xi}(t))dt+$$

$$+\int\limits_{t_0}^{t_1}\left\{\left(\frac{\partial\mathcal{H}(\psi_0,\psi(t),\mu(t),\widehat{x}(t),\ldots,u(t-\vartheta_q))}{\partial x}|\xi(t)\right)+\right.$$

$$+\left(\frac{\partial\mathcal{H}(\psi_0,\psi(t),\mu(t),\widehat{x}(t),\widehat{x}(t-\tau_1),\ldots,\widehat{x}(t-\tau_p),\widehat{u}(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial z_1}|\right.$$

$$|\xi(t-\tau_1)\Big)+$$

$$+\cdots+\left(\frac{\partial\mathcal{H}(\psi_0,\psi(t),\mu(t),\widehat{x}(t),\widehat{x}(t-\tau_1),\ldots,\widehat{x}(t-\tau_p),\widehat{u}(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial z_p}|\right.$$

$$\left.|\xi(t-\tau_p)\Big)\right\}dt+$$

$$+([\lambda_{(1)}-\lambda_{(2)}]|\xi(t_1))+\left(\lambda_{(2)}|\int\limits_{t_0}^{t_1}\dot{\xi}(t)dt\right)+\int\limits_{t_0}^{t_1}(\psi(t)|\dot{\xi}(t))dt=0 \quad (74)$$

for $\forall\xi(t)\in W_{1,1}^n[t_0-\tau_p,t_1]$.

The condition (74) can be represented in the form

$$(a_0|\xi(t_0-\tau_p))+\int\limits_{t_0-\tau_p}^{t_0}(\sigma(t)|\dot{\xi}(t))dt+$$

$$+\int\limits_{t_0}^{t_1}(\psi(t)|\dot{\xi}(t))dt+\left(\lambda_{(2)}|\int\limits_{t_0}^{t_1}\dot{\xi}(t)dt\right)+([\lambda_{(1)}-\lambda_{(2)}]|\xi(t_1))+$$

$$+\int\limits_{t_0-\tau_p}^{t_0-\tau_{p-1}}\left(\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_p),\mu(t+\tau_p),\widehat{x}(t+\tau_p),\ldots,\widehat{u}(t+\tau_p-\vartheta_q))}{\partial z_p}|\xi(t)\right)dt+$$

$$+\cdots+\int\limits_{t_0-\tau_1}^{t_0}\left(\left\{\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_p),\ldots,\widehat{u}(t+\tau_p-\vartheta_q))}{\partial z_p}+\right.\right.$$

$$\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_{p-1}),\ldots,u(t+\tau_{p-1}+\vartheta_q))}{\partial z_{p-1}}+$$

$$+\cdots+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}\Big\}|\xi(t)\Big)dt+$$

$$+\int\limits_{t_0}^{t_1-\tau_p}\left(\left[\frac{\partial\mathcal{H}(\psi_0,\psi(t),\mu(t),\widehat{x}(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}+\right.\right.$$

$$\left.\left.+\cdots+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_p),\ldots,\widehat{u}(t+\tau_p-\vartheta_q))}{\partial z_p}\right]|\xi(t)\right)dt+$$

$$\int\limits_{t_1-\tau_p}^{t_1-\tau_{p-1}}\left(\left[\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}+\right.\right.$$

$$\left.\left.+\cdots+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_{p-1}),\ldots,\widehat{u}(t+\tau_{p-1}-\vartheta_q))}{\partial z_{p-1}}\right]|\xi(t)\right)dt+$$

$$+\cdots+\int\limits_{t_1-\tau_2}^{t_1-\tau_1}\left(\left[\frac{\partial\mathcal{H}(\psi_0,\psi(t),\widehat{u}(t-\vartheta_q))}{\partial x}+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}\right]|\xi(t)\right)dt+$$

$$+\int\limits_{t_1-\tau_1}^{t_1}\left(\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}|\xi(t)\right)dt=0 \tag{75}$$

for $\forall\xi(t)\in W_{1,1}^n[t_0-\tau_p,t_1]$.

From (75), integrating by parts the summands involving $\xi(t)$ under the integral symbol, after elementary transformations we obtain complete analogues of the conditions (38)–(43) from which it follows that the function $\psi(t)$ is absolutely continuous on $[t_0,t_1]$ and almost everywhere on that interval satisfies the conjugate system of equations

$$\frac{d\psi}{dt}=\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}+$$

$$+\cdots+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_p),\ldots,\widehat{u}(t+\tau_p-\vartheta_q))}{\partial z_p},\quad t\in[t_0,t_1-\tau_p],$$

$$\frac{d\psi}{dt}=\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1}+$$

$$+\cdots+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_{p-1}),\ldots,\widehat{u}(t+\tau_{p-1}-\vartheta_q))}{\partial z_{p-1}},\quad t\in[t_1-\tau_p,t_1-\tau_{p-1}],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{76}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d\psi}{dt}=\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x}+$$

$$+\frac{\partial\mathcal{H}(\psi_0,\psi(t+\tau_1),\ldots,\widehat{u}(t+\tau_1-\vartheta_q))}{\partial z_1},\quad t\in[t_1-\tau_2,t_1-\tau_1],$$

$$\frac{d\psi}{dt}=\frac{\partial\mathcal{H}(\psi_0,\psi(t),\ldots,\widehat{u}(t-\vartheta_q))}{\partial x},\quad t\in[t_1-\tau_1,t_1].$$

The condition $\mathcal{L}_u(\widehat{w}(\cdot))(v(\cdot)) = 0$ for the function $\mathcal{L}$ from (71) makes it possible to get a complete analogue of the condition (61), whence it follows that the relations

$$\frac{\partial \mathcal{H}(\psi_0,\psi(t),\mu(t),\widehat{x}(t),...,\widehat{u}(t),...,\widehat{u}(t-\vartheta_q))}{\partial u} + \frac{\partial \mathcal{H}(\psi_0,\psi(t+\vartheta_1),...,\widehat{u}(t+\vartheta_1-\vartheta_q))}{\partial v_1} +$$

$$+ \cdots + \frac{\partial \mathcal{H}(\psi_0,\psi(t+\vartheta_q),\mu(t+\vartheta_q),...,...,\widehat{u}(t+\vartheta_q-\vartheta_{q-1}),\widehat{u}(t))}{\partial v_q} = 0, \quad t \in [t_0, t_1 - \vartheta_q],$$

$$\frac{\partial \mathcal{H}(\psi_0,\psi(t),\mu(t),...,\widehat{u}(t),...,\widehat{u}(t-\vartheta_q))}{\partial u} + \frac{\partial \mathcal{H}(\psi_0,\psi(t+\vartheta_1),...,\widehat{u}(t+\vartheta_1-\vartheta_q))}{\partial v_1} +$$

$$+ \cdots + \frac{\partial \mathcal{H}(\psi_0,\psi(t+\vartheta_{q-1}),...,\widehat{u}(t),\widehat{u}(t+\vartheta_{q-1}-\vartheta_q))}{\partial v_{q-1}} = 0, \quad t \in [t_1 - \vartheta_q, t_1 - \vartheta_{q-1}],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{77}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial \mathcal{H}(\psi_0,\psi(t),\mu(t),...,\widehat{u}(t),...,\widehat{u}(t-\vartheta_q))}{\partial u} +$$

$$+ \frac{\partial \mathcal{H}(\psi_0,\psi(t+\vartheta_1),\mu(t+\vartheta_1),...,\widehat{u}(t+\vartheta_1-\vartheta_q))}{\partial v_1} = 0, \quad t \in [t_1 - \vartheta_2, t_1 - \vartheta_1],$$

$$\frac{\partial \mathcal{H}(\psi_0,\psi(t),\mu(t),...,\widehat{u}(t),...,\widehat{u}(t-\vartheta_q))}{\partial u} = 0, \quad t \in [t_1 - \vartheta_1, t_1]$$

hold for almost all $t \in [t_0, t_1]$

Owing to (77), it follows as in the previous section that $(\psi_0, \psi(\cdot))$ is non-trivial.

Thus, the above relations provide us with the following

Theorem 4. *Let $(\widehat{x}(t), \widehat{u}(t))$ be a solution of the problem (66)–(70). Then there exist non-trivial multipliers $\psi_0 \geq 0$, $\psi(t) \in W_{1,1}^n[t_0, t_1]$ and $\mu(t) \times \times L_\infty^m[t_0, t_1]$, satisfying (17), (18), (76) and (77) as well as the "minimum" conditions (72) and (73).*

Note that the type of the conjugate equation and of the "minimum" condition for the function H is connected with the choice of the sign of the multiplier ψ_0. If the sign of ψ_0 is taken to be negative and if we denote $\mathcal{H} = \psi_0 f + (\mu|f) - (\mu|g)$, then the conjugate equation will be written in a standard form, while the "minimum" condition for the function $H = \psi_0 f + (\psi|f)$ will take the form of the "maximum" condition.

4. The case of optimal problems with the free right end. If in the optimal problems considered in sections 2 and 3 the right end is free, then the corresponding Lagrange functions $\mathcal{L}$ for these problems will involve no summand $(\lambda_{(1)}|[x(t_1) - x_1])$ and hence, the condition (43) whose form is common for all the above-considered optimal problems will also involve no summand $\lambda_{(1)}$. Because of this fact, we obtain from (43) the condition $\psi(t_1) = 0$. This circumstance, in combination with the linearity of the problems under consideration, enables us to construct a scheme of solving these problems

which is based on the application of the above-obtained necessary conditions of optimality. For the sake of definiteness let us consider the scheme of solving the problem (12)–(16). (Clearly, the same scheme can be used for investigation of the remaining above-indicated linear optimal problems with the free right end, in the presence of mixed restrictions and delays).

I. From the condition (21), define uniquely almost everywhere on $[t_0, t_1]$ the function $u(t)$ as a function of $\psi(t)$.

II. From the conditions (22), (18) and (17), define almost everywhere on $[t_0, t_1]$ the functions $\mu_j(t)$, $j = \overline{1, m}$ as functions of $\psi(t)$.

III. Define an absolutely continuous function $\psi(t)$ for all $t \in [t_0, t_1]$ from the equations (20) and (19) as well as from the finite condition $\psi(t_1) = 0$.

IV. Knowing $\psi(t)$, from the conditions (21), define finally $u(t)$ almost everywhere on $[t_0, t_1]$.

V. Knowing $u(t)$ almost everywhere on $[t_0, t_1]$, define $x(t)$, $t \in [t_0, t_1]$, from (13) and (16).

Clearly, the determination of the unknown functions at Steps I and II is not possible for all types of the above-mentioned linear optimal problems. But if one succeeds in carrying out Steps I and II, then the problem of finding the optimal process can be solved to the very end.

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