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ON EXACTNESS OF UPPER ESTIMATES OF THE CHARACTERISTIC EXPONENT OF A LINEAR SYSTEM WITH EXPONENTIALLY DECREASING PERTURBATIONS

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Consider a linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1_A}$$

with piecewise continuous bounded coefficients and a binormal [1, p. 49] system of solutions $X_A(t)$ ordered in increasing exponents. Let $\lambda_i(A)$ be the characteristic exponent of the i-th column of the matrix $X(t)$ and δ_i be the characteristic exponent of the i-th row of the matrix $X_A^{-1}(t)$. By means of the sums $\sigma_i(A) = \lambda_i(A) + \delta_i(A)$, we introduce (see [2]) the number $\sigma_0(A) = \frac{\sigma_m(A)+\sigma_l(A)}{2}$, in which the indices $m \in \{1, \dots, n\}$ and $l \in \{1, \dots, n\}$, $l \neq m$, are defined by the equalities $\sigma_m(A) = \max_i\{\sigma_i(A)\}$, $\sigma_l(A) = \max_{i \neq m}\{\sigma_i(A)\}$.

In [2] it is established that the characteristic exponents $\lambda_1(A+Q) \leq \dots \leq \lambda_n(A+Q)$, of the perturbed system (1_{A+Q}) with a piecewise continuous perturbation $Q(\cdot)$ whose Lyapunov exponent $\lambda[Q]$ satisfies $\lambda[Q] < -\sigma_0(A)$, admit the estimates

$$\lambda_{k(i)}(A+Q) \leq \lambda_i(A) + \frac{\sigma_k(A) - \sigma_i(A)}{2}, \quad i = 1, \dots, n. \tag{2}$$

The question of attainability of these estimates arises. The following theorem gives the positive answer to it in a rather general case.

Theorem. *For any numbers $2 \leq n \in \mathbb{N}$, $m \in \{1, \dots, n\}$, $\lambda_1 \leq \dots \leq \lambda_n$, $0 < \sigma_1 < \sigma_2$ and any $\varepsilon \in (0, (\sigma_2 - \sigma_1)/2)$ satisfying the additional condition $\varepsilon < \lambda_p - \lambda_m + (\sigma_2 - \sigma_1)/2$ if there is (the least) $p \in \{1, \dots, m-1\}$ for which $\lambda_m < \lambda_p + (\sigma_2 - \sigma_1)/2$, there exist: (i) a system (1_A) with infinitely differentiable bounded coefficients such that $\lambda_i(A) = \lambda_i$, $i = 1, \dots, n$, $\sigma_m(A) = \sigma_2$ and $\sigma_i(A) = \sigma_1$ for $m \neq i \in \{1, \dots, n\}$; (ii) an analytical perturbation $Q(\cdot)$ with the Lyapunov exponent $\lambda[Q] < -\sigma_0 = -\frac{\sigma_1+\sigma_2}{2}$ such that the perturbed system (1_{A+Q}) has all different characteristic exponents and: 1) in the absence of the aboves pecified p, $\lambda_m(A+Q) = \lambda_m$ and*

$$\lambda_i(A+Q) = \lambda_i + \frac{\sigma_2 - \sigma_1}{2} - \frac{1+n-i}{n}\varepsilon, \tag{3}$$

for $m \neq i = 1, \dots, n$; 2) in the presence of such p, $\lambda_p(A+Q) = \lambda_m$, $\lambda_i(A+Q) = \lambda_{i-1} + \frac{\sigma_2-\sigma_1}{2} - \frac{1+n-i}{n}\varepsilon$, $i = p+1, \dots, m$, and other exponents, determined by the formula (3).

Proof. We will construct not the system (1_A) itself but its fundamental system of solutions $X(t) = \operatorname{diag}[\exp tf_1(t), \dots, \exp tf_n(t)]$. Fix $\theta > 1$ and a rather small $\varepsilon > 0$

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satisfying the necessary conditions. On the segments $\nabla_k(\gamma) \equiv [\theta^{k+\gamma}, \theta^{k+1}]$, $k \geq 0$, with the determined below number $\gamma \in (0,1)$, we define the functions $f_i(t)$ by

$$f_i(t) = \begin{cases} \lambda_i, & t \in \nabla_{2k}(\gamma), \\ -\delta_i, & t \in \nabla_{2k+1}(\gamma), \end{cases} \quad m \neq i = 1, \dots, n,$$
$$f_m(t) = \begin{cases} \lambda_m, & t \in \nabla_{2k+1}(\gamma), \\ -\delta_m, & t \in \nabla_{2k}(\gamma). \end{cases}$$

On the initial segment [0, 1] let's assume $f_i(t) = -\delta_i$, $m \neq i = 1, \dots, n$, $f_m(t) = \lambda_m$. On other intervals $\Delta_k(\gamma) \equiv (\theta^k, \theta^{k+\gamma})$, $k \geq 0$, the functions $f_i(t)$, $i = 1, \dots, n$, are defined by means of a special infinitely differentiable function

$$f(t;\eta_1,a;\eta_2,b) = a + (b-a)\exp\{-\ln^{-2}(t/\eta_1) \times$$
$$\times \exp[-\ln^{-2}(t/\eta_2)]\}, \quad \eta_1 < t < \eta_2,$$

updating [3] the standard function from [4, p. 54]. The functions $f_i(t)$ on the interval $\Delta_k(\gamma)$, $k \geq 0$, are defined by the equality

$$f_i(t) = f(t; t_k, f_i(t_k); t_{k+\gamma}, f_i(t_{k+\gamma})), \quad i = 1, \dots, n,$$

where $t_\alpha \equiv \theta^\alpha$. It is easy to see that the system (1_A) so constructed has infinitely differentiable coefficients with all derivatives bounded.

We construct an n-th order matrix of perturbation $Q(\cdot)$ as having nonzero elements only in the m-th line, except $q_{mm}(t) = 0$, $t \geq 0$. These elements look like

$$q_{mi}(t,\varepsilon_i) = \exp(-\sigma_0 - \varepsilon_i)t, \quad i \neq m, \quad t \geq 0, \tag{4}$$

with specially determined below constant ε_i, $i \neq m$.

We choose $\gamma > 0$ involved in the definition of the system (1_A) so small that

$$[2(\lambda_n - \lambda_1) + \sigma_2 - \sigma_1](\theta^\gamma - 1) < 2\varepsilon/n. \tag{5}$$

Denote the i-th solutionof the system (1_{A+Q}), by $Y_i(t,\varepsilon_i)$, $i \neq m$. Its components are

$$Y_{ji}(t,\varepsilon_i) = 0, \quad j \neq i, m; \quad Y_{ii}(t,\varepsilon_i) = x_i(t) \equiv \exp t f_i(t);$$
$$Y_{mi}(t,\varepsilon_i) = x_m(t)\left[Y_{mi}(\varepsilon_i) + \int\limits_0^t q_{mi}(\tau,\varepsilon_i)x_i(\tau)x_m^{-1}(\tau)d\tau\right],$$

where the constant $Y_{mi}(\varepsilon_i) = 0$, if the Lyapunov exponent $\lambda[q_{mi}x_i/x_m]$ of the integrand is not less than zero, and $Y_{mi}(\varepsilon_i) = -\int\limits_0^{+\infty} q_{mi}x_i x_m^{-1} d\tau$. It is obvious that the m-th solution of the system (1_{A+Q}) is the vector-function $Y_m(t)$ with the unique different from zero component $Y_{mm}(t) = x_m(t)$.

For any fixed $i \neq m$, let's establish now the existence of a constant $\varepsilon_i = \tilde{\varepsilon}_i > 0$ that the corresponding solution $Y_i(t,\tilde{\varepsilon}_i)$ has an exponent $\lambda[Y_i] = \lambda_i + (\sigma_2 - \sigma_1)/2 - (1+n-i)\varepsilon/n$. For this purpose at first we shall establish existence of a constant $\varepsilon_i^{(1)} > 0$ such that the inequality $\lambda[Y_i] > \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon/n$ be true. Really, in the case $\lambda_i + \delta_m - \sigma_0 > 0$, due to the condition (5) a constant $\varepsilon_i^{(1)} > 0$ exists such that $\lambda_i + \delta_m - \sigma_0 - \varepsilon_i > 0$ and for sequence $\{t_{2k+1+\gamma}\}$ the following estimates are fulfilled

$$\lambda[Y_i] \geq \overline{\lim_{k\to\infty}}\, t_{2k+1+\gamma}^{-1} ln Y_{mi}(t_{2k+1+\gamma}, \varepsilon_i^{(1)}) \geq \lambda_m + (\lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(1)})\theta^{-\gamma} \geq$$
$$\geq \lambda_i + (\sigma_2 - \sigma_1)/2 - [\lambda_n - \lambda_1 + (\sigma_2 - \sigma_1)/2](1 - \theta^{-\gamma}) - \varepsilon_i^{(1)}\theta^{-\gamma} >$$
$$> \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon/n. \tag{6_1}$$

In the case $\lambda_i + \delta_m - \sigma_0 \le 0$, on the basis of the same condition (5), there exists a constant $\varepsilon_i^{(1)} > 0$ such that for a sequence $\{t_{2k}\}$ the inequalities

$$\lambda[Y_i] \ge \overline{\lim_{k\to\infty}}\, t_{2k}^{-1} ln|Y_{mi}(t_{2k}, \varepsilon_i^{(1)})| \ge \lambda_m + \overline{\lim_{k\to\infty}}\, t_{2k}^{-1} \int\limits_{t_{2k+\gamma}}^{t_{2k+\gamma}+1} q_{mi} x_i x_m^{-1} d\tau =$$

$$= \lambda_m + (\lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(1)})\theta^\gamma \ge \lambda_i + (\sigma_2 - \sigma_1)/2 + [\lambda_1 - \lambda_n -$$

$$-(\sigma_2 - \sigma_1)/2](\theta^\gamma - 1) - \varepsilon_i^{(1)}\theta^\gamma > \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon/n. \qquad (6_2)$$

Let's establish now the existence of a constant $\varepsilon_i^{(2)} > \varepsilon_i^{(1)}$ such that $\lambda[Y_{mi}(\cdot, \varepsilon_i^{(2)})] < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon$. In the first place, choose this constant $\varepsilon_i^{(2)}$ so large, that the $\lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(2)} < 0$. Then, due to the Lyapunov lemma concerning the exponent of the integral, we have

$$\lambda[Y_{mi}(\cdot, \varepsilon_i^{(2)})] \le \lambda_m + \lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(2)} < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon \qquad (7)$$

under the second additional condition $\varepsilon_i^{(2)} > \varepsilon$ on $\varepsilon_i^{(2)}$. Thus, the inequality $\lambda[Y_i(\cdot, \varepsilon_i^{(2)})] < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon$ becomes obvious. Due to established in [5] continuous dependence of the exponent $\lambda[Y_i(\cdot, \varepsilon_i])$ on the parameter $\varepsilon_i > 0$, from (6_1), (6_2) and (7) it follows the existence of the required $\tilde{\varepsilon}_i \in (\varepsilon_i^{(1)}, \varepsilon_i^{(2)})$.

For the completion of the proof of the theorem it is necessary to order the solutions $Y_i(t, \tilde{\varepsilon}_i)$, $i \ne m$, and $Y_m(t)$ according to increase of their exponents. In the first case $\lambda_m \ge \lambda_{m-1} + (\sigma_2 - \sigma_1)/2$, mentioned in the formulation of the theorem, we have obvious inequalities $\lambda[Y_1] < \dots < \lambda[Y_{m-1}] < \lambda[Y_m] = \lambda_m < \lambda[Y_{m+1}] < \dots < \lambda[Y_n]$ and so all characteristic exponents of the perturbed system (1_{A+Q}) are different. In the second case of existence of (the least) $p \in \{1, \dots, m-1\}$ for which $\lambda_m < \lambda_p + (\sigma_2 - \sigma_1)/2$, the fundamental system, ordered in decreasing of the exponents, looks like $Y(t) = [Y_1(t), \dots, Y_{p-1}(t), Y_m(t), Y_p(t), \dots, Y_{m-1}(t), Y_{m+1}(t), \dots, Y_n(t)]$, and the exponents of its solutions are all various, because of the choice, in this case, of the number $\varepsilon > 0$, and for the obtained exponents $\lambda[Y_i]$ the inequalities

$$\begin{gathered}
\lambda[Y_{p-1}] = \lambda_{p-1} + (\sigma_2 - \sigma_1)/2 - (2 + n - p)\varepsilon/n < \\
< \lambda_{p-1} + (\sigma_2 - \sigma_1)/2 \le \lambda_m = \lambda[Y_m] < \lambda_p + (\sigma_2 - \sigma_1)/2 - \varepsilon \le \\
\le \lambda[Y_p] = \lambda_p + (\sigma_2 - \sigma_1)/2 - (1 + n - p)\varepsilon/n < \\
< \dots < \lambda[Y_{m-1}] < \lambda[Y_{m+1}] < \dots < \lambda[Y_n]
\end{gathered}$$

are true. Thus, the fundamental matrix $Y(t)$ is normal and the system (1_{A+Q}), in this case, has all different characteristic exponents specified in the formulation of the theorem. ■

Remark. For the characteristic exponents of the systems (1_A) and (1_{A+Q}) constructed in the proof of the theorem, the attainability of the estimates (2) is shown by the inequalities $\lambda_{k(i)}(A + Q) \ge \lambda_i(A) + (\sigma_2 - \sigma_1)/2 - \varepsilon$, $\quad i = 1, \dots, n$, which are valid for the permutation

$$k(i) = \begin{cases} i, & \text{if } \; i = 1, \dots, p-1, \dots, m+1, \dots, n, \\ p, & \text{if } \; i = m, \\ i+1, & \text{if } \; i = p, \dots, m-1. \end{cases}$$

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