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ON SUFFICIENT CONDITIONS OF EXISTENCE AND UNIQUENESS OF PERIODIC IN A STRIP SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS

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Let $b > 0$ and $\mathcal{D}_b = \mathbb{R} \times [0, b]$. In the strip $\mathcal{D}_b$, consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial y}\right), \tag{1}$$

where $f : \mathcal{D}_b \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the local Carathéodory conditions, i.e. $f(\cdot, \cdot, z_0, z_1) : \mathcal{D}_b \to \mathbb{R}$ is measurable for every $(z_0, z_1) \in \mathbb{R}^2$, $f(x, y, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for almost every $(x, y) \in \mathcal{D}_b$ and the function

$$\sup\{|f(\cdot, \cdot, z_0, z_1)| : |z_0| + |z_1| \le \rho\}$$

is summable on the rectangle $[-a, a] \times [0, b]$ for any $\rho > 0$ $a > 0$.

By solution of the equation (1) we understand a locally absolutely continuous function $u : \mathcal{D}_b \to \mathbb{R}$ (see [2]) satisfying the equation (1) almost everywhere in $\mathcal{D}_b$.

We study the case where f is ω-periodic in the first argument for some $\omega > 0$, i.e.,

$$f(x + \omega, y, z_0, z_1) = f(x, y, z_0, z_1) \quad \text{for} \quad (x, y) \in \mathcal{D}_b,\ (z_0, z_1) \in \mathbb{R}^2.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous ω-periodic function. Below we formulate sufficient conditions of existence and uniqueness of the solution of the equation (1) satisfying

$$u(x, 0) = \varphi(x), \quad u(x + \omega, y) = u(x, y) \quad \text{for} \quad (x, y) \in \mathcal{D}_b. \tag{2}$$

Note that earlier the problem (1),(2) has been investigated when f is linear or quasilinear with respect to the last two arguments (see. [1,4-10]).

Theorem 1. *Let the inequalities*

$$|f(x, y, z_0, 0)| \le p_0(x, y)(1 + |z_0|), \tag{3}$$

$$\sigma(y)[f(x, y, z_0, z_1) - f(x, y, z_0, \overline{z_1})] \operatorname{sign}(z_1 - \overline{z_1}) \le p(x, y)|z_1 - \overline{z_1}| \tag{4}$$

take place on $\mathcal{D}_b \times \mathbb{R}^2$, *where* $p_0 : [0, \omega] \times [0, b] \to \mathbb{R}_+$ *is summable,* $\sigma : [0, b] \to \{-1, 1\}$ *is measurable and* $p : [0, \omega] \times [0, b] \to \mathbb{R}$ *is a measurable function such that* $p(x, \cdot) : [0, b] \to \mathbb{R}$ *is continuous almost for every* x, $\max\{|p(\cdot, t)| : t \in [0, b]\} : [0, \omega] \to \mathbb{R}_+$ *is summable and*

$$\int_0^\omega p(s, y)\, ds < 0 \quad \textit{for} \quad y \in [0, b]. \tag{5}$$

Then the problem (1), (2) *is solvable.*

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Remark 1. The conditions (3) and (5) are essential and cannot be weakened. Violation of the condition (3) may result in the loss of global solvability in the whole strip $\mathcal{D}_b$. As an example, consider the problem

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y} - u^{1+\varepsilon} - 1, \quad u(x,0) = 0, \quad u(x+\omega, y) = u(x,y),$$

in the strip $\mathcal{D}_b$, where $b = 2\int_0^{+\infty} \frac{d\xi}{1+\xi^{1+\varepsilon}}$, $\varepsilon > 0$ is an arbitrary constant. The given problem may have at most one solution (see Theorem 2 below). Therefore if $u(x,y)$ is a solution of the given problem, then $u(x,y) = u(y)$ and it is simultaneously the solution of the Cauchy problem

$$\frac{du}{dy} = u^{1+\varepsilon} + 1, \quad u(0) = 0,$$

defined on the segment $[0,b]$. But it is impossible since $\lim_{y \to \frac{b}{2}} u(y) = +\infty$.

On the other hand, violation of the condition (5) may result in the loss of solvability of the problem (1),(2). To convince ourselves that is so, consider the problem

$$\frac{\partial^2 u}{\partial x \partial y} = y\frac{\partial u}{\partial y} + 1, \quad u(x,0) = 0, \quad u(x+\omega, y) = u(x,y)$$

for which all conditions of Theorem 1, except of (5), are fulfilled. Nevertheless, the above problem has no solution. In fact, otherwise we should have

$$\frac{\partial u(x,y)}{\partial y} = -\frac{1}{y} \quad \text{for} \quad 0 < y \le b.$$

But this contradicts the absolute continuity of u.

Theorem 2. *Let the conditions* (4) *and* (5) *hold and let there exist nonnegative summable functions* $c_0 : [0,\omega] \times [0,b] \to \mathbb{R}_+$ *and* $c_1 : [0,b] \to \mathbb{R}_+$ *such that the inequality*

$$|f(x,y,z_0,z_1) - f(x,y,\overline{z_0},\overline{z_1})| \le c_0(x,y)|z_0 - \overline{z_0}| + c_1(x)|z_1 - \overline{z_1}|$$

holds on $\mathcal{D}_b \times \mathbb{R}^2$. *Then the problem* (1), (2) *has at most one solution.*

Finally, consider the case where $f(x,y,z_0,z_1) \equiv f(x,y,z_0)$, i.e., where the equation (1) has the form

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u). \tag{6}$$

In addition, assume that f has partial derivatives in the second and the third arguments satisfying local Carathéodory conditions. Put

$$g_0(x,y,z) = \frac{\partial f(x,y,z)}{\partial y}, \quad g_1(x,y,z) = \frac{\partial f(x,y,z)}{\partial z}.$$

Theorem 3. *Let there exist a positive constant* l *and a measurable function* $\sigma : [0,b] \to \{-1,1\}$ *such that the inequalities*

$$|g_0(x,y,z)| \le l(1+|z|), \quad |g_1(x,y,z)| \le l,$$

and

$$\int_0^b g(s)\,ds > 0 \tag{7}$$

hold on $[0,\omega]\times[0,b]\times\mathbb{R}$, *where* $g(x)=\operatorname{ess\,inf}\{\sigma(y)g_1(x,t,z): t\in[0,b],\ z\in\mathbb{R}\}$. *Then the problem* (6), (2) *is solvable if and only if*

$$\int_0^\omega f(s,0,\varphi(s))\,ds=0.$$

If, besides, g_i $(i=0,1)$ *are locally Lipschitz continuous in* z, *then the problem* (6), (2) *is uniquely solvable.*

Remark 2. The condition (7) in Theorem 3 is essential and cannot be neglected. For example, it is obvious that the problem

$$\frac{\partial^2 u}{\partial x\partial y}=(1+u^2)^{\frac{1}{2}}-1+\sin(\frac{2\pi}{\omega}x)$$
$$u(x,0)=0,\quad u(x+\omega,y)=u(x,y)$$

has no solution, although all conditions of Theorem 3, except (7), hold since

$$g_1(x,y,z)=(1+z^2)^{-\frac{1}{2}}z.$$

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