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PSEUDO BL-ALGEBRAS AND $DR\ell$-MONOIDS

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Abstract. It is shown that pseudo BL-algebras are categorically equivalent to certain bounded $DR\ell$-monoids. Using this result, we obtain some properties of pseudo BL-algebras, in particular, we can characterize congruence kernels by means of normal filters. Further, we deal with representable pseudo BL-algebras and, in conclusion, we prove that they form a variety.

Keywords: pseudo BL-algebra, $DR\ell$-monoid, filter, polar, representable pseudo BL-algebra

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1. Connections between pseudo BL-algebras and $DR\ell$-monoids

Recently, pseudo BL-algebras were introduced by A. Di Nola, G. Georgescu and A. Iorgulescu in [3] as a noncommutative extension of Hájek's BL-algebras (see [6]).

An algebra $\mathfrak{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2,2,2,2,2,0,0\rangle$ is called a *pseudo BL-algebra* iff $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a monoid and the following conditions are satisfied for all $x, y, z \in A$:

(1) $x \odot y \leqslant z$ iff $x \leqslant y \rightarrow z$ iff $y \leqslant x \rightsquigarrow z$,

(2) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$,

(3) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

By [3, Corollary 3.29], pseudo BL-algebras satisfying the identity

$$(x \rightsquigarrow 0) \rightarrow 0 = (x \rightarrow 0) \rightsquigarrow 0 = x$$

are the duals of pseudo MV-algebras.

In the same way, (noncommutative) $DR\ell$-monoids extend Swamy's $DR\ell$-semigroups which were introduced in [12] as a common generalization of abelian ℓ-groups and Brouwerian algebras.

An algebra $\mathfrak{A} = (A, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$ is a *dually residuated lattice ordered monoid*, or simply a *$DR\ell$-monoid*, iff

(1) $(A, +, 0, \vee, \wedge)$ is an ℓ-monoid, that is, $(A, +, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and, for any $x, y, s, t \in A$, the following distributive laws are satisfied:

$$s + (x \vee y) + t = (s + x + t) \vee (s + y + t),$$
$$s + (x \wedge y) + t = (s + x + t) \wedge (s + y + t);$$

(2) for any $x, y \in A$, $x \rightharpoonup y$ is the least $s \in A$ such that $s + y \geqslant x$, and $x \leftharpoondown y$ is the least $t \in A$ such that $y + t \geqslant x$;

(3) $\mathfrak{A}$ fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leqslant x \vee y,\ y + ((x \leftharpoondown y) \vee 0) \leqslant x \vee y,$$
$$x \rightharpoonup x \geqslant 0,\ x \leftharpoondown x \geqslant 0.$$

Note that the inequalities $x \rightharpoonup x \geqslant 0$ and $x \leftharpoondown x \geqslant 0$ can be omitted, and the condition (2) is equivalent to the system of identities (see [10])

$$(x \rightharpoonup y) + y \geqslant x,\ y + (x \leftharpoondown y) \geqslant x,$$
$$x \rightharpoonup y \leqslant (x \vee z) \rightharpoonup y,\ x \leftharpoondown y \leqslant (x \vee z) \leftharpoondown y,$$
$$(x + y) \rightharpoonup y \leqslant x,\ (y + x) \leftharpoondown y \leqslant x.$$

In [11], mutual relationships between BL-algebras and bounded representable commutative $DR\ell$-monoids are described.

Theorem 1.1. *Let $\mathfrak{A} = (A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo BL-algebra. If we set*

$$x + y := x \odot y,\ x \vee_d y := x \wedge y,\ x \wedge_d y := x \vee y,$$
$$x \rightharpoonup y := y \rightarrow x,\ x \leftharpoondown y := y \rightsquigarrow x,\ 0_d := 1,\ 1_d := 0$$

for any $x, y \in A$, then $\mathfrak{A}_d = (A, +, 0_d, \vee_d, \wedge_d, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid with the greatest element 1_d. In addition, this $DR\ell$-monoid satisfies the identities

$$(*)\qquad \begin{aligned} (x \rightharpoonup y) \wedge_d (y \rightharpoonup x) &= 0_d,\\ (x \leftharpoondown y) \wedge_d (y \leftharpoondown x) &= 0_d. \end{aligned}$$

Proof. Since $(A, \odot, 1, \vee, \wedge)$ is an ℓ-monoid, by [3, Propositions 3.3, 3.9], so is $(A, +, 0_d, \vee_d, \wedge_d)$. The rest follows directly by the definitions. Note that if a $DR\ell$-monoid $\mathfrak{A}_d$ contains the greatest element 1_d then 0_d is its least element, by [8, Theorem 1.2.3]. □

In view of Theorem 1.1, it is easily seen that in the definition of a pseudo BL-algebra, the condition (1) can be equivalently replaced by the following identities:

$$\begin{gathered}(x \to y) \odot x \leqslant y,\ x \odot (x \rightsquigarrow y) \leqslant y,\\ x \to y \geqslant x \to (y \wedge z),\ x \rightsquigarrow y \geqslant x \rightsquigarrow (y \wedge z),\\ y \to (x \odot y) \geqslant x,\ y \rightsquigarrow (y \odot x) \geqslant x.\end{gathered}$$

Consequently, pseudo BL-algebras form a variety of algebras of type $\langle 2,2,2,2,2,0,0\rangle$. This variety is arithmetical; in accordance with [8, Theorem 3.1.1], the Pixley term of the variety of pseudo BL-algebras can be taken as follows:

$$p(x,y,z) = ((x \rightsquigarrow y) \to z) \wedge ((z \rightsquigarrow y) \to x) \wedge (x \vee z).$$

Theorem 1.2. *Let* $\mathfrak{A} = (A, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ *be a* $DR\ell$*-monoid with the greatest element* 1. *For any* $x, y \in A$ *set*

$$\begin{gathered}x \odot y := x + y,\ x \vee_d y := x \wedge y,\ x \wedge_d y := x \vee y,\\ x \to y := y \rightharpoonup x,\ x \rightsquigarrow y := y \leftharpoondown x,\ 0_d := 1,\ 1_d := 0.\end{gathered}$$

Then $\mathfrak{A}_d = (A, \vee_d, \wedge_d, \odot, \to, \rightsquigarrow, 0_d, 1_d)$ *is a pseudo* BL*-algebra if and only if* $\mathfrak{A}$ *satisfies* $(*)$.

Proof. In any $DR\ell$-monoid we have

$$x \vee y = ((y \rightharpoonup x) \vee 0) + x = x + ((y \leftharpoondown x) \vee 0).$$

Since $\mathfrak{A}$ is bounded, that is, $0 \leqslant x \leqslant 1$ for any $x \in A$, it follows that

$$x \wedge_d y = (x \to y) \odot x = x \odot (x \rightsquigarrow y).$$

The rest is obvious. □

Let $\mathcal{PBL}$ be the category of pseudo BL-algebras, that is, the category whose objects are pseudo BL-algebras and morphisms are homomorphisms of pseudo BL-algebras. Let $\mathcal{DRL}_{1(*)}$ be the category of bounded $DR\ell$-monoids satisfying $(*)$. Its morphisms are homomorphisms of $DR\ell$-monoids which preserve also 1, thus in the sequel, bounded $DR\ell$-monoids are regarded as algebras $(A, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown, 1)$ of type $\langle 2,0,2,2,2,2,0\rangle$.

Theorem 1.3. *The categories* $\mathcal{PBL}$ *and* $\mathcal{DRL}_{1(*)}$ *are equivalent.*

Proof. Theorems 1.1 and 1.2 enable us to define a functor $\mathcal{F}\colon \mathcal{PBL} \to \mathcal{DRL}_{1(*)}$ as follows: (i) $\mathcal{F}(\mathfrak{A}) = \mathfrak{A}_d$ for any pseudo BL-algebra $\mathfrak{A}$, and (ii) $\mathcal{F}(h) = h$ for any pseudo BL-homomorphism h. It is easy to see that $\mathcal{F}$ is really a categorical equivalence. □

2. FILTERS

According to [3], a subset F of a pseudo BL-algebra $\mathfrak{A}$ with the following properties is said to be a *filter of* $\mathfrak{A}$:
(F1) $1 \in F$;
(F2) $\forall x, y \in F; x \odot y \in F$;
(F3) $\forall x \in F\ \forall y \in A; x \leqslant y \Longrightarrow y \in F$.

For any subset $M \subseteq A$, the intersection of all filters containing M is called a *filter generated by* M and denoted by $[M)$. It is clear that

$$[M) = \{x \in A;\ x \geqslant a_1 \odot ... \odot a_n \text{ for some } a_1, ..., a_n \in M \text{ and } n \geqslant 1\},$$

and if we write briefly $[a)$ for $[\{a\})$ then

$$[a) = \{x \in A;\ x \geqslant a^n \text{ for some } n \geqslant 1\}.$$

In Section 1, we have already proved that $DR\ell$-monoids include the duals of pseudo BL-algebras. It is obvious that $F \subseteq A$ is a filter of a pseudo BL-algebra $\mathfrak{A}$ iff it is an ideal of the induced bounded $DR\ell$-monoid $\mathfrak{A}_d$, that is,
(I1) $0_d \in F$;
(I2) $\forall x, y \in F; x + y \in F$;
(I3) $\forall x \in F\ \forall y \in A; x \geqslant_d y \Longrightarrow y \in F$.

Ideals of noncommutative $DR\ell$-monoids were studied in [9]. Considering the above facts, we immediately obtain the following results.

Proposition 2.1. *The set of all filters of any pseudo BL-algebra $\mathfrak{A}$, ordered by set inclusion, is an algebraic Brouwerian lattice. For any filters F, G of $\mathfrak{A}$, the relative pseudocomplement of F with respect to G is given by*

$$F * G = \{a \in A;\ a \vee x \in G \text{ for all } x \in F\}.$$

Let $\mathfrak{A}$ be a pseudo BL-algebra and $X \subseteq A$. The set

$$X^{\perp} = \{a \in A;\ a \vee x = 1 \text{ for any } x \in X\}$$

is called the *polar of* X. For any $x \in A$ we write $x^{\perp}$ instead of $\{x\}^{\perp}$.

A subset X of A is a *polar in* $\mathfrak{A}$ iff $X = Y^{\perp}$ for some $Y \subseteq A$.

Proposition 2.2 [3, Propositions 4.38, 4.39]**.** *For all subsets X, Y of a pseudo BL-algebra $\mathfrak{A}$,* (i) *$X^{\perp}$ is a filter of $\mathfrak{A}$,* (ii) *$X \subseteq X^{\perp\perp}$,* (iii) *$X \subseteq Y$ implies $Y^{\perp} \subseteq X^{\perp}$,* (iv) *$X^{\perp} = X^{\perp\perp\perp}$.*

Proposition 2.3. *For any subset X of a pseudo BL-algebra $\mathfrak{A}$, X is a polar in $\mathfrak{A}$ iff $X = X^{\perp\perp}$.*

Proof. Let $X = Y^{\perp}$; then $X^{\perp\perp} = Y^{\perp\perp\perp} = Y^{\perp} = X$. □

By Proposition 2.1, the pseudocomplement of a filter F is

$$F^{*} = \{a \in A;\ a \vee x = 1 \text{ for any } x \in F\}.$$

Moreover, it is clear that $F^{\perp} = F^{*}$ whenever F is a filter, and conversely, any polar is the pseudocomplement of some filter; in fact, $X = (X^{\perp})^{*}$. Thus the polars in any pseudo BL-algebra are precisely the pseudocomplements in the lattice of its filters. Therefore, by the Glivenko-Frink Theorem, we directly obtain

Theorem 2.4. *The set of all polars in any pseudo BL-algebra, ordered by set inclusion, is a complete Boolean algebra.*

A filter F of a pseudo BL-algebra $\mathfrak{A}$ is said to be *normal* iff it satisfies the following condition for each $x, y \in A$:

$$x \to y \in F \iff x \rightsquigarrow y \in F.$$

Proposition 2.5. *For any filter F, the following conditions are equivalent:*
(i) *F is normal;*
(ii) *$x \odot F = F \odot x$ for each $x \in A$.*

Proposition 2.6. *If F and G are normal filters of $\mathfrak{A}$ then*

$$F \vee G = \{x \in A;\ x \geqslant a \odot b \text{ for some } a \in F, b \in G\}.$$

In addition, $F \vee G$ is a normal filter. Consequently, normal filters of any pseudo BL-algebra form a complete sublattice of the lattice of all its filters.

Theorem 2.7. *In any pseudo BL-algebra, there is a one-to-one correspondence between the normal filters and the congruence relations. In fact, F corresponds to $\Theta(F)$ defined by*

$$\langle x, y\rangle \in \Theta(F) = \Theta_1(F) \iff (x \to y) \wedge (y \to x) \in F,$$

or equivalently,

$$\langle x, y\rangle \in \Theta(F) = \Theta_2(F) \iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F.$$

As proved in [3], and in general for noncommutative $DR\ell$-monoids in [9], if F is not a normal filter then the binary relations defined in the previous theorem, $\Theta_1(F)$ and $\Theta_2(F)$, are two distinct congruence relations on the distributive lattice $\mathfrak{L}(\mathfrak{A}) = (A, \vee, \wedge, 0, 1)$. In the quotient lattices $\mathfrak{L}(\mathfrak{A})/\Theta_1(F)$ and $\mathfrak{L}(\mathfrak{A})/\Theta_2(F)$ we have

$$[x]\Theta_1(F) \leqslant [y]\Theta_1(F) \iff x \to y \in F \tag{2.1}$$

and

$$[x]\Theta_2(F) \leqslant [y]\Theta_2(F) \iff x \rightsquigarrow y \in F, \tag{2.2}$$

respectively.

Let $\mathfrak{A}$ be a pseudo BL-algebra. A filter F of $\mathfrak{A}$ is said to be *prime* if it is a finitely meet-irreducible element in the lattice of filters of $\mathfrak{A}$.

By [3, Theorem 4.28], for any filter F of a pseudo BL-algebra $\mathfrak{A}$ and for each ideal I of the lattice $\mathfrak{L}(\mathfrak{A})$, if $F \cap I = \emptyset$ then there exists a prime filter P of $\mathfrak{A}$ with $F \subseteq P$ and $P \cap I = \emptyset$. Consequently, every proper filter is the intersection of all prime filters including it. In particular, the intersection of all prime filters is equal to $\{1\}$.

Theorem 2.8. *For any filter F of a pseudo BL-algebra $\mathfrak{A}$, the following conditions are equivalent:*

(i) *F is prime;*
(ii) *for all filters G, H of $\mathfrak{A}$, $G \cap H \subseteq F$ implies $G \subseteq F$ or $H \subseteq F$;*
(iii) *for any $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$;*
(iv) *for any $x, y \in A$, $x \vee y = 1$ implies $x \in F$ or $y \in F$;*
(v) *for any $x, y \in A$, $x \to y \in F$ or $y \to x \in F$;*
(vi) *for any $x, y \in A$, $x \rightsquigarrow y \in F$ or $y \rightsquigarrow x \in F$;*
(vii) *$\mathfrak{L}(\mathfrak{A})/\Theta_1(F)$ is totally ordered;*
(viii) *$\mathfrak{L}(\mathfrak{A})/\Theta_2(F)$ is totally ordered;*
(ix) *the set of all filters including F is totally ordered under set inclusion.*

Remark. The equivalence of (iii), (v), (vi), (vii) and (viii) is due to [3, Proposition 4.25].

Proof. (i) ⇒ (ii): Using the distributivity of the lattice of filters, $G \cap H \subseteq F$ implies $F = F \vee (G \cap H) = (F \vee G) \cap (F \vee H)$, whence $F = F \vee G$ or $F = F \vee H$, that is, $F \supseteq G$ or $F \supseteq H$.

(ii) ⇒ (iii): Obviously, $x \vee y \in F$ yields $[x) \cap [y) = [x \vee y) \subseteq F$. Hence, by (ii), $[x) \subseteq F$ or $[y) \subseteq F$ and thus $x \in F$ or $y \in F$.

(iii) ⇒ (iv): This is evident since $1 \in F$.

(iv) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi): By the definition of a pseudo BL-algebra,

$$(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1,$$

which implies the assertion by (iv).

(v) $\Rightarrow$ (vii) and (vi) $\Rightarrow$ (viii): This is obvious from (2.1) and (2.2), respectively.

(vii) $\Rightarrow$ (ix): If $F \subseteq G, H$ and neither $G \subseteq H$ nor $H \subseteq G$ then there exist $a, b \in A$ with $a \in G \setminus H$ and $b \in H \setminus G$. For instance, let $a \to b \in F$. Then $b \geqslant a \wedge b = (a \to b) \odot a \in G$, whence $b \in G$; a contradiction. Similarly (viii) $\Rightarrow$ (ix).

(ix) $\Rightarrow$ (i): $F = G \cap H$ entails $F = G$ or $F = H$, because either $G \subseteq H$ or $H \subseteq G$. ☐

3. REPRESENTABLE PSEUDO BL-ALGEBRAS

Proposition 3.1. *If P is a minimal prime filter of a pseudo BL-algebra $\mathfrak{A}$ then $A \setminus P$ is a maximal ideal of the lattice $\mathfrak{L}(\mathfrak{A})$.*

Proof. By Zorn's Lemma, there is a maximal ideal I of $\mathfrak{L}(\mathfrak{A})$ with $A \setminus P \subseteq I$. (Since P is also a prime filter of $\mathfrak{L}(\mathfrak{A})$, it follows that $A \setminus P$ is a prime ideal of $\mathfrak{L}(\mathfrak{A})$ which is included in some maximal (prime) ideal.) We will show that $I = A \setminus P$. Denote $Q = \bigcup\{a^{\perp};\ a \in I\}$. We claim that $P = Q$.

If $x \in a^{\perp}$ for some $a \in I$, then $x \vee a = 1$ and $x \notin I$. Indeed, if $x \in I$ then $x \vee a \neq 1$ since $x \vee a = 1$ would mean $I = A$. Thus $x \in A \setminus I \subseteq A \setminus (A \setminus P) = P$, whence $a^{\perp} \subseteq A \setminus I \subseteq P$ and consequently, $Q \subseteq A \setminus I \subseteq P$.

We shall now prove that Q is a prime filter of $\mathfrak{A}$. (F1): Since any principal polar $a^{\perp}$ contains 1, so does Q. (F2): If $x, y \in Q$, that is, $x \in a^{\perp}, y \in b^{\perp}$ for some $a, b \in I$, then $a \vee b \in I$ and

$$(x \odot y) \vee a \vee b \geqslant (x \vee a \vee b) \odot (y \vee a \vee b) = 1 \odot 1 = 1.$$

Therefore $x \odot y \in (a \vee b)^{\perp} \subseteq Q$. (F3): It is obvious since $a^{\perp}$ is a filter of $\mathfrak{A}$ for each $a \in I$.

To prove that Q is prime, suppose $x \vee y = 1$ and $x \notin Q$, that is, $x \vee a \neq 1$ for all $a \in I$. If $x \notin I$ then the ideal in the lattice $\mathfrak{L}(\mathfrak{A})$ generated by $I \cup \{x\}$, $(I \cup \{x\}]$, is proper, i.e., $A \setminus P \subseteq I \subset (I \cup \{x\}] \neq A$, since $(I \cup \{x\}] = A$ would entail $1 \leqslant x \vee a$ for some $a \in I$; a contradiction. Hence $x \in I$ and thus $y \in x^{\perp} \subseteq Q$, proving that Q is prime.

However, P is a minimal prime filter of $\mathfrak{A}$; thus $Q \subseteq A \setminus I \subseteq P$ yields $Q = A \setminus I = P$ as claimed. Therefore $I = A \setminus P$. ☐

Corollary 3.2. *If P is a minimal prime filter then*

$$P = \bigcup\{a^\perp;\ a \notin P\}.$$

Proof. By the proof of the previous proposition, $P = \bigcup\{a^\perp;\ a \in I\}$, where $I = A \setminus P$. ☐

A pseudo BL-algebra is said to be *representable* if it is a subdirect product of linearly ordered pseudo BL-algebras.

By Theorems 2.7 and 2.8, subdirect representations by totally ordered pseudo BL-algebras are associated with families of normal prime filters whose intersections are precisely $\{1\}$. Therefore it is obvious that every BL-algebra is representable (see also [11]). In contrast, for pseudo BL-algebras, this assertion fails.

The following results generalize the similar properties of pseudo MV-algebras, [4, Theorem 2.20], [1, Theorem 5.9], and [2, Theorem 6.11].

Theorem 3.3. *For any pseudo BL-algebra $\mathfrak{A}$, the following statements are equivalent.*

(i) *$\mathfrak{A}$ is representable.*

(ii) *There exists a family $\{P_i\}_{i\in I}$ of normal prime filters of $\mathfrak{A}$ such that*

$$\bigcap_{i\in I} P_i = \{1\}.$$

(iii) *Any polar of $\mathfrak{A}$ is a normal filter of $\mathfrak{A}$.*

(iv) *Any principal polar is a normal filter.*

(v) *Any minimal prime filter is normal.*

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) ⇒ (iii): Suppose that $\mathfrak{A}$ is a subdirect product of linearly ordered pseudo BL-algebras $\{\mathfrak{A}_i\}_{i\in I}$. Observe that

$$x \vee y = 1 \text{ iff } \{i \in I;\ x_i \neq 1_i\} \cap \{i \in I;\ y_i \neq 1_i\} = \emptyset \tag{3.1}$$

for all $x, y \in A$, since $\mathfrak{A}_i$ are totally ordered.

Let now P be a polar in $\mathfrak{A}$, i.e. $P = P^{\perp\perp}$. Let $x \in A, a \in P$ and $y \in P^\perp$. Then $x \odot a \leqslant x$ implies $x \odot a = (x \odot a) \wedge x = (x \to (x \odot a)) \odot x$. Further, $\{i \in I;\ x_i \to (x_i \odot a_i) \neq 1_i\} \subseteq \{i \in I;\ a_i \neq 1_i\}$. Indeed, if $a_i = 1_i$ then $x_i \to (x_i \odot a_i) = x_i \to (x_i \odot 1_i) = x_i \to x_i = 1_i$. Hence we obtain

$$\{i \in I;\ x_i \to (x_i \odot a_i) \neq 1_i\} \cap \{i \in I;\ y_i \neq 1_i\} \subseteq \{i \in I;\ a_i \neq 1_i\} \cap \{i \in I;\ y_i \neq 1_i\} = \emptyset$$

by (3.1), since $a \in P$ and $y \in P^{\perp}$. Therefore $(x \to (x \odot a)) \vee y = 1$, and thus $x \to (x \odot a) \in P^{\perp\perp} = P$. Hence $x \odot a = (x \to (x \odot a)) \odot x \in P \odot x$, proving $x \odot P \subseteq P \odot x$.

(iii) ⇒ (iv): Obvious.

(iv) ⇒ (v): By Corollary 3.2, $P = \bigcup\{a^{\perp};\ a \notin P\}$ for any minimal prime filter P. If $x \to y \in P$ then there is $a \notin P$ with $x \to y \in a^{\perp}$ which is a normal filter, and hence $x \rightsquigarrow y \in a^{\perp} \subseteq P$. Summarizing, $x \to y \in P$ iff $x \rightsquigarrow y \in P$.

(v) ⇒ (i): Since any prime filter contains a minimal prime filter and the intersection of all prime filters of $\mathfrak{A}$ is obviously $\{1\}$, so does the intersection of minimal prime filters. Thus, by (ii), $\mathfrak{A}$ is representable. □

Theorem 3.4. *A pseudo BL-algebra is representable if and only if it satisfies the identities*

$$(y \to x) \vee (z \rightsquigarrow ((x \to y) \odot z)) = 1, \tag{3.2}$$

$$(y \rightsquigarrow x) \vee (z \to (z \odot (x \rightsquigarrow y))) = 1. \tag{3.3}$$

Consequently, the class of representable pseudo BL-algebras is a variety.

Proof. In any linearly ordered pseudo BL-algebra $\mathfrak{A}$, either $y \to x = 1$ or $x \to y = 1$ (and also $y \rightsquigarrow x = 1$ or $x \rightsquigarrow y = 1$), and so it is easy to verify that $\mathfrak{A}$ fulfils (3.2) and (3.3). Therefore the part "only if" is obvious.

Conversely, suppose that (3.2) and (3.3) are satisfied by $\mathfrak{A}$. In view of Theorem 3.3 (iv), it suffices to prove that any principal polar $x^{\perp}$ is a normal filter of $\mathfrak{A}$.

Let $y \in x^{\perp}$, that is, $y \vee x = 1$. Observe that in this case

$$x = 1 \to x = (y \vee x) \to x = (y \to x) \wedge (x \to x) = (y \to x) \wedge 1 = y \to x$$

and similarly $y = x \to y$. Hence, by (3.2),

$$x \vee (z \rightsquigarrow (y \odot z)) = (y \to x) \vee (z \rightsquigarrow ((x \to y) \odot z)) = 1,$$

thus $z \rightsquigarrow (y \odot z) \in x^{\perp}$. Further, $y \odot z \leqslant z$ implies $y \odot z = (y \odot z) \wedge z = z \odot (z \rightsquigarrow (y \odot z)) \in z \odot x^{\perp}$, which shows $x^{\perp} \odot z \subseteq z \odot x^{\perp}$. The other inclusion follows similarly by (3.3). □

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