# THE INTERVAL FUNCTION OF A CONNECTED GRAPH AND A CHARACTERIZATION OF GEODETIC GRAPHS

Ladislav Nebeský, Praha

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*Abstract.* The interval function (in the sense of H. M. Mulder) is an important tool for studying those properties of a connected graph that depend on the distance between vertices. An axiomatic characterization of the interval function of a connected graph was published by Nebeský in 1994. In Section 2 of the present paper, a simpler and shorter proof of that characterization will be given. In Section 3, a characterization of geodetic graphs will be established; this characterization will utilize properties of the interval function.

*Keywords*: graphs, distance, interval function, geodetic graphs

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By a graph we mean here a finite undirected graph (without loops and multiple edges). If $G$ is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively. Moreover, if $G$ is connected and $u, v \in V(G)$, then $d_G(u,v)$ denotes the distance between $u$ and $v$ in $G$.

The letters $f, g, \dots, n$ will be used for denoting integers.

1. Let $G$ be a connected graph. Put $W = V(G)$. Following Mulder's book [4], by the *interval function* of $G$ we mean the mapping $I_G$ of $W \times W$ into $2^W$ (i.e. into the set of all subsets of $W$) defined as follows:

$$I_G(r,s) = \{t \in W;\ d_G(r,s) = d_G(r,t) + d_G(t,s)\}$$

for each ordered pair $r, s \in W$.

**Lemma 1.** *Let $G$ be a connected graph. Put $W = V(G)$ and $J = I_G$. Then $J$ satisfies the following axioms* (i1)–(i7)*:*

(i1) $J(v,u) = J(u,v)$ *for all* $u, v \in W$*;*

(i2) $u \in J(u,v)$ *for all* $u, v \in W$;

(i3) *if* $x \in J(u,v)$, *then* $|J(u,x) \cap J(x,v)| = 1$ *for all* $u, v, x \in W$;

(i4) *if* $x \in J(u,v)$ *and* $y \in J(x,v)$, *then* $y \in J(u,v)$ *for all* $u, v, x, y \in W$;

(i5) *if* $x \in J(u,v)$ *and* $y \in J(x,v)$, *then* $x \in J(u,y)$ *for all* $u, v, x, y \in W$;

(i6) *if* $|J(u,x)| = 2 = |J(v,y)|$, $x, y \in J(u,v)$ *and* $u \in J(x,y)$, *then* $v \in J(x,y)$ *for all* $u, v, x, y \in W$;

(i7) *if* $|J(u,x)| = 2 = |J(v,y)|$, $x \in J(u,v)$, $x \notin J(u,y)$ *and* $y \notin J(u,v)$, *then* $v \in J(x,y)$ *for all* $u, v, x, y \in W$.

P r o o f. Axioms (i1)–(i5) can be verified easily. It is also not difficult to verify axioms (i6) and (i7); their verification can be found in [6]. □

R e m a r k 1. Properties of the interval function of a connected graph that are very similar to axioms (i1)–(i5) were presented in [4, 1.1.2.].

Let $W$ be a finite nonempty set, and let $J$ be a mapping of $W \times W$ into $2^W$. We denote by $\mathbb{G}_J$ the graph $H$ with $V(H) = W$ and

$$E(H) = \{rs;\ r, s \in W,\ r \neq s \text{ and } J(r,s) = \{r,s\} = J(s,r)\}.$$

If $\mathbb{G}_J$ is connected and $n \geqslant 0$, then we denote by $J_n$ the mapping of

$$Z_n \overset{\text{df}}{=} \{(u,v) \in W \times W;\ d_{\mathbb{G}_J}(u,v) = n\}$$

into $2^W$ such that $J_n(x,y) = J(x,y)$ for each $(x,y) \in Z_n$.

**Lemma 2.** *If $G$ is a connected graph and $J = I_G$, then $G = \mathbb{G}_J$.*

P r o o f is obvious. □

In Lemmas 3–5 and in Corollary 1 we will assume that a finite nonempty set $W$ and a mapping $J$ of $W \times W$ into $2^W$ are given.

**Lemma 3.** *Assume that $J$ satisfies axioms* (i1), (i2) *and* (i3). *Let $u_0, \ldots, u_n \in W$, where $n \geqslant 1$, and let*

$$|J(u_0,u_1)| = \ldots = |J(u_{n-1},u_n)| = 2. \tag{1}$$

*Then $u_0u_1, \ldots, u_{n-1}u_n \in E(\mathbb{G}_J)$.*

P r o o f is very easy. □

**Lemma 4.** *Assume that $J$ satisfies axioms* (i1)–(i4). *Consider arbitrary distinct $u, v \in W$. Then* (a) *$u$ and $v$ belong to the same component of $\mathbb{G}_J$ and* (b) *there exists $w \in J(u, v)$ such that $|J(u, w)| = 2$.*

P r o o f. We proceed by induction on $|J(u, v)|$. By (i1) and (i2), $|J(u, v)| \geqslant 2$. If $|J(u, v)| = 2$, then, by virtue of (i3), $u$ and $v$ are adjacent and thus they belong to the same component; we put $w = v$. Now, let $|J(u, v)| > 2$. There exists $x \in J(u, v), u \neq x \neq v$. By (i1) and (i2), $u \in J(u, x)$ and $v \in J(x, v)$. According to (i1) and (i4), $J(u, x), J(x, v) \subseteq J(u, v)$. By virtue of (i3), $|J(u, x)|, |J(x, v)| < |J(u, v)|$. By the induction hypothesis, (a′) $x$ belongs to the same component as $u$ and to the same component as $v$, and (b′) there exists $w \in J(u, x)$ such that $|J(u, w)| = 2$. Obviously, $u$ and $v$ belong to the same component. Combining (i1) and (i4), we get $w \in J(u, v)$. Hence (a) and (b) hold. □

**Corollary 1.** *If $J$ satisfies axioms* (i1)–(i4), *then $\mathbb{G}_J$ is connected.*

**Lemma 5.** *Assume that $J$ satisfies axioms* (i2), (i4) *and* (i5). *Let $u_0, \ldots, u_n, v \in W$, where $n \geqslant 1$, and let*

$$u_{i+1} \in J(u_i, v) \tag{$2_i$}$$

*for each $i$, $0 \leqslant i \leqslant n-1$. Then*

$$u_j \in J(u_0, v) \text{ and } u_{j-1} \in J(u_0, u_j) \tag{$3_j$}$$

*for each $j$, $1 \leqslant j \leqslant n$.*

P r o o f. We proceed by induction on $j$. The case $j = 1$ is trivial. Let $j \geqslant 2$. By the induction hypothesis, $u_{j-1} \in J(u_0, v)$. By ($2_{j-1}$), $u_j \in J(u_{j-1}, v)$. As follows from (i4), $u_j \in J(u_0, v)$. As follows from (i5), $u_{j-1} \in J(u_0, u_j)$. □

2. The interval function of a connected graph $G$ plays a very important role in studying those structural properties of $G$ that depend on distance between vertices. Cf. Mulder [4] or, for example, Bandelt and Mulder [1] and [2], and Bandelt, Mulder and Wilkeit [3].

However, the concept of the interval function of a connected graph is not only well-motivated; it is also transparently characterizable. Nebeský [6] proved a theorem which can be reformulated as follows: If $W$ is a finite nonempty set, $J$ is a mapping of $W \times W$ into $2^W$ and $\mathbb{G}_J$ is connected, then $J$ is the interval function of $\mathbb{G}_J$ if and only if $J$ satisfies axioms (i1)–(i7).

The proof given in [6] was unnecessarily complicated. A new proof will be presented here. It will utilize some ideas of the original proof but it will be shorter and significantly simpler. We will formulate a theorem slightly stronger than the one mentioned above:

**Theorem 1.** *Let $W$ be a finite nonempty set, and let $J$ be a mapping of $W \times W$ into $2^W$. Then $J$ is the interval function of a connected graph if and only if $J$ satisfies axioms* (i1)–(i7).

P r o o f. If $J$ is the interval function of a connected graph $G$, then, by virtue of Lemma 2, $V(G) = W$ and thus, by Lemma 1, $J$ satisfies axioms (i1)–(i7).

Conversely, let $J$ satisfy axioms (i1)–(i7). Put $G = \mathbb{G}_J$. By Corollary 1, $G$ is connected. Put $d = d_G$ and $I = I_G$. We will prove that $J = I$.

Suppose, to the contrary, that $I \neq J$. Then there exists $n \geqslant 0$ such that $J_n \neq I_n$ and

$$J_f = I_f \text{ for all } f,\ 0 \leqslant f < n. \tag{4}$$

It is easy to see that $n \geqslant 2$. We distinguish two cases.

C a s e 1. Let $I_n \setminus J_n \neq \emptyset$. There exist $u, v, w \in W$ such that $d(u, v) = n$ and $w \in I(u, v) \setminus J(u, v)$. Thus, there exist $v_0, \dots, v_n \in W$ and $g$, $0 < g < n$, such that $v_0 = v, v_n = u, v_g = w$ and $(v_n, \dots, v_0)$ is a path from $u$ to $v$ of length $d(u, v)$ in $G$. Let $v_{n-1} \in J(u, v)$; clearly, $d(v_{n-1}, v) = n - 1$ and $v_g \in I(v_{n-1}, v)$; by (4), $v_g \in J(v_{n-1}, v)$ and by (i4), $v_g \in J(u, v)$; a contradiction. Hence $v_{n-1} \notin J(u, v)$. By (i1) and (i2), $v \in J(u, v)$. Recall that $d(u, v) = n$. Lemmas 3 and 4 imply that there exist $u_0, \dots, u_n \in W$ such that $u_0 = u$, (1) holds and ($2_i$) holds for each $i$, $0 \leqslant i \leqslant n - 1$. By Lemma 5, ($3_n$) holds. As follows from Lemma 3,

$$d(u_j, v_j) \leqslant n \text{ for each } j,\ 0 \leqslant j \leqslant n. \tag{5}$$

Put $u_{-1} = v_{n-1}$. The following three statements hold for $i = 0$:

$$d(u_i, v_i) = n, \tag{$6_i$}$$

$$v \in J(u_i, v_i), \tag{$7_i$}$$

$$u_{i-1} \notin J(u_i, v_i). \tag{$8_i$}$$

By ($3_n$) and (i1), $u_{n-1} \in J(u_n, u_0)$. Since $v_n = u_0$, ($8_n$) does not hold.

There exists $h$, $0 \leqslant h \leqslant n - 1$, such that ($6_h$), ($7_h$) and ($8_h$) hold but at least one of ($6_{h+1}$), ($7_{h+1}$) and ($8_{h+1}$) does not. Combining ($2_h$), ($7_h$) and (i1) with (i4) and (i5), we get

$$u_{h+1} \in J(u_h, v_h), \tag{9}$$

$$v \in J(u_{h+1}, v_h). \tag{10}$$

As follows from $(6_h)$,

$$d(u_h, v_{h+1}) = n - 1. \tag{11}$$

Clearly, $u_{h-1} \in I(u_h, v_{h+1})$. By (11) and (4), $u_{h-1} \in J(u_h, v_{h+1})$. Let $v_{h+1} \in J(u_h, v_h)$; by (i1) and (i4) we get $u_{h-1} \in J(u_h, v_h)$, which contradicts $(8_h)$. Hence

$$v_{h+1} \notin J(u_h, v_h). \tag{12}$$

Let $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$. Thus $d(u_{h+1}, v_{h+1}) = n - 2$. As follows from $(6_h)$, $d(u_{h+1}, v_h) = n - 1$ and $v_{h+1} \in I(u_{h+1}, v_h)$. By (4), $v_{h+1} \in J(u_{h+1}, v_h)$. Combining (9) and (i4), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). Hence $u_{h+1} \notin J(u_h, v_{h+1})$. Thus, combining (9), (12) and (i7), we get

$$v_h \in J(u_{h+1}, v_{h+1}). \tag{13}$$

Let $d(u_{h+1}, v_{h+1}) < n$. By (13) and (4), $v_h \in I(u_{h+1}, v_{h+1})$. Therefore, $d(u_{h+1}, v_h) < n - 1$. This means that $d(u_h, v_h) < n$, which contradicts $(6_h)$. Thus, by virtue of (5), we get $(6_{h+1})$.

Combining (10), (13), (i1) and (i4), we get $(7_{h+1})$.

Assume that $u_h \in J(u_{h+1}, v_{h+1})$. Combining (9), (13) and (i6), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). We get $(8_{h+1})$, which is a contradiction with the definition of $h$.

Case 2. Let $I_n \subseteq J_n$. Then $J_n \setminus I_n \neq \emptyset$. There exist $u, v, z \in W$ such that $d(u, v) = n$ and $z \in J(u, v) \setminus I(u, v)$. By (i2), $z \neq u$. By Lemma 3, there exists $t \in J(u, z)$ such that $|J(u, t)| = 2$. By (i1), (i4) and (i5), $t \in J(u, v)$ and $z \in J(t, v)$. If $d(t, v) < n$, then $d(t, v) = n - 1$, $t \in I(u, v)$ and, by (4), $z \in I(t, v) \subseteq I(u, v)$; a contradiction. Hence $d(t, v) \geqslant n$. Lemmas 3 and 4 imply that there exist $u_0, \ldots, u_n \in W$ such that $u_0 = u$, $u_1 = t$, (1) holds and $(2_i)$ holds for each $i, 0 \leqslant i \leqslant n - 1$. Since $d(u_1, v) \geqslant n$ and $u_1u_2, \ldots, u_{n-1}u_n \in E(G)$, we have

$$d(u_{i+1}, v) \geqslant n - i \tag{$14_i$}$$

for each $i,\ 0 \leqslant i \leqslant n - 1$. Thus $u_n \neq v$. By Lemma 5, $(3_n)$ holds. Since $d(u, v) = n$, there exist $v_0, \ldots, v_n \in W$ such that $v_0 = v, v_n = u$ and $(v_n, \ldots, v_0)$ is a path from $u$ to $v$ of length $d(u, v)$ in $G$. Thus

$$d(v, v_i) = i \tag{$15_i$}$$

for each $i,\ 0 \leqslant i \leqslant n - 1$. Moreover, (5) holds.

Obviously, both $(6_0)$ and $(7_0)$ hold. By $(3_n)$, $u_n \in J(v_n, v)$. Since $u_n \neq v$, (i1), (i2) and (i3) imply that $(7_n)$ does not hold.

There exists $h$, $0 \leqslant h \leqslant n-1$, such that $(6_h)$ and $(7_h)$ hold but at least one of $(6_{h+1})$ and $(7_{h+1})$ does not. Similarly as in Case 1, we get (9), (10) and (11).

Let $d(u_{h+1}, v_h) < n$. Combining (4) and (10), we get $v \in I(u_{h+1}, v_h)$. By virtue of $(14_h)$, we have $d(v, v_h) < h$, which contradicts $(15_h)$. Hence $d(u_{h+1}, v_h) \geqslant n$.

Let $d(u_{h+1}, v_{h+1}) < n$. Since $d(u_{h+1}, v_h) \geqslant n$, we have $d(u_{h+1}, v_h) = n$ and $d(u_{h+1}, v_{h+1}) = n-1$. Therefore, $v_{h+1} \in I(u_{h+1}, v_h)$. Since $I_n \subseteq J_n$, we get $v_{h+1} \in J(u_{h+1}, v_h)$. Thus, combining (9) and (i5), we see that $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$ and therefore, $d(u_{h+1}, v_{h+1}) = n-2$; a contradiction. Thus, by virtue of (5), $(6_{h+1})$ holds.

By virtue of $(6_h)$, $v_{h+1} \in I(u_h, v_h)$; by $(6_{h+1})$, $u_h \in I(u_{h+1}, v_{h+1})$. Recall that $I_n \subseteq J_n$. We have $v_{h+1} \in J(u_h, v_h)$ and $u_h \in J(u_{h+1}, v_{h+1})$. Thus, (9) and (i6) imply (13). Similarly as in Case 1, we get $(7_{h+1})$, which is a contradiction with the definition of $h$.

Thus $J = I$, which completes the proof. $\square$

R e m a r k 2. An extension of Theorem 1 (with a different and rather long proof) was presented in Nebeský [8].

3. A graph $G$ is said to be *geodetic* if it is connected and for each pair $r, s \in V(G)$, there exists exactly one path from $r$ to $s$ of length $d_G(r, s)$. The problem to characterize geodetic graphs was stated in Ore's book [10].

The next theorem gives a characterization of geodetic graphs based on properties of the interval function.

**Theorem 2.** *Let $G$ be a graph. Put $W = V(G)$. Then $G$ is geodetic if and only if there exists a mapping $J$ of $W \times W$ into $2^W$ such that $G = \mathbb{G}_J$ and $J$ satisfies axioms* (i1), (i2), (i3) *and the following axioms* (g1) *and* (g2)*:*

(g1) *if $x \in J(u, v)$, then $J(u, v) = J(u, x) \cup J(x, v)$ for all $u, v, x \in W$;*

(g2) *if $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$, then $x \in J(u, y)$ or $v \in J(x, y)$ for all $u, v, x, y \in W$.*

P r o o f. I. Assume that $G$ is geodetic. Then $G$ is connected. Let $J$ denote its interval function. By Lemma 1, $J$ satisfies (i1), (i2) and (i3). As immediately follows from the definition of a geodetic graph, $J$ satisfies (g1). We will show that $J$ satisfies (g2).

Consider arbitrary $u, v, x, y \in W$. Assume that $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$. If $y = u$, then $x = v$ and thus $v \in J(x, y)$. Let $y \neq u$. Since $G$ is geodetic, there exists exactly one path $P$ from $u$ to $v$ of length $d_G(u, v)$ in $G$. If $y \in J(u, v)$, then $y$ belongs to $P$ and thus $x \in J(u, y)$. Let $y \notin J(u, v)$. Then

$d_G(x,v) \leqslant d_G(x,y) \leqslant d_G(x,v)+1$. If $d_G(x,y) = d_G(x,v)+1$, then $v \in J(x,y)$. If $d_G(x,y) = d_G(x,v)$, then $x \in J(u,y)$. Thus $J$ satisfies (g2).

II. Conversely, assume that $J$ satisfies (i1), (i2), (i3), (g1) and (g2). The fact that $J$ satisfies (g2) implies that $J$ satisfies (i7). First, we will show that $J$ also satisfies (i4), (i5) and (i6).

Consider arbitrary $u, v, x, y \in W$.

Let $x \in J(u,v)$ and $y \in J(x,v)$. By (g1),

$$J(u,v) = J(u,x) \cup J(x,v) \text{ and } J(x,v) = J(x,y) \cup J(y,v). \tag{16}$$

Combining (i2) and (16), we get $y \in J(u,v)$. By (g1) again,

$$J(u,v) = J(u,y) \cup J(y,v). \tag{17}$$

If $x = y$, then (i1), (i2) and (17) imply that $x \in J(u,y)$. Let $x \neq y$. By (i2), $x \in J(x,y)$. Since $y \in J(x,v)$, it follows from (i3) that $J(x,y) \cap J(y,v) = \{y\}$, and therefore, $x \notin J(y,v)$. Since $x \in J(u,v)$, (17) implies that $x \in J(u,y)$. We see that $J$ satisfies (i4) and (i5).

Let $|J(u,x)| = 2 = |J(v,y)|$ and let $x, y \in J(u,v)$. As follows from (i1), (i2) and (i3), we have $x \neq u \neq v \neq y$, $J(u,x) = \{u,x\}$, $J(v,y) = \{v,y\}$ and $u, v \in J(u,v)$. Hence $|J(u,v)| \geqslant 2$. First, let $|J(u,v)| = 2$. Then $u = y$ and $v = x$. We have $u, v \in J(u,v)$. Now, let $|J(u,v)| > 2$. Then $x \neq v$ and $y \neq u$. It follows from (i3) and (g1) that $|\{u,x\} \cap J(x,v)| = 1$ and $J(u,v) = J(u,x) \cup J(x,v)$. Thus $u \notin J(x,v)$. As follows from (i3) and (g1) again, $|J(x,y) \cap \{y,v\}| = 1$ and $J(x,v) = J(x,y) \cup \{y,v\}$. By (i1) and (i2), $y \in J(x,y)$, Thus $v \notin J(x,y)$. Since $u \notin J(x,v)$, we get $u \notin J(x,y)$. We see that $J$ satisfies also (i6).

Put $G = \mathbb{G}_J$. By Corollary 1, $G$ is connected. By Theorem 1 and Lemma 2, $J$ is the interval function of $G$. Recall that $J$ satisfies (g1). We will show that

$$\text{there exists exactly one path from } u \text{ to } v \text{ of length } d_G(u,v) \text{ in } G \tag{18}$$

for each pair of distinct $u, v \in W$.

Consider arbitrary distinct $u, v \in W$. Put $n = d_G(u,v)$. To prove (18), we will proceed by induction on $n$. Obviously, $n \geqslant 1$. There exist $u_0, u_1, \ldots, u_n \in W$ such that $u_0 = u, u_n = v$ and $(u_0, u_1, \ldots, u_n)$ is a path in $G$. Since $d_G(u,v) = n$, we see that $u_i$ and $u_j$ are adjacent in $G$ if and only if $|i-j| = 1$ for all $i$ and $j$, $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$. Clearly, if $n = 1$, then (18) holds. Let $n > 1$. Then $u_1 \neq v$. Since $d_G(u_1,v) = n-1$, the induction hypothesis implies that $(u_1, \ldots, u_n)$ is the only path from $u_1$ to $v$ of length $n-1$ in $G$. Thus $J(u_1,v) = \{u_1, \ldots, u_n\}$. By virtue of (g1), $J(u,v) = \{u_0, u_1, \ldots, u_n\}$. This means that $(u_0, u_1, \ldots, u_n)$ is the only path from $u$ to $v$ of length $n$ in $G$.

Thus $G$ is geodetic. □

R e m a r k 3. A characterization of geodetic graphs utilizing properties of the set of all shortest paths was given in Nebeský [5] and [7]. A characterization of geodetic graphs based on a binary operation on the vertex set was given in Nebeský [9].

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*Author's address*: *Ladislav Nebeský*, Univerzita Karlova v Praze, Filozofická fakulta, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic, e-mail: `ladislav.nebesky@ff.cuni.cz`.