GRAPHS WITH THE SAME PERIPHERAL AND CENTER ECCENTRIC VERTICES

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(Received July 27, 1998)

Abstract. The eccentricity $e(v)$ of a vertex v is the distance from v to a vertex farthest from v, and u is an eccentric vertex for v if its distance from v is $d(u,v) = e(v)$. A vertex of maximum eccentricity in a graph G is called peripheral, and the set of all such vertices is the peripherian, denoted $\operatorname{Peri} G$). We use $\operatorname{Cep}(G)$ to denote the set of eccentric vertices of vertices in $C(G)$. A graph G is called an S-graph if $\operatorname{Cep}(G) = \operatorname{Peri}(G)$. In this paper we characterize S-graphs with diameters less or equal to four, give some constructions of S-graphs and investigate S-graphs with one central vertex. We also correct and generalize some results of F. Gliviak.

Keywords: graph, radius, diameter, center

MSC 2000: 05C12, 05C20

1. Introduction

We consider nonempty and finite graphs without loops and multiple edges. All terminology as well as notation except that given here is taken from [1].

The set of vertices of a graph G is denoted by $V(G)$, and the set of edges by $E(G)$. Let G be a connected graph with vertices u and v. The *distance* $d(u,v)$ between u and v is the length of a shortest $u-v$ path in G. The *eccentricity* $e(v)$ of a vertex v is the distance from v to a vertex farthest from v, and u is an *eccentric vertex* for v if $d(u,v) = e(v)$. The *radius* $r(G)$ of G is $\min\{e(v);\ v \in V(G)\}$, while the *diameter* $d(G)$ of G is $\max\{e(v);\ v \in V(G)\}$. A *diametral path* is a path of length $d(G)$ joining a pair of vertices of the graph G that are at distance $d(G)$ from one another. A vertex with minimum eccentricity is called a *central vertex* and the set of all such vertices is the *center* of G denoted by $C(G)$. A graph is *self-centered* if its every vertex is in the center. The neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$. For any nonempty subset S of vertices in G, the *induced subgraph* $\langle S\rangle$ is the maximal subgraph of G with the vertex set S. A vertex u is a *center eccentric vertex* of G if it is an eccentric vertex of some central vertex of G. A vertex v is *peripheral* if $e(v) = d(G)$, and the set of such vertices is the *peripherian* of G. We use $\mathrm{Cep}(G)$ to denote the set of all center eccentric vertices and $\mathrm{Peri}(G)$ to denote the peripherian of G. A connected nontrivial graph G is called an S-*graph* if $\mathrm{Cep}(G) = \mathrm{Peri}(G)$.

Buckley and Lewinter [2] proved the existence of an S-graph G with $r(G) = a$ and $d(G) = b$ for every positive integers a, b, $a \leqslant b \leqslant 2a$, and showed how to embed a graph G into an S-graph. They also proved that the cartesian product of two graphs is an S-graph if and only if both these graphs are S-graphs.

2. Main results

As follows from the definition of an S-graph, any self-centered graph as well as any tree are S-graphs. In particular, the complete graph K_n, $n \geqslant 1$ is an S-graph. Further we will investigate only S-graphs that are not self-centered.

Let G_1, G_2 be two disjoint connected graphs. Let $x \in V(G_1)$, $y \in V(G_2)$. We say that a graph G arose from G_1 and G_2 by the identification of the vertices x, y with a new vertex t ($t \notin V(G_1)$, $t \notin V(G_2)$), if

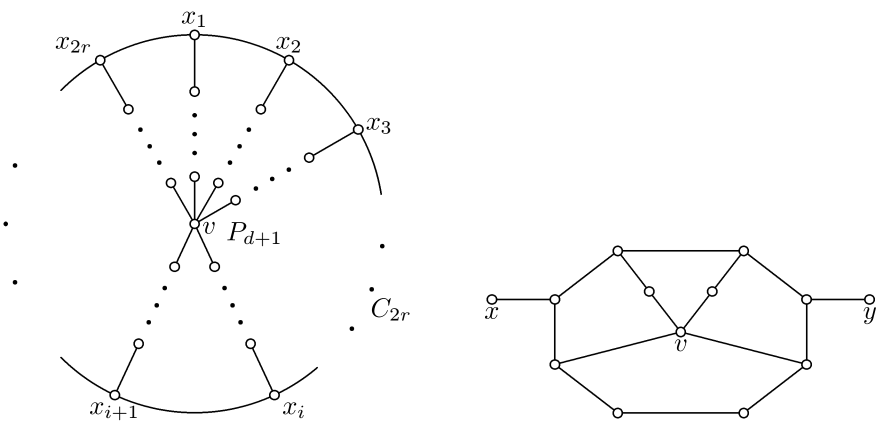
$$\begin{aligned} V(G) &= V(G_1) \cup V(G_2) - \{x, y\} \cup \{t\}, \\ E(G) &= E(G_1) \cup E(G_2) - \{xu, yu;\ xu \in E(G_1), yu \in E(G_2)\} \\ &\qquad \cup \{tu;\ xu \in E(G_1) \text{ or } yu \in E(G_2)\}. \end{aligned}$$

Gliviak [4] gave the following construction of S-graphs with one central vertex.

C o n s t r u c t i o n. Let $r \geqslant 1$, $n \geqslant 2$ be natural numbers. Let G_i, $i = 1, 2, \ldots, n$ be vertex disjoint graphs having at least one vertex v_i of eccentricity r. Let the graph H arise from graphs G_i by identification of all vertices v_i with one common vertex w.

He claims (Theorem 2) that a graph Q is an S-graph of radius one if and only if Q is either a complete graph K_n, $n \geqslant 2$ or can be constructed according to the previous construction, as well as (Theorem 3) that for any $r \geqslant 2$ a graph Q is an S-graph of radius r with one central vertex if and only if Q can be constructed according to this construction.

This construction gives S-graphs with one central vertex, but not all such S-graphs. In any S-graph G which we get by the construction the central vertex is a cut vertex of G and the diameter of such a graph is equal to $2r(G)$. As follows from Fig. 1 there exist S-graphs with one central vertex, which cannot be constructed according to the above construction. In Fig. 1a there is an S-graph with radius $r \geqslant 1$, diameter



a) b)

Fig. 1

$2r$, and one central vertex v. The peripherian of this graph consists of all vertices belonging to C_{2r}. In Fig. 1b there is an S-graph with radius 3, diameter 5 and one central vertex v. The peripherian of this graph is $\{x, y\}$.

Theorem 2.1. *Let G be a graph with $r(G) = 1$ and $d(G) = 2$. Then G is an S-graph if and only if $|C(G)| = 1$.*

Proof. If $C(G) = \{c\}$, then the eccentricity of any vertex from $V(G) - \{c\}$ is equal to two. Thus $\mathrm{Peri}(G) = \mathrm{Cep}(G) = V(G) - \{c\}$, and G is an S-graph.

Conversely, let G be an S-graph. Let $|C(G)| \geqslant 2$ and $x, y \in C(G)$, $x \neq y$. Since $x, y \in \mathrm{Cep}(G)$ and no vertex from $C(G)$ can belong to $\mathrm{Peri}(G)$, we have a contradiction. This completes the proof. □

Corollary 2.2. *A graph G with $r(G) = 1$ is an S-graph if and only if it is a complete graph K_p, $p \geqslant 2$, or has one central vertex.*

As mentioned above, for any two positive integers a and b, $a \leqslant b \leqslant 2a$, there exists an S-graph with radius a and diameter b. The following theorem shows that there exists no S-graph G with $d(G) = r(G) + 1$ and $|C(G)| = 1$.

Theorem 2.3. *Let G be an S-graph with $r(G) \neq d(G) \geqslant 3$ and $|C(G)| = 1$. Then $r(G) < d(G) - 1$.*

Proof. Let G be an S-graph with $C(G) = \{c\}$ and $d(G) \geqslant 3$. Let $r(G) = d(G) - 1$. The eccentricity of any vertex $t \in N_G(c)$ is greater than $r(G)$, but $e_G(t) \leqslant d(G) = r(G) + 1$. Therefore $e_G(t) = d(G)$ and $t \in \mathrm{Peri}(G) = \mathrm{Cep}(G)$, which is a contradiction with $d_G(t, c) = 1 < r(G)$. This completes the proof. □

Lemma 2.4. *Let G be an S-graph, $r(G) \geqslant 2$ and $r(G) \neq d(G)$. Then the distance of any two vertices from $C(G)$ in G is less than $r(G)$ and $d(\langle V(G)-C(G)\rangle) \geqslant d(G)$.*

Proof. The distance of no two vertices from $C(G)$ can be equal to $r(G)$, since no vertex from $C(G)$ can belong to $\mathrm{Peri}(G)$.

If $e_{\langle V(G)-C(G)\rangle}(x) < d(G)$ for any vertex $x \in V(G) - C(G)$, then the eccentricity of any vertex from $V(G)$ in G is less than $d(G)$, which is a contradiction. □

Theorem 2.5. *Let G be a graph with $d(G) = r(G) + 1$ and $r(G) \geqslant 2$. Then G is an S-graph if and only if any vertex from $V(G) - C(G)$ is a center eccentric vertex of G.*

Proof. Let G be an S-graph. Then $\mathrm{Peri}(G) = V(G) - C(G)$, because the eccentricity of any vertex from $V(G) - C(G)$ is equal to $d(G)$. Since G is an S-graph, every vertex from $\mathrm{Peri}(G)$ is eccentric for some vertex from $C(G)$.

Conversely, let any vertex of $V(G) - C(G)$ be a center eccentric vertex of G. Since $\mathrm{Peri}(G) = V(G) - C(G)$, the graph G is an S-graph. This completes the proof. □

Corollary 2.6. *Let G be a graph with $d(G) = 3$ and $r(G) = 2$. Then G is an S-graph if and only if no vertex from $V(G) - C(G)$ is joined to all central vertices of G.*

Theorem 2.7. *Let G be a graph with diameter four and radius two. Let $Q = \langle V(G) - C(G)\rangle$, where $C(G)$ is the center of G. Then G is an S-graph if and only if*

1) *$\langle C(G)\rangle$ is a complete graph and*
2) *the cardinality of the set $T = \{x \in V(Q);\ N_G(x) \cap C(G) = \emptyset\}$ is at least two, and for every vertex $x \in T$ at least one vertex $y \in V(Q)$ such that $d_Q(x, y) \geqslant 4$ belongs to T.*

Proof. Let G be an S-graph. As follows from Lemma 2.4, $\langle C(G)\rangle$ is a complete graph, and $d(Q) \geqslant 4$. Any vertex $t \in V(Q) - T$ has eccentricity $e_G(t) = 3$. Thus any vertex with eccentricity four belongs to T, i.e. $\mathrm{Peri}(G)$ is a subset of T. Since G is an S-graph, $\mathrm{Peri}(G) = \mathrm{Cep}(G)$. But $T = \mathrm{Cep}(G)$, because T consists of all center eccentric vertices of G. Obviously, $|\mathrm{Cep}(G)| \geqslant 2$. Let $x \in T$. If there is no vertex $y \in V(Q)$ such that $d_Q(x, y) \geqslant 4$ belongs to T, then $e_G(x) \leqslant 3$, which is a contradiction with $x \in \mathrm{Peri}(G)$.

Conversely, let 1) and 2) hold. As follows from 2), $d(Q) \geqslant 4$ and T consists of all center eccentric vertices of G. Thus $\mathrm{Cep}(G) = T$. Let $x \in T$ and $y \in T$ be such that

$d_Q(x,y) \geqslant 4$. Then $e_G(x) = 4$, and T consists of all diametral vertices of G. Thus $\mathrm{Peri}(G) = T$.

This completes the proof. □

There exists no self-centered S-graph with one central vertex. As follows from Theorem 2.3, there exists no S-graph G with one central vertex and $r(G) = d(G) - 1$. Next we prove that for all other possible cases of radius and diameter there does exist an S-graph with one central vertex.

Theorem 2.8. *For each pair of positive integers d and r, $r \geqslant 2$, $r+2 \leqslant d \leqslant 2r$, there exists an S-graph G with $r(G) = r$, $d(G) = d$ and $|C(G)| = 1$.*

Proof. If $d = 2r$, then we can use the graph formed by P_{d+1}.

Let $d = r + k$, $2 \leqslant k < r$. To construct an S-graph G with the required properties we can proceed as follows.

Let $T_1, T_2, \ldots, T_m$, where $m = 4k - 2$, be disjoint copies of the trees from Fig. 2, such that T_1 and T_{2k} are copies of the tree H, T_k and T_{3k-1} are copies of S and all the other T_i, $1 \leqslant i \leqslant m$ are copies of Q.

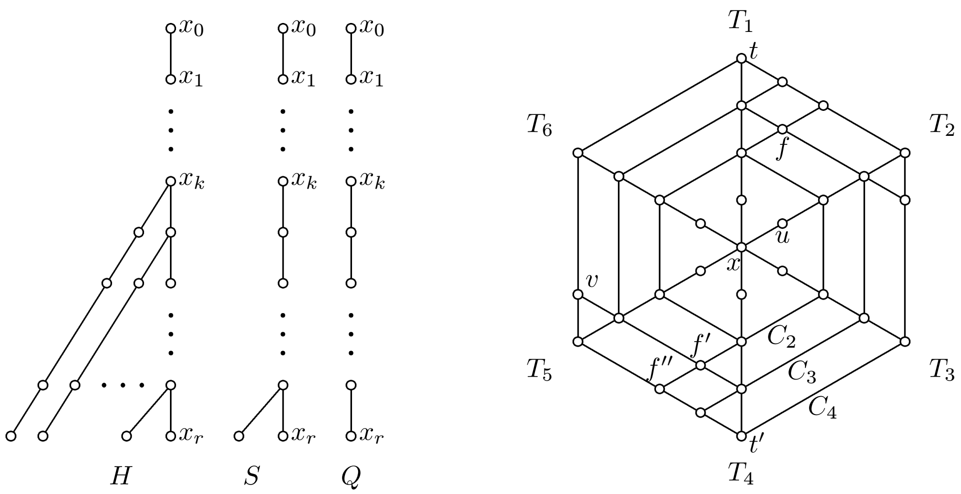


Fig. 2

Let the tree T arise from $T_1, T_2, \ldots, T_m$ by the identification of all the corresponding copies of the vertex x_0 from $T_1, T_2, \ldots, T_m$ in this order, with one common vertex x.

Let G be the graph which we get by forming a cycle C_j on all vertices with the distance j from x, for $j = k, k+1, \ldots, r$. (The corresponding graph G with the cycles C_2, C_3, C_4 for $r = 4$ and $d = 6$ is in Fig. 2). There are $l = r - k + 1$ such cycles C_j

with lengths $2d - 2(r - j + 1) = 2(k + j - 1)$ for $j = k, k+1, \dots, r-1$, and $2d$ for $j = r$.

It is easy to check that the eccentricity of any vertex t of C_{2d} is equal to d and this eccentricity is attained for the vertex t' of C_{2d} whose distance in C_{2d} from t is d (Fig. 2).

Let f be a vertex from C_j, $k \leqslant j \leqslant r-1$ and let f' be the vertex from C_j with distance $d - r + j - 1 = k + j - 1$ from f in C_j. Let f'' be a nearest vertex from C_{2d} for the vertex f' in G. Then $d_G(f, f'') = d - 1 = e_G(f)$.

Let u be a vertex with a distance $i \leqslant k - 1$ from x, and let u belong to T_n, $1 \leqslant n \leqslant m$. Let v be a vertex from C_{2d} belonging to $T_{(n+2k-1) \bmod (4k-2)}$. Then $d_G(u, v) = r + i = e_G(u)$.

Thus $\mathrm{Peri}(G)$ consists of all vertices of C_l, $\mathrm{Cep}(G) = \mathrm{Peri}(G)$ and $C(G) = \{x\}$, which completes the proof. □

For any vertex v of a connected graph G there exists a spanning tree T that is distance preserving from v, i.e. $d_T(v, u) = d_G(v, u)$ for any vertex $u \in V(G)$. If G is an S-graph with one central vertex x, then its distance preserving spanning tree T_x from x is also an S-graph with $d(T_x) = 2r(G)$, $C(T_x) = C(G) = \{x\}$ and $\mathrm{Peri}(T_x) = \mathrm{Cep}(T_x) = \mathrm{Peri}(G) = \mathrm{Cep}(G)$. Therefore any S-graph G with one central vertex can be constructed by adding new edges to a proper tree of radius $r(G)$ and diameter $2r(G)$. On the other hand, not every tree T with radius r and one central vertex, is a distance preserving spanning tree of an S-graph with radius r, diameter $d(G) < d(T)$ and one central vertex. For example, P_7 cannot be supplemented (by adding new edges) to an S-graph G with one central vertex, of diameter 5 and radius 3. As follows from the proof of Theorem 2.8 for the case $r(G) = 3$ and $d(G) = 5$ there exists such a distance preserving spanning tree given by the construction of T as in Fig. 2.

Given an S-graph G we will say that an edge $e \in E(G)$ is *superfluous* in G, if $G - e$ is also an S-graph with the same radius, diameter and peripherian as in G. An S-graph G is said to be *critical*, if it has no superfluous edge. For example, G is a critical S-graph with one central vertex, radius $r(G)$ and diameter $d(G) = 2r(G)$ if and only if it is a tree with one central vertex, radius $r(G)$ and diameter $2r(G)$. We suggest to investigate critical S-graphs with one central vertex.

Buckley and Lewinter [2] studied two other interesting classes of graphs called F-graphs and L′-graphs. A connected graph G is an F-graph if it has at least two central vertices, and for each pair of central vertices u and v, $d(u, v) = r(G)$ holds. They showed that the only graph that is both an F-graph and an S-graph is K_n, $n \geqslant 2$.

If no diametral path of a connected graph G contains a central vertex, G is called an L$'$-graph.

Let G be an S-graph with one central vertex c and $d(G) < 2r$. Since G is an S-graph, $\mathrm{Peri}(G) = \mathrm{Cep}(G)$ and the distance between c and any diametral vertex is equal to $r(G)$. Since no diametral path in G can contain the central vertex c, G is an L$'$-graph. The opposite is not true as there are L$'$-graphs that are not S-graphs. One such graph G with diameter 5 and radius 3 is shown in Fig. 3. The center of this graph is $C(G) = \{t\}$, $\mathrm{Cep}(G) = \{x, v, q, y\}$ and $\mathrm{Peri}(G) = \{x, y\}$.

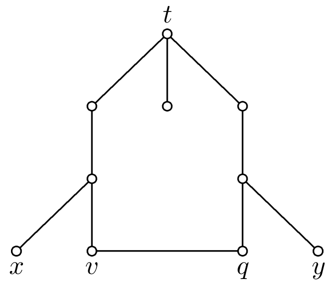


Fig. 3

Let H and Q be two graphs and let s be an arbitrary vertex from $V(H)$. We say that the graph G arose from H by the substitution of s by Q if

$$V(G) = V(H) \cup V(Q) - \{s\}$$

and

$$E(G) = E(H) \cup E(Q) \cup \{xy;\ x \in V(Q) \text{ and } sy \in E(H)\} - \{sy;\ sy \in E(H)\}.$$

Gliviak [4] used this substitution to prove the existence of an S-graph G with a prescribed radius a and diameter b containing a given graph Q as an induced subgraph of G. To prove this assertion for the case $a \geqslant 2$ and $b = 2a - 1$ it is suggested to construct such an S-graph by the substitution of any central vertices of P_{2a} by G and for $a \geqslant 2$ and $b = 2a$ by the substitution of the central vertex of P_{2a+1} by G. If $a = 2$ and $d(G) \geqslant 2$ then the resulting graph is not an S-graph. This can be corrected by substituting not a central but a diametral vertex of P_{2a} and P_{2a-1}, respectively.

Next we formulate a more general theorem based on the substitution of a vertex of an S-graph.

Theorem 2.9. *Let G be an S-graph with $r(G) \geqslant 3$ and $x \in V(G)$. Let H be a graph disjoint with G. Let G' be the graph, arising from G by the substitution of x by H. Then G' is an S-graph, and*

1) $r(G) = r(G')$, $d(G) = d(G')$
2) $\mathrm{Cep}(G') = \mathrm{Peri}(G') = \mathrm{Cep}(G) = \mathrm{Peri}(G)$ *if* $x \notin \mathrm{Cep}(G)$
3) $\mathrm{Cep}(G') = \mathrm{Peri}(G') = \mathrm{Cep}(G) \cup V(H) - \{x\}$, *if* $x \in \mathrm{Cep}(G)$.

P r o o f. As follows from the substitution, $e_G(s) = e_{G'}(s)$ for any $s \in V(G)$ and $e_G(x) = e_{G'}(t)$ for any $t \in V(H)$. Thus $r(G) = r(G')$ and $d(G) = d(G')$.

Let $x \notin \mathrm{Cep}(G)$.Then $d_G(x, c) < r(G)$ for any central vertex $c \in C(G)$, and also $d_{G'}(t, c) < r(G)$ for any $t \in V(H)$. Any $s \in V(G)$ is a center eccentric vertex in G if and only if it is a center eccentric vertex in G'. Thus $\mathrm{Cep}(G) = \mathrm{Cep}(G')$. Since $x \notin \mathrm{Cep}(G) = \mathrm{Peri}(G)$, the eccentricity of x in G is less than $d(G)$. Then also $e_{G'}(t) < d(G)$ for any $t \in V(H)$, and no vertex from $V(H)$ is a diametral vertex in G'. Any $s \in V(G)$ is a diametral vertex in G if and only if it is a diametral vertex in G'. Thus $\mathrm{Peri}(G) = \mathrm{Peri}(G')$.

Let $x \in \mathrm{Cep}(G)$. Then x is an eccentric center vertex in G, and any $t \in V(H)$ is an eccentric vertex in G'. Thus the set of the center eccentric vertices in G' is $\mathrm{Cep}(G) \cup V(H) - \{x\} = \mathrm{Cep}(G')$. Since G is an S-graph, $x \in \mathrm{Peri}(G)$. Then any vertex $t \in V(H)$ is a diametral vertex in G'. Therefore the set of diametral vertices in G' is $\mathrm{Peri}(G') = \mathrm{Peri}(G) \cup V(H) - \{x\} = \mathrm{Cep}(G) \cup V(H) - \{x\}$. This completes the proof. □

Theorem 2.10. *Let H be a graph. For any positive integers a,b such that $a + 2 \leqslant b \leqslant 2a$, $a \neq b$ and $a \geqslant 3$ there exists an S-graph G with $r(G) = a$, $d(G) = b$ and $\langle C(G) \rangle = H$.*

P r o o f. Let Q be an S-graph with radius a and diameter b with one center vertex c. Let G be the graph which we get by the substitution of c by H in G. As follows from Theorem 2.9, G is an S-graph with $r(G) = r(Q)$, $d(Q) = d(G)$, $\mathrm{Peri}(Q) = \mathrm{Cep}(Q) = \mathrm{Peri}(G) = \mathrm{Cep}(G)$ and $\langle C(G) \rangle = H$. □

Theorem 2.11. *Let $G_1, G_2, \ldots, G_n$, $n \geqslant 2$, be disjoint S-graphs with $r(G_i) = r \geqslant 2$ and $C(G_i) = \{x_i\}$ for $i = 1, 2, \ldots, n$. Let H be a self-centered graph disjoint with G_i, $1 \leqslant i \leqslant n$ and $V(H) = \{t_1, t_2, \ldots, t_n\}$. Let G be the graph that we construct by identifying the corresponding pairs of vertices t_i and x_i with a new vertex u_i, for $i = 1, 2, \ldots, n$. Then the graph G is an S-graph with $d(G) = 2r + r(H)$, $r(G) = r + r(H)$, $C(G) = V(H)$, $\mathrm{Peri}(G) = \bigcup_{i=1}^{n} \mathrm{Peri}(G_i)$ and $\mathrm{Cep}(G) = \bigcup_{i=1}^{n} \mathrm{Cep}(G_i)$*

P r o o f. For every $i = 1, 2, \ldots, n$ the eccentricity $e_G(u_i) = r + r(H)$ and the eccentricity of any vertex $z \in V(G)$, $z \neq u_i$ for $i = 1, 2, \ldots, n$ is $e_G(z) > r + r(H)$. Thus $r(G) = r + r(H)$ and $C(G) = \{u_1, u_2, \ldots, u_n\}$.

The distance of any two vertices $x, y \in V(G)$ is $d_G(x,y) \leqslant 2r + r(H)$. The equality holds only if x and y are center eccentric vertices of the graphs G_i and G_j, respectively, for which $d_G(u_i, u_j) = r(H)$. Thus $\operatorname{Peri}(G) = \bigcup_{i=1}^{n} \operatorname{Peri}(G_i)$.

Similarly, the distance of a vertex $x \in V(G)$ from u_i, $i \in \langle 1, n \rangle$ is $d_G(x, u_i) = r + r(H)$ only if x is a center eccentric vertex of G_j for which $d_G(u_i, u_j) = r + r(H)$. Therefore $\operatorname{Cep}(G) = \bigcup_{i=1}^{n} \operatorname{Cep}(G_i)$. □

In the conclusion we give an estimate of the number of edges of an S-graph.

Theorem 2.12. *Let G be an* S-*graph with p vertices and q edges. Let $d(G) = d \geqslant 3$. Then $p - 1 \leqslant q \leqslant d + 1/2(p - d + 1)(p - d + 4)$.*

P r o o f. Ore [5] proved this upper bound for any connected graph with diameter $d \geqslant 3$. As follows from the example in Fig. 4, this upper bound is attained also for S-graphs. The lower bound is obvious and it is attained for P_{d+1}. □

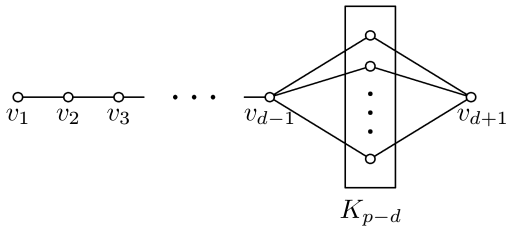


Fig. 4

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