NEARLY DISJOINT SEQUENCES IN CONVERGENCE ℓ-GROUPS

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Abstract. For an abelian lattice ordered group G let $\operatorname{conv} G$ be the system of all compatible convergences on G; this system is a meet semilattice but in general it fails to be a lattice. Let α_{nd} be the convergence on G which is generated by the set of all nearly disjoint sequences in G, and let α be any element of $\operatorname{conv} G$. In the present paper we prove that the join $\alpha_{nd} \vee \alpha$ does exist in $\operatorname{conv} G$.

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All ℓ-groups (= lattice ordered groups) considered in the present paper are assumed to be abelian.

For a convergence ℓ-group we apply the same notation and definitions as in [2].

Let G be an ℓ-group. A sequence (a_n) in G^+ is said to be nearly disjoint if there exists a positive integer m such that $a_{n(1)} \wedge a_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers with $n(i) \geqslant m$ for $i = 1, 2$.

We prove that for each ℓ-group G there exists a convergence α on G such that, whenever (x_n) is a nearly disjoint sequence in G^+, then $x_n \to_\alpha 0$.

This yields that there exists a convergence α_{nd} on G such that α_{nd} is generated by the set of all nearly disjoint sequences in G^+.

We denote by $\operatorname{conv} G$ the system of all convergences on G; this system is partially ordered by the set-theoretical inclusion. Each interval of $\operatorname{conv} G$ is a complete lattice, but if α_1 and α_2 are elements of $\operatorname{conv} G$, then the join $\alpha_1 \vee \alpha_2$ need not exist in $\operatorname{conv} G$.

We show that the join $\alpha_{nd} \vee \alpha$ does exist in $\operatorname{conv} G$ for each element α of $\operatorname{conv} G$.

For a similar result concerning disjoint sequences in a Boolean algebra cf. [3] (the distinction is in the point that in the present paper we do not assume the Urysohn property for a convergence, while in [3] the Urysohn property was supposed to be valid).

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A convergence ℓ-group (G,α) will be called strong if for each $g \in G$ with $g > 0$ there exists a sequence (x_n) in the interval $[0,g]$ such that $x_n \to_\alpha 0$ and $x_{n(1)} \neq x_{n(2)}$ whenever $n(1), n(2)$ are distinct positive integers. We use nearly disjoint sequences to construct a proper class of nonisomorphic types of archimedean strong convergence ℓ-groups.

1. Convergences generated by nearly disjoint sequences

In this section we assume that G is an ℓ-group. The symbol $\mathbb{N}$ denotes the set of all positive integers.

For the sake of completeness, we recall the following notation and definition concerning the notion of convergence in G as applied in [2].

Let $g \in G$ and $(g_n) \in G^{\mathbb{N}}$. If $g_n = g$ for each $n \in \mathbb{N}$, then we write $(g_n) = \operatorname{const} g$. For $(h_n) \in G^{\mathbb{N}}$ we put $(h_n) \sim (g_n)$ if there is $m \in \mathbb{N}$ such that $h_n = g_n$ for each $n \in \mathbb{N}$ with $n \geqslant m$.

A convex subsemigroup α of the lattice ordered semigroup $(G^{\mathbb{N}})^+ = (G^+)^{\mathbb{N}}$ is said to be a convergence on G if it satisfies the following conditions:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II′) Let $(g_n) \in \alpha$ and $(h_n) \in (G^+)^{\mathbb{N}}$. If $(h_n) \sim (g_n)$, then $(h_n) \in \alpha$.

(III) Let $g \in G$. Then $\operatorname{const} g \in \alpha$ if and only if $g = 0$.

We denote by $D(G)$ the system of all nearly disjoint sequences in G^+. Consider the following condition for a sequence (a_n) in G^+:

$(*)$ For each $m \in \mathbb{N}$, the relation $\bigwedge\limits_{n \geqslant m} a_n = 0$ is valid.

1.1. Lemma. *Let (b_n) be a sequence in G^+ satisfying the condition $(*)$. Further, let $k \in \mathbb{N}$ and for each $i \in \{1, 2, \dots, k\}$ let (x_n^i) be an element of $D(G)$. Then the sequence*

$$(x_n^1 + x_n^2 + \ldots + x_n^k + b_n)$$

satisfies the condition $(*)$.

Proof. We put

$$u_n = x_n^1 + x_n^2 + \ldots + x_n^k + b_n$$

for each $n \in \mathbb{N}$. We proceed by induction with respect to k.

Let $k = 1$. By way of contradiction, suppose that (u_n) does not satisfy the condition $(*)$. Hence there are $m \in \mathbb{N}$ and $0 < c \in G$ such that the relation

$$c \leqslant x_n^1 + b_n$$

is valid for each $n \in \mathbb{N}$ with $n \geqslant m$.

We shall repeatedly use Riesz Decomposition Theorem. It yields that for each $n \geqslant m$ there are c_n^1 and c_n^2 such that

$$c_n^1 \in [0, x_n^1], \quad c_n^2 \in [0, b_n], \quad c = c_n^1 + c_n^2.$$

If $c_n^1 = 0$ for each $n \geqslant m$, then $c = c_n^2$ for each $n \geqslant m$. This is impossible since (b_n) satisfies the condition $(*)$.

Hence there is $n(1) \geqslant m$ such that $c_{n(1)}^1 > 0$. Let $n \geqslant n(1) + 1$. Then

$$c_n^1 \wedge c_{n(1)}^1 \leqslant x_n^1 \wedge x_{n(1)}^1 = 0, \quad c_{n(1)}^1 \leqslant c,$$

thus $c_{n(1)}^1 \leqslant c_n^2 \leqslant b_n$. We have arrived at a contradiction with the condition $(*)$ for (b_n). Hence the assertion is valid for $k = 1$.

Let $k > 1$ and suppose that the assertion holds for $k - 1$. By way of contradiction, suppose that it does not hold for k. Hence there exist $m \in \mathbb{N}$ and $0 < c \in G$ such that

$$c \leqslant x_n^1 + x_n^2 + \ldots + x_n^k + b_n$$

is valid for each $n \geqslant m$. Thus for each such n there are c_n^1, c_n^2 and c_n^3 in G such that

$$c = c_n^1 + c_n^2 + c_n^3,$$

$$c_n^1 \in [0, x_n^1 + x_n^2 + \ldots + x_n^{k-1}], \quad c_n^2 \in [0, x_n^k], \quad c_n^3 \in [0, b_n].$$

If $c_n^2 = 0$ for each $n \geqslant m$, then

$$c \leqslant x_n^1 + x_n^2 + \ldots + x_n^{k-1} + b_n$$

for each $n \geqslant m$, which is a contradiction with the induction assumption.

Thus there exists $n(1) \geqslant m$ with $c_{n(1)}^2 > 0$. Put $m_1 = n(1) + 1$ and let $n \geqslant m_1$. Then

$$c_{n(1)}^2 \leqslant c, \quad c_{n(1)}^2 \wedge c_n^2 \leqslant a_{n(1)}^k \wedge a_n^k = 0,$$

whence $c_{n(1)}^2 \leqslant c_n^1 + c_n^3$. Therefore

$$c_{n(1)}^2 \leqslant x_n^1 + x_n^2 + \ldots + x_n^{k-1} + b_n$$

for each $n \geqslant m_1$. This is again a contradiction with the induction assumption. $\square$

1.2. Corollary. *Let $k \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, k\}$ let (x_n^i) be an element of $D(G)$. Then the sequence $(x_n^1 + x_n^2 + \ldots + x_n^k)$ satisfies the condition $(*)$.*

A nonempty subset X of $(G^+)^{\mathbb{N}}$ is said to be regular if there exists $\alpha \in \operatorname{conv} G$ such that $X \subseteq \alpha$.

1.3. Lemma. *Let X be a nonempty subset of $(G^+)^{\mathbb{N}}$. Then the following conditions are equivalent:*

(i) *X is regular.*

(ii) *Whenever $0 \leqslant c \in G$, $(x_n^1), (x_n^2), \ldots, (x_n^k) \in X$, (y_n^i) is a subsequence of (x_n^i) for $i = 1, 2, \ldots, k$, and $K, m \in \mathbb{N}$ satisfy*

$$c \leqslant K(y_n^1 + y_n^2 + \ldots + y_n^k)$$

for each $n \in \mathbb{N}$ with $n \geqslant m$, then $c = 0$.

Proof. This is a consequence of Proposition 2.3 in [2]. □

1.4. Lemma. *The set $D(G)$ is regular.*

Proof. Let $(x_n^1), (x_n^2), \ldots, (x_n^k)$ be elements of $D(G)$. For each $i \in \{1, 2, \ldots, k\}$ let (y_n^i) be a subsequence of (x_n^i). If $K \in \mathbb{N}$, then (Ky_n^i) belongs to $D(G)$ for $i = 1, 2, \ldots, k$. Now it suffices to apply 1.2 and 1.3. □

Let $\emptyset \neq X \subseteq (G^+)^{\mathbb{N}}$ and $\alpha \in \operatorname{conv} G$. Suppose that

(i) $X \subseteq \alpha$;

(ii) whenever $\beta \in \operatorname{conv} G$ and $X \subseteq \beta$, then $\alpha \subseteq \beta$.

Under these conditions the convergence α is said to be generated by the set X.

We denote by $D_1(G)$ the set of all sequences (u_n) which satisfy the following condition: there exist $(x_n^1), (x_n^2), \ldots, (x_n^k)$ in $D(G)$ such that

$$u_n = x_n^1 + x_n^2 + \ldots + x_n^k$$

for each $n \in \mathbb{N}$.

From Proposition 2.3 in [2] we obtain

1.5. Lemma. *Let X be a regular subset of $(G^+)^{\mathbb{N}}$ and let (z_n) be a sequence in G^+. Then the following conditions are equivalent:*

(i) *(z_n) belongs to the convergence on G which is generated by X.*

(ii) *There exist $(x_n^1), (x_n^2), \ldots, (x_n^k) \in X$, $K \in M$, $m \in \mathbb{N}$ and $(y_n^1), (y_n^2), \ldots, (y_n^k) \in (G^+)^{\mathbb{N}}$ such that (y_n^i) is a subsequence of (x_n^i) $(i = 1, 2, \ldots, k)$ and*

$$z_n \leqslant K(y_n^1 + y_n^2 + \ldots + y_n^k)$$

is valid for each $n \in \mathbb{N}$ with $n \geqslant m$.

Let the meaning of α_{nd} be as in the introduction; in view of 1.4, α_{nd} does exist.

1.6. Proposition. $D_1(G) = \alpha_{nd}$.

P r o o f. It is clear that $D(G) \subseteq \alpha_{nd}$ and hence $D_1(G) \subseteq \alpha_{nd}$. Let $(z_n) \in \alpha_{nd}$. We apply 1.5 for $X = D(G)$. Then (under the notation as in 1.5) $(Ky_n^i) \in D(G)$ for $i = 1, 2, \ldots, k$, and for each $n \geqslant m$ the element z_n can be written in the form

$$z_n = t_n^1 + t_n^2 + \ldots + t_n^k$$

with $t_n^i \in [0, Ky_n^i]$, $i = 1, 2, \ldots, k$. Thus $(t_n^i) \in D(G)$ for $i = 1, 2, \ldots, k$ and hence $(z_n) \in D_1(G)$. □

1.7. Lemma. *Let $\alpha \in \operatorname{conv} G$, $X = \alpha \cup \alpha_{nd}$. Then X is regular.*

P r o o f. This is a consequence of 1.1 and 1.6. □

From 1.7 and from Proposition 2.1 in [2] we obtain

1.8. Theorem. *Let $\alpha \in \operatorname{conv} G$. Then the join $\alpha \vee \alpha_{nd}$ does exist in $\operatorname{conv} G$.*

2. Strong convergence ℓ-groups

We apply the notion of strong convergence ℓ-group as defined in the introduction.

2.1. E x a m p l e. Let $\mathbb{R}$ be the set of all reals with the usual topology and let H be the additive group of all continuous real functions on $\mathbb{R}$. The set H is partially ordered coordinate-wise. Then H is an archimedean ℓ-group. Put $\alpha = D_1(H)$. In view of 1.6, (H, α) is a convergence ℓ-group. Let $0 < f \in H$. There exist $f_n \in [0, f]$ $(n \in \mathbb{N})$ such that $f_n > 0$ for each $n \in \mathbb{N}$ and $f_{n(1)} \wedge f_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers. Thus $f_n \to_\alpha 0$. Therefore the convergence ℓ-group (H, α) is strong.

2.2. E x a m p l e. Let I be a nonempty set and for each $i \in I$ let $H_i = H$, where H is as in 2.1. Put

$$H(I) = \prod_{i \in I} H_i.$$

Then H is an archimedean ℓ-group.

For $i \in I$ and $f \in H(I)$ let f^i be the component of f in H_i. Let $0 < f \in H(I)$. Thus there is $i \in I$ such that $f^i > 0$. Then in view of the properties of H (cf. 2.1) there exist $f_n \in [0, f]$ $(n \in \mathbb{N})$ such that $f_n > 0$ for each $n \in \mathbb{N}$ and $f_{n(1)} \wedge f_{n(2)} = 0$ whenever $n(1), n(2)$ are distinct positive integers. Thus $f_n \to_\alpha 0$, where $\alpha = D_1(H(I))$. Hence $(H(I), \alpha)$ is a strong convergence ℓ-group.

Let I_1 and I_2 be nonempty sets such that

$$\operatorname{card} I_1 \neq \operatorname{card} I_2. \tag{1}$$

It is easy to verify that the ℓ-group H is directly indecomposable. If an ℓ-group has a direct product decomposition with nonzero directly indecomposable direct factors, then this direct decomposition is uniquely determined (this is a consequence of the well-known Shimbireva's theorem [4] on the existence of a common refinement of any two direct product decompositions of a directed group; cf. also Fuchs [1]). Hence the number of nonzero directly indecomposable direct factors of $H(I_k)$ is equal to $\operatorname{card} I_k$ $(k = 1, 2)$. This yields that whenever (1) holds, then $H((I_1)$ and $(H(I_2)$ are not isomorphic. Therefore the convergence ℓ-groups $(H(I_1), D_1(H(I_1)))$ and $(H(I_2), D_1(H(I_2)))$ are not isomorphic.

From this we conclude

2.3. Proposition. *There exists a proper class of nonisomorphic types of archimedean strong convergence ℓ-groups.*

Let us denote by S the class of all ℓ-groups G having the property that there is $\alpha \in \operatorname{conv} G$ such that (G, α) is a strong convergence ℓ-group.

It is easy to verify that the class S is closed with respect to ℓ-subgroups and with respect to direct products. The following example shows that S is not closed with respect to homomorphisms. Hence S fails to be a variety.

2.4. Example. Let $\mathbb{Z}$ and $\mathbb{R}$ be the additive group of all integers or of all reals, respectively, with the natural linear order. Put

$$G = \mathbb{Z} \circ \mathbb{R},$$

where the symbol $\circ$ denotes the lexicographic product. Then $G \in S$, but the factor ℓ-group $G/\mathbb{R}$ (being isomorphic to $\mathbb{Z}$) does not belong to S.

We remark without proof that S is a radical class of ℓ-groups.

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