INCOMPARABLY CONTINUABLE SETS OF SEMILATTICES

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Abstract. A finite set of finite semilattices is said to be incomparably continuable if it can be extended to an infinite set of pairwise incomparable (with respect to embeddability) finite semilattices. After giving some simple examples we show that the set consisting of the four-element Boolean algebra and the four-element fork is incomparably continuable.

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By a semilattice we mean a meet semilattice. Two semilattices A and B are said to be comparable if either A can be embedded into B or B can be embedded into A. A finite set S of pairwise incomparable finite semilattices is called incomparably continuable if there exists an infinite set of pairwise incomparable finite semilattices containing S.

This paper does not attempt to develop a general theory of incomparably continuable sets. We just bring a few examples, showing that the general theory might be quite intricate. Of course, the concept could be introduced in the more general framework of universal algebra.

For the terminology and basic concepts of universal algebra, the reader is referred to [1].

For every $n \geqslant 1$ denote by $\mathscr{C}_n$ the n-element chain. Denote by $\mathscr{B}_4$ the four-element Boolean algebra and by $\mathscr{F}$ the fork, i.e., the semilattice with elements a, b, c, d and relations $a < b < c$ and $b < d$.

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Theorem 1. *If $n \leqslant 3$, then $\{\mathscr{C}_n\}$ is not incomparably continuable. If $n \geqslant 4$, then $\{\mathscr{C}_n\}$ is incomparably continuable.*

P r o o f. Every semilattice incomparable with $\mathscr{C}_3$ consists of the zero element and a set of atoms. Any two such semilattices are comparable. Consequently, $\{\mathscr{C}_3\}$ is not incomparably continuable.

For every $n \geqslant 3$ let S_n be the semilattice with elements $0, a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $0 < a_i < b_i$ for all i, $a_i < b_{i+1}$ for $i < n$, $a_n < b_1$ (and there are no other relations $x < y$). Then $\{\mathscr{C}_4, S_3, S_4, S_5, \ldots\}$ is an infinite set of pairwise incomparable finite semilattices. □

Lemma 2. *Let $f_0, f_1, \ldots$ be an infinite sequence of non-increasing functions from n^ω (where $n \in \omega$ is fixed). Then there exists an increasing sequence $i_0 < i_1 < i_2 < \ldots$ of nonnegative integers with $f_{i_0} \leqslant f_{i_1} \leqslant \ldots$.*

P r o o f. By induction on n. For $n = 0$ it is clear. Let $n > 0$.

Suppose first that for some $k \geqslant 0$, there are infinitely many numbers i with $f_i(j) < n-1$ for all $j \geqslant k$. Then for some $k \geqslant 0$, there are infinitely many numbers i with $f_i(j) = n-1$ for $j < k$ and $f_i(j) < n-1$ for $j \geqslant k$. Evidently, we can use induction.

Now suppose that there is no such k. Then there is an infinite increasing sequence $j_0 < j_1 < j_2 < \ldots$ such that whenever $f_{j_l}(i) = n-1$ then $f_{j_{l+1}}(i) = n-1$. We can assume $j_l = l$, and hence: whenever $f_l(i) = n-1$ then $f_{l+1}(i) = f_{l+2}(i) = \ldots = n-1$. Define g_k by $g_k(i) = f_k(i)$ if $f_k(i) < n-1$ and $g_k(i) = n-2$ if $f_k(i) = n-1$. By induction, there is an infinite sequence $k_0 < k_1 < k_2 < \ldots$ with $g_{k_0} \leqslant g_{k_1} \leqslant g_{k_2} \leqslant \ldots$. But then evidently $f_{k_0} \leqslant f_{k_1} \leqslant f_{k_2} \leqslant \ldots$. □

Lemma 3. *Let $a_0, a_1, \ldots$ be an infinite sequence of nonnegative integers. Then there is a sequence $i_0 < i_1 < i_2 < \ldots$ such that $a_{i_0} \leqslant a_{i_1} \leqslant a_{i_2} \leqslant \ldots$.*

P r o o f. If for every k there is an i with $a_i \geqslant k$, it is evident. In the opposite case there is a number k with $a_i \leqslant k$ for all i. Then there is a number k with $a_i = k$ for infinitely many numbers i and everything is evident. □

Lemma 4. *Let $f_0, f_1, f_2, \ldots$ be an infinite sequence of non-increasing functions from ω^n (where $n \in \omega$ is fixed). Then there exists an infinite sequence $i_0 < i_1 < i_2 < \ldots$ with $f_{i_0} \leqslant f_{i_1} \leqslant f_{i_2} \ldots$.*

P r o o f. By induction on n. For $n = 0$ it is evident. Let $n > 0$. By induction, there is a sequence $j_0 < j_1 < \ldots$ such that $f_{j_0} \restriction n-1 \leqslant f_{j_1} \restriction n-1 \leqslant f_{j_2} \restriction n-1 \leqslant \ldots$. By Lemma 3 there is a sequence $k_0 < k_1 < k_2 < \ldots$ such that $f_{j_{k_0}}(n-1) \leqslant f_{j_{k_1}}(n-1) \leqslant f_{j_{k_2}}(n-1) \leqslant \ldots$. Then evidently $f_{j_{k_0}} \leqslant f_{j_{k_1}} \leqslant f_{j_{k_2}} \leqslant \ldots$. □

Lemma 5. *Let $f_0, f_1, f_2, \dots$ be an infinite sequence of non-increasing functions from ω^ω. Then there exist i, j with $i \neq j$ and $f_i \leqslant f_j$.*

P r o o f. Suppose that there are no such i, j. Denote by m the minimum of all $f_i(j)$. Let us fix a pair c, d with $f_c(d) = m$. For every $i \neq c$ there exists a number j with $f_i(j) < f_c(j)$; necessarily, $j < d$. Hence there is a number $j_0 < d$ such that $f_i(j_0) < f_c(j_0)$ for infinitely many numbers i. It follows from Lemma 2 that there is a sequence $k_0 < k_1 < k_2 < \dots$ such that $f_{k_0} \restriction (\omega - j_0) \leqslant f_{k_0} \restriction (\omega - j_0) \leqslant f_{k_2} \restriction (\omega - j_0) \leqslant \dots$. It follows from Lemma 4 that there is a sequence $l_0 < l_1 < l_2 < \dots$ such that $f_{k_{l_0}} \restriction j_0 \leqslant f_{k_{l_1}} \restriction j_0 \leqslant f_{k_{l_2}} \restriction j_0 \leqslant \dots$. Now, $f_{k_{l_0}} \leqslant f_{k_{l_1}}$ and $k_{l_0} \neq k_{l_1}$. $\square$

Theorem 6. *The set $\{\mathscr{F}, \mathscr{B}_4\}$ is not incomparably continuable.*

P r o o f. Suppose that there is an infinite sequence $\mathscr{F}, \mathscr{B}_4, A_1, A_2, \dots$ of pairwise incomparable finite semilattices. For every nonincreasing sequence f from ω^ω which is non-zero on only a finite subset of ω define a finite semilattice S_f with zero 0 and elements $0 < a_{0,1} < \dots < a_{0,f(0)}$, $0 < a_{1,1} < \dots < a_{1,f(1)}$, …, $0 < a_{k,1} < \dots < a_{k,f(k)}$ where k is the greatest number with $f(k) \neq 0$. Then every A_i is some S_f. Since S_f is embeddable into S_g if $f \leqslant g$, the result follows from Lemma 5. $\square$

It would be possible to formulate various open problems concerning incomparably continuable sets of semilattices (or other kinds of algebraic systems). Let us point out just one open problem:

P r o b l e m. Is there an algorithm deciding, for any finite set of finite semilattices, whether the set is incomparably continuable?

References

[1] *R. McKenzie, G. McNulty, W. Taylor*: Algebras, Lattices, Varieties, Vol. I. Wadsworth & Brooks/Cole, Monterey, CA, 1987.

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