CONCRETE QUANTUM LOGICS WITH GENERALISED COMPATIBILITY

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(Received April 16, 1997)

Abstract. We present three results stating when a concrete (= set-representable) quantum logic with covering properties (generalization of compatibility) has to be a Boolean algebra. These results complete and generalize some previous results [3, 5] and answer partially a question posed in [2].

Keywords: Boolean algebra, concrete quantum logic, covering, Jauch-Piron state, orthocompleteness

MSC 1991: 03G12, 81P10

1. Basic notions

Let us recall the main notion we shall deal with in this paper.

Definition 1.1. A *concrete logic* is a pair (X, L), where $X \neq \emptyset$ and $L \subset \exp X$ such that

(1) $\emptyset \in L$;
(2) $A^c = X \setminus A \in L$ whenever $A \in L$;
(3) $\bigcup M \in L$ whenever $M \subset L$ is a finite set of mutually disjoint elements.

A *concrete σ-logic* is a concrete logic (X, L) such that

(3σ) $\bigcup M \in L$ whenever $M \subset L$ is a countable set of mutually disjoint elements.

Let us note that the above definition is not given in the most efficient way. Indeed, since $\emptyset$ is a finite set of mutually disjoint elements and $\bigcup \emptyset = \emptyset$, condition (1) follows from condition (3). Moreover, it is obvious that condition (3) follows from condition (3σ).

The author acknowledges the support by the grant 201/96/0117 of the Czech Grant Agency.

The following lemma will be useful in the sequel. First, let us observe that if A, $B \in L$ and $A \subset B$, then $B \setminus A = (A \cup B^c)^c \in L$ for every concrete logic (X, L).

Lemma 1.2. *Let (X, L) be a concrete σ-logic and let $A_i \in L$ $(i = 1, 2, \ldots)$ be such that $A_1 \supset A_2 \supset \cdots$. Then $\bigcap\limits_{i=1}^{\infty} A_i \in L$.*

Proof. The elements $A_i \setminus A_{i+1} \in L$ $(i = 1, 2, \ldots)$ are mutually orthogonal, hence $\bigcup\limits_{i=1}^{\infty} (A_i \setminus A_{i+1}) \in L$ and $\bigcap\limits_{i=1}^{\infty} A_i = A_1 \setminus \bigcup\limits_{i=1}^{\infty} (A_i \setminus A_{i+1}) \in L$. □

2. Covering properties

Definition 2.1. Let (X, L) be a concrete logic, $Y \subset X$ and let n be a natural number. A *covering of* Y is a set $M \subset L$ such that $Y = \bigcup M$. A covering M is an *n-covering* if $\operatorname{card} M \leqslant n$.

We say that (X, L) has the *n-covering property* (*finite covering property*, resp.) if for every A, $B \in L$ there is an n-covering (finite covering, resp.) of $A \cap B$.

It is well-known that a concrete logic (X, L) is a Boolean algebra if and only if $A \cap B \in L$ for every A, $B \in L$, i. e. if and only if (X, L) has the 1-covering property. Thus, the notions of n-covering property (finite covering property), introduced in [3], are generalizations of compatibility in Boolean algebras.

The next lemma will be used in the sequel.

Lemma 2.2. *Let (X, L) be a concrete logic with the finite covering property. Then for every finite set $F \subset L$ there is a finite covering $G \subset L$ of $\bigcap F$.*

Proof. Let us proceed by induction. First, if F is a one-element subset of L (empty set, resp.), then we can put $G = F$ ($G = \{X\}$, resp.).

Now, let us suppose that there is a natural number $n \geqslant 1$ such that the lemma holds for every $F \subset L$ with $\operatorname{card} F = n$. Let $F \subset L$ with $\operatorname{card} F = n + 1$ and let $A \in F$. According to the previous assumption, there is a finite covering $G \subset L$ of $\bigcap (F \setminus \{A\})$. According to the finite covering property, for every $B \in G$ there is a finite covering $G_B \subset L$ of $A \cap B$. Thus, $\bigcup\limits_{B \in G} G_B \subset L$ is a finite covering of $\bigcap F$.

□

Before we present the main result of this section, let us prove the following technical lemma.

Lemma 2.3. *Let (X, L) be a concrete σ-logic and let $m, n \geqslant 2$ be natural numbers such that $m \leqslant n+1$. Let us suppose that for every set $F \subset L$ with* $\operatorname{card} F \leqslant n$ *there is an m-coverig $G \subset L$ of $\bigcap F$. Then for every set $F \subset L$ with* $\operatorname{card} F \leqslant n$ *there is an $(m-1)$-covering $G \subset L$ of $\bigcap F$.*

Proof. Let $F \subset L$ with $\operatorname{card} F \leqslant n$. Let us define by induction sequences $(A_{i1}, \dots, A_{in}) \in L^n$, $(B_{i0}, \dots, B_{in}) \in L^{n+1}$ $(i = 1, 2, \dots)$ as follows: Let $(A_{11}, \dots, A_{1n})$ be such that $F = \{A_{11}, \dots, A_{1n}\}$. If $(A_{i1}, \dots, A_{in}) \in L^n$ is defined for a natural number $i \geqslant 1$ then let us take $(B_{i0}, \dots, B_{in}) \in L^{n+1}$ such that $B_{ij} = \emptyset$ for $j \geqslant m$ and $\bigcap\limits_{j=1}^{n} A_{ij} = \bigcup\limits_{j=0}^{n} B_{ij}$ and let us put $A_{i+1,j} = A_{ij} \setminus B_{ij}$ $(j \in \{1, \dots, n\})$.

Let us denote

$$B_0 = \bigcap_{i=1}^{\infty} B_{i0}, \qquad B_j = \bigcup_{i=1}^{\infty} B_{ij}, \quad j \in \{1, \dots, n\}.$$

It is easy to see that the elements $B_{1j}, B_{2j}, \dots$ $(j \in \{1, \dots, n\})$ are mutually disjoint, hence $B_j \in L$ for every $j \in \{1, \dots, n\}$. Moreover, $B_m = \dots = B_n = \emptyset$. Further,

$$B_{i0} \supset \bigcap_{j=1}^{n} A_{i+1,j} \supset B_{i+1,0} \qquad (i = 1, 2, \dots).$$

Hence, according to Lemma 1.2, $B_0 \in L$, too. Since

$$\bigcap F = B_0 \cup B_1 \cup \dots \cup B_{m-1}$$

and since $B_0 \cup B_1 \in L$ $(B_0 \cap B_1 = \emptyset)$, the proof is complete. □

Theorem 2.4. *Let (X, L) be a concrete σ-logic. Let us suppose that there is a natural number $n \geqslant 2$ such that for any set $F \subset L$ with* $\operatorname{card} F \leqslant n$ *there is an $(n+1)$-covering of $\bigcap F$. Then (X, L) is a Boolean algebra.*

Proof. Using Lemma 2.3 n-times, we obtain that (X, L) has the 1-covering property, i. e. (X, L) is a Boolean algebra. □

Corollary 2.5. *Every concrete σ-logic with the 3-covering property is a Boolean algebra.*

This corollary generalizes [3, Proposition 4.6], where an analogous result is stated for concrete σ-logics with the 2-covering property.

Definition 3.1. Let (X, L) be a concrete logic. A *state* on (X, L) is a mapping $s\colon L \to [0, 1]$ such that

(1) $s\big(X\big) = 1$;

(2) $s\big(\bigcup M\big) = \sum\limits_{A \in M} s(A)$ whenever $M \subset L$ is a finite set of mutually disjoint elements.

A state s on (X, L) is called *Jauch-Piron* if for every $A, B \in L$ with $s(A) = s(B) = 1$ there is a $C \in L$ such that $C \subset A \cap B$ and $s(C) = 1$.

It is easy to see that $s(\emptyset) = 0$ and $s(A^c) = 1 - s(A)$ for every state s on a concrete logic (X, L) and for every $A \in L \setminus \{\emptyset\}$. Further, for every concrete logic (X, L), every point $x \in X$ *carries* a two-valued state s_x on (X, L) defined by

$$s_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Before we present the main result of this section, we need the following definition.

Definition 3.2. Let (X, L) be a concrete logic and let $M, N \subset L$ be two coverings of $Y \subset X$. We say that N is a *coarsing of* M if for every $A \in M$ there is a $B \in N$ such that $A \subset B$.

Theorem 3.3. *Let (X, L) be a concrete logic such that every state on it is Jauch-Piron. Let us suppose that for every A, $B \in L$ every covering of $A \cap B$ admits a countable coarsing. Then L is a Boolean algebra.*

Proof. It suffices to prove that $A \cap B \in L$ for every A, $B \in L$. Let A, $B \in L$. If $A \cap B = \emptyset$, the proof is complete. Let us suppose that $A \cap B \neq \emptyset$. Then $S_{A,B} = \big\{s;\ s \text{ is a state on } (X, L) \text{ with } s(A) = s(B) = 1\big\}$ is nonempty (every point $x \in A \cap B$ carries a two-valued state $s_x \in S_{A,B}$). Since every state on (X, L) is Jauch-Piron, for every $s \in S_{A,B}$ there is a $C_s \in L$ such that $s(C_s) = 1$. Let us take a countable coarsing M of the covering $\{C_s;\ s \in S_{A,B}\}$ of $A \cap B$, a countable set $Y \subset A \cap B$ such that $Y \cap (C \setminus D) \neq \emptyset$ for every $C, D \in M$ with $C \setminus D \neq \emptyset$ and, finally, a state s that is a σ-convex combination (with non-zero coefficients) of all s_y $(y \in Y)$. Since $s \in S_{A,B}$, there is a $D_s \in M$ such that $s(D_s) = 1$. Thus, $D_s \supset Y$ and therefore $A \cap B = \bigcup M = D_s \in L$. □

Theorem 3.3 seems to be independent of the previous results in [3, 4, 7], nevertheless it has corollaries that were obtained using quite a different techniques. (Let us note that a unifying look at these attempts is presented in [8].) The following corollary was obtained (in a more general form) in [4].

Corollary 3.4. *Every countable concrete logic such that every state on it is Jauch-Piron is a Boolean algebra.*

The next corollary of Theorem 3.3 was obtained (in a more general form) in [7].

Corollary 3.5. *Let (X, L) be a concrete logic such that every state on it is Jauch-Piron. Let us suppose that (X, L) contains only countably many maximal Boolean subalgebras and these are complete. Then (X, L) is a Boolean algebra.*

P r o o f. It is easy to see that for every A, $B \in L$ every covering of $A \cap B$ admits a countable coarsing. $\square$

4. Covering properties and orthocompleteness

Definition 4.1. Let α be a cardinal number. A concrete logic (X, L) is called *α-orthocomplete* if $\bigvee M \in L$ (supremum with respect to inclusion) whenever $M \subset L$ is a set of mutually disjoint elements with $\operatorname{card} M \leqslant \alpha$.

It is obvious that condition (3σ) of Definition 1.1 implies that a concrete σ-logic is ω_0-orthocomplete (ω_0 denotes the countable cardinal)—this is usually denoted as *σ-orthocompleteness.*

The following theorem generalizes a result from [5] and answers partially a question posed in [2].

Theorem 4.2. *Every c-orthocomplete (c denotes the cardinality of real numbers) concrete σ-logic with the finite covering property is a Boolean algebra.*

P r o o f. Let (X, L) be a concrete σ-logic with the finite covering property and let A, $B \in L$. It suffices to prove that $A \cap B \in L$. Let us define by induction finite subsets F_i $(i = 1, 2, \ldots)$ of L as follows: First, $F_1 \subset L$ is a finite covering of $A \cap B$. Now, let a finite set $F_i = \{A_1, \ldots, A_n\} \subset L$ be defined for a natural number $i \geqslant 1$. Let us denote by G_i the set of all intersections of the form $A_1^{e_1} \cap \cdots \cap A_n^{e_n}$, where $(e_1, \ldots, e_n) \in \{-1, 1\}^n \setminus \{-1\}^n$ and $A_j^1 = A_j$, $A_j^{-1} = X \setminus A_j$ $(j = 1, \ldots, n)$. G_i is a finite set of mutually disjoint subsets of X such that $\bigcap F_i = \bigcup G_i$. According to Lemma 2.2, for every $Y \in G_i$ there is a finite covering $G_Y \subset L$ of Y. Let us put $F_{i+1} = \bigcup\limits_{Y \in G_i} G_Y$.

Let us consider all sequences $C_1, C_2, \ldots$ such that $C_i \in F_i$ $(i = 1, 2, \ldots)$ and $C_1 \supset C_2 \supset \cdots$. According to Lemma 1.2, $\bigcap\limits_{i=1}^{\infty} C_i \in L$ for each such sequence. Hence, we have obtained at most the continuum of mutually disjoint elements of L such that their union is $A \cap B$. Since their supremum exists, it is equal to $A \cap B$. Thus, $A \cap B \in L$. $\square$

Before we present a corollary of Theorem 4.2, let us recall a result connecting the covering properties with Jauch-Piron states [3, Theorem 3.5].

Theorem 4.3. *Let (X, L) be a concrete logic such that every two-valued state on it is Jauch-Piron. Then (X, L) has the finite covering property.*

Corollary 4.4. *Every c-orthocomplete concrete σ-logic such that every two-valued state on it is Jauch-Piron is a Boolean algebra.*

Proof. It follows from Theorem 4.3 and Theorem 4.2. □

Remark 4.5. The above corollary can be stated in the following (more general) way: Every c-orthocomplete quantum σ-logic with a closed full set of two-valued Jauch-Piron σ-states is a Boolean algebra. Indeed, concrete σ-logics are exactly representations of quantum σ-logics with a full set of two-valued σ-states (see e. g. [1, 6]) and Theorem 4.3 can be stated for quantum logics with a closed full set of two-valued Jauch-Piron states (the set of two-valued states is closed in the product topology in $[0, 1]^L$).

The following question (posed in [2]) remains open. Here we have given the negative answer in the case that the concrete logic in question is also c-orthocomplete.

Question 4.6. Is there a concrete σ-logic that is not a Boolean algebra such that every state on it is Jauch-Piron?

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