PROPERTY (A) OF n-TH ORDER ODE'S

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Abstract. The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the ordinary differential equation

$$L_n u(t) + p(t)u(t) = 0.$$

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Consider the n-th order ($n \geqslant 2$) differential equation

$$L_n u(t) + p(t)u(t) = 0, \tag{1}$$

where

$$L_n u(t) = \left(\frac{1}{r_{n-1}(t)}\left(\frac{1}{r_{n-2}(t)}\cdots\left(\frac{1}{r_1(t)}u'(t)\right)'\cdots\right)'\right)',$$

p and $r_i : (t_0, \infty) \to \mathbb{R}^+ = (0, \infty)$ are continuous, $1 \leqslant i \leqslant n-1$. In the sequel we will suppose that $\int_{t_0}^{\infty} r_i(s)\,\mathrm{d}s = \infty$ for $1 \leqslant i \leqslant n-1$. It is usual to denote

$$\begin{aligned} D_0 u(t) &= u(t), \\ D_i u(t) &= \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}D_{i-1}u(t), \quad 1 \leqslant i \leqslant n-1, \\ D_n u(t) &= \frac{\mathrm{d}}{\mathrm{d}t}D_{n-1}u(t). \end{aligned} \tag{2}$$

By a solution of Eq. (1) we mean a function $u : (T_u, \infty) \to \mathbb{R}$ such that

(i) $D_i u(t)$, $0 \leqslant i \leqslant n$ exist and are continuous on $[T_u, \infty)$;

(ii) $u(t)$ satisfies (1);

(iii) $\sup\{|u(s)| : t \leqslant s < \infty\} > 0$ for any $t \geqslant T_u$.

Such a solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

It is well known (see e.g. [2] or [3]) that the set $\mathcal{N}$ of all nonoscillatory solutions of (1) can be divided into the following classes:

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ odd},$$
$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ even},$$

where $u(t) \in \mathcal{N}_\ell$ if and only if

$$\begin{aligned} u(t)D_i u(t) > 0, &\quad 0 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t) D_i u(t) > 0, &\quad \ell \leqslant i \leqslant n \end{aligned} \tag{3}$$

for all large t. Following Foster and Grimmer [3] we say that $u(t)$ is a function of degree ℓ if $u(t)$ satisfies (3).

For the class $\mathcal{N}_0$ of (1), it is shown in [4] that $\mathcal{N}_0 \neq \emptyset$ if n is odd. Therefore, we are interested in the following particular situation:

Definition 1. Equation (1) is said to have property (A) if for n even $\mathcal{N} = \emptyset$ (i.e. (1) is oscillatory) and for n odd $\mathcal{N} = \mathcal{N}_0$.

This definition can be found in [6]. There is much literature regarding property (A) of (1) (see enclosed references). Integral conditions have been given under which (1) enjoys property (A). The following result is due to Trench [18].

Define for $1 \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$

$$\begin{aligned} I_0 &= 1, \\ I_k(t, s; r_1, \ldots, r_k) &= \int_s^t r_1(x) I_{k-1}(x, s; r_2, \ldots, r_k)\,\mathrm{d}x, \\ J_k(t) &= I_k(t_0, t; r_1, \ldots, r_k), \\ N_k(t) &= I_k(t_0, t; r_{n-1}, \ldots, r_{n-k}). \end{aligned}$$

Theorem A. *Let n be even. Assume that for all $i \in \{1, 3, \ldots, n-1\}$*

$$\int^\infty J_{i-1}(t) N_{n-i-1}(t) p(t)\,\mathrm{d}t = \infty. \tag{4}$$

Then (1) has property (A).

A question naturally arises what will happen when conditions (4) are violated. In fact, Theorem A cannot cover an important class of Euler's equation of the form

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} t^{\alpha+m} \frac{\mathrm{d}^m x}{\mathrm{d}t^m} + c t^{\alpha-m} x = 0, \quad t \geqslant 1, \tag{5}$$

where α and $c > 0$ are constants with $\alpha + m \leqslant 1$, since in this case the integrals in (4) converge.

Trench's result has been later improved by Kusano, Naito and Tanaka in [6] and [7], where (1) is compared with a set of second order differential equations and property (A) of (1) is reduced to the oscillation of a set of second order differential equations. On the other hand, Chanturia and Kiguradze [1] have improved (4) for the particular case of (1), namely for the differential equation

$$y^{(n)}(t) + p(t)y(t) = 0. \tag{6}$$

They have compared (1) with Euler's equation $t^n y^{(n)} + cy = 0$ to obtain the integral criterion

$$\liminf_{t\to\infty} t^{n-1} \int_t^\infty p(s)\,\mathrm{d}s = \frac{M^*}{n-1}$$

for property (A) of (6).

Our concern in this paper is to replace condition (4) by a similar one that is applicable also to (5). Our results complement and extend the above-mentioned results and also some other ones given in [16], [14], [10] and [8].

We consider a set of ℓ-th order ($n-1 \geqslant \ell \geqslant 1$) differential inequalities

$$\{M_{\ell+1}u(t) + q_{\ell+1}(t)u(t)\} \operatorname{sgn} u(t) \leqslant 0, \tag{$E_{\ell+1}$}$$

where $q_{\ell+1}$ is positive and continuous and

$$M_{\ell+1}u(t) = \left(\frac{1}{r_\ell(t)}\left(\frac{1}{r_{\ell-1}(t)} \cdots \left(\frac{1}{r_1(t)}u'(t)\right)' \cdots\right)\right)',$$

that is $M_{\ell+1}u(t) = r_{\ell+1}(t)D_{\ell+1}u(t)$ for $\ell < n$, and $M_n u(t) = D_n u(t)$.

Let us put

$$J_{1,\ell}(t) = J_\ell(t) \quad \text{and} \quad J_{2,\ell}(t) = I_{\ell-1}(t, t_0; r_2, \ldots, r_\ell).$$

Our main results are based on the following theorem:

Theorem 1. *Let $1 \leqslant \ell \leqslant n-1$. Assume that*

$$\int^{\infty}\left(J_{1,\ell}(t)q_{\ell+1}(t)-\frac{r_1(t)J_{2,\ell}(t)}{4J_{1,\ell}(t)}\right)\,\mathrm{d}t=\infty. \tag{7_ℓ}$$

Then $(E_{\ell+1})$ has no solutions of degree ℓ.

Proof. Assume that $(E_{\ell+1})$ possesses a positive nonoscillatory solution $u(t)$ such that $u(t)$ is of degree ℓ, that is

$$D_0u(t)>0,\ D_1u(t)>0,\dots,D_\ell u(t)>0,\ \big(D_\ell u(t)\big)'<0,\quad t\geqslant t_0.$$

Let

$$z(t)=\frac{J_{1,\ell}(t)D_\ell u(t)}{u(t)},\qquad t\geqslant t_0.$$

Then $z(t)>0$ and

$$z'(t)=\frac{r_1(t)J_{2,\ell}(t)}{J_{1,\ell}(t)}z(t)+\frac{J_{1,\ell}(t)(D_\ell u(t))'}{u(t)}-z(t)\frac{r_1(t)D_1u(t)}{u(t)}. \tag{8}$$

Assume that $\ell>1$. The identity $D_\ell u(t)=\dfrac{1}{r_\ell(t)}\big(D_{\ell-1}u(t)\big)'$ implies that

$$\begin{aligned}D_{\ell-1}u(t)&=D_{\ell-1}u(t_0)+\int_{t_0}^{t}r_\ell(s)D_\ell u(s)\,\mathrm{d}s\\&\geqslant D_\ell u(t)\int_{t_0}^{t}r_\ell(s)\,\mathrm{d}s.\end{aligned}$$

Hence, after $(\ell-3)$-fold integration, we arrive at

$$D_1u(t)\geqslant J_{2,\ell}(t)D_\ell u(t),\qquad t\geqslant t_0.$$

Therefore, combining (8) with the last inequality, one gets

$$\frac{J_{1,\ell}(t)(D_\ell u(t))'}{u(t)}\geqslant z'(t)+\frac{r_1(t)J_{2,\ell}(t)}{J_{1,\ell}(t)}\big(z^2(t)-z(t)\big). \tag{9}$$

Note that $z^2(t)-z(t)\geqslant-\frac{1}{4}$. Multiplying $(E_{\ell+1})$ by $J_{1,\ell}(t)$ and dividing the resulting equality by $u(t)$, we see in view of (9) that $z(t)$ is a positive solution of the differential inequality

$$z'(t)-\frac{r_1(t)J_{2,\ell}(t)}{4J_{1,\ell}(t)}+J_{1,\ell}(t)q_{\ell+1}(t)\leqslant 0. \tag{10}$$

That (10) also holds for $\ell=1$ follows from (8) and (M_2) (note that $J_{2,1}(t)\equiv 1$). An integration of (10) yields

$$z(t)+\int_{t_0}^{t}\left(J_{1,\ell}(s)q_{\ell+1}(s)-\frac{r_1(s)J_{2,\ell}(s)}{4J_{1,\ell}(s)}\right)\,\mathrm{d}s\leqslant z(t_0).$$

Letting $t\to\infty$, we get a contradiction with (7_ℓ). The proof is complete. □

The following result can be found in [5, Corollary 1].

Theorem B. *The equation* (1) *has a solution of degree* $n-1$ *if and only if the inequality* (E_n) *has a solution of degree* $n-1$.

For the particular case of (1) with $n=2$ and $n=3$ we have the following corollaries.

Corollary 1. *Denote* $R(t)=\int_{t_0}^{t} r(s)\,\mathrm{d}s$. *Assume that*

$$\int^{\infty}\left(R(s)p(s)-\frac{r(s)}{4R(s)}\right)\mathrm{d}s=\infty. \tag{11}$$

Then the second order differential equation

$$\left(\frac{1}{r(t)}u'\right)'+p(t)u=0 \tag{12}$$

is oscillatory.

Proof. By Theorem B, Eq. (12) is oscillatory if and only if (E_2) with $q_2=p$ and $r_1=r$ has no solution of degree 1. Since (7_1) reduces to (11), the assertion of this corollary follows from Theorem 1. □

Corollary 2. *Assume that*

$$\int^{\infty}\left(J_{1,2}(s)p(s)-\frac{r_1(s)J_{2,2}(s)}{4J_{1,2}(s)}\right)\mathrm{d}s=\infty. \tag{13}$$

Then the third order differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'\right)'\right)'+p(t)u=0$$

has property (A).

Proof. The proof of this corollary is analogous to that of Corollary 1 (noting that (7_2) reduces to (13)) and can be omitted. □

Example 1. Consider the equation

$$\left(\frac{1}{t}u''\right)'+\frac{a}{t^4}u=0,\qquad a>0,\quad t\geqslant 1.$$

By Corollary 2, this equation has property (A) provided $a>4.5$.

Now we extend our previous results to (1) with $n > 3$. For all large t and $i \in \{1, \dots, n-1\}$ define

$$\begin{aligned}
K_1(t;p) &= \int_t^\infty p(s)\,\mathrm{d}s,\\
K_2(t;r_{n-1},p) &= \int_t^\infty r_{n-1}(x)K_1(x;p)\,\mathrm{d}x,\\
K_i(t;r_{n-i+1},\dots,r_{n-1},p) &= \int_t^\infty r_{n-i+1}(x)K_{i-1}(x;r_{n-i+2},\dots,r_{n-1},p)\,\mathrm{d}x,\\
q_n(t) &= p(t),\\
q_i(t) &= r_i(t)K_{n-i}(t;r_{i+1},\dots,r_{n-1},p).
\end{aligned}$$

Theorem 2. *Assume that for all $\ell \in \{1, \dots, n-1\}$ with $n+\ell$ odd, conditions (7_ℓ) are satisfied. Then (1) has property (A).*

P r o o f. Since (7_1) with $n = 2$ reduces to (11) and (7_2) with $n = 3$ reduces to (13) the assertion of the theorem for $n = 2$ and $n = 3$ follows from Corollaries 1 and 2.

Now assume that $n > 3$. We want to show that $\mathcal{N}_\ell = \emptyset$ for all $\ell \in \{1, \dots, n-1\}$ with $n+\ell$ odd. Note that by Theorem 1, condition (7_{n-1}) implies that differential inequality (E_n) has no solution of degree $n-1$. By Theorem (B), Eq. (1) has no solution of degree $n-1$, either (i.e. $\mathcal{N}_{n-1} = \emptyset$).

Let $1 \leqslant \ell \leqslant n-2$. Assume that (1) has a positive nonoscillatory solution $u(t)$ and $u(t)$ is of degree ℓ. From (1) and $u'(t) > 0$ it follows that

$$D_{n-1}u(\infty) - D_{n-1}u(t) + \int_t^\infty p(s)u(s)\,\mathrm{d}s = 0, \quad t \geqslant t_0.$$

That is,

$$-D_{n-1}u(t) + u(t)\int_t^\infty p(s)\,\mathrm{d}s \leqslant 0.$$

Hence, after $(n-\ell-2)$-fold integration we arrive at

$$M_{\ell+1}u(t) + q_{\ell+1}(t)u(t) \leqslant 0.$$

That is, $u(t)$ is a solution of $(E_{\ell+1})$, but as $u(t)$ is of degree ℓ, it contradicts the assertions of Theorem 1. The proof is complete. □

Corollary 3. *Assume that*

$$\int^\infty \left((t-t_0)^{n-1}p(t) - \frac{(n-1)(n-1)!}{4(t-t_0)}\right)\mathrm{d}t = \infty. \tag{14}$$

Then the n-th order differential equation

$$u^{(n)} + p(t)u = 0 \tag{15}$$

has property (A).

P r o o f. To prove that (15) has property (A), it suffices (see Theorem 1.1 in [1]) to show that (15) has no solution of degree $n-1$. This fact follows from Theorems A and 1. □

It is interesting to compare Corollary 3 with the following result which is due to Chanturia and Kiguradze [1].

Lemma A. *The condition*

$$\int^{\infty} t^{n-1} p(t)\,\mathrm{d}t = \infty \tag{16}$$

is necessary for (15) *to have property* (A).

Note that the stronger condition (13) guarantees property (A) of (14), while (16) is not enough.

R e m a r k. Using suitable comparison theorems, our results can be extended to more general differential equations. In fact, it is known [5] that the delay differential equation

$$L_n u(t) + p(t)u(\tau(t)) = 0, \tag{17}$$

where L_n and p are the same as in (1) and τ satisfies

$$\tau \in C^1, \quad \tau(t) \leqslant t, \quad \tau(t) \to \infty \text{ as } t \to \infty, \tag{18}$$

has property (A) if so does the differential equation without delay

$$L_n u(t) + \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} u(t) = 0, \tag{19}$$

where τ^{-1} is the inverse function to τ. Applying Theorem 2 to (19) we immediately have a sufficient condition for (17) to have property (A). We illustrate this by the following result.

Corollary 4. *Assume that* (18) *holds. Further assume that*

$$\int^{\infty} \left((\tau(t) - t_0)^{n-1} p(t) - \frac{(n-1)(n-1)!}{4(t-t_0)} \tau'(t) \right) \mathrm{d}t = \infty.$$

Then the delay differential equation

$$u^{(n)}(t) + p(t)u(\tau(t)) = 0$$

has property (A).

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