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ON RECURRENCE PROPERTY OF RIESZ-RAIKOV SUMS

(submitted by D. Kh. Mushtari)

Abstract. The Riesz-Raikov sums $\sum f(\theta^k x)$ are recurrent in most cases.

1. Introduction

In the theory of lacunary series, some probabilistic limit theorems are proved for gap series $\sum f(n_k x)$. Among these, we focus on the recurrence property.

Hawkes [5] proved that $\{\sum_{k=1}^{N} \exp(in_k x)\}_{N\in\mathbf{N}}$ is dense in complex plain for a.e. x assuming very strong gap condition $\sum n_k/n_{k+1} < \infty$. Anderson and Pitt [1] used the theory of Bloch function and weaken the gap condition to $n_{k+1}/n_k \to \infty$ or $n_k = a^k$, where $a \geq 2$ is an integer. These results imply that $\{\sum_{k=1}^{N} \cos n_k x\}_{N\in\mathbf{N}}$ is dense in the real line. As to this one-dimensional recurrence, Ullich, Grubb and Moore [10], [4] succeeded in weakening the gap condition to the Hadamard's $n_{k+1}/n_k > q > 1$. The purpose of this paper is to show that their real analytic proof is also effective for general gap series $\sum f(n_k x)$ where f is not necessarily analytic function, which seems difficult to treat by the method of Anderson and Pitt.

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We assume that f is a real-valued function on $\mathbf{R}$ with period 1 satisfying

(1) $$\int_0^1 f(x)\,dx = 0, \quad 0 < \int_0^1 f^2(x)\,dx < \infty, \quad |f(x+h)-f(x)| \le Mh^\alpha$$

for some $\alpha > 0$ and $M > 0$.

We first consider the Riesz-Raikov sum $\sum f(\theta^k x)$, where $\theta > 1$ is a real number. In the case when the condition

(2)
$$\theta = \sqrt[\rho]{\nu/\mu} \text{ and } f(x) = g(\nu x) - g(\mu x) \text{ for some } \rho, \nu, \mu \in \mathbf{N} \text{ and } g \in L^2$$

is satisfied, partial sum of $\sum f(\theta^k x)$ always reduced to sum of at most 2ρ terms, and may not be recurrent on the whole line. Except for this trivial case, we have the recurrence property.

Theorem 1. *If we exclude the case (2), then $\{\sum_{k=1}^n f(\theta^k x)\}_{n\in\mathbf{N}}$ is dense in $\mathbf{R}$ for a.e. x, i.e., the Riesz-Raikov sum is recurrent a.e. except for the trivial case.*

By noting the case (2) we see that the Hadamard's gap condition is not enough to assure the recurrence of $\sum f(n_k x)$. It is, by the way, possible to prove the recurrence by assuming a stronger gap condition:

Theorem 2. *Let $\{\beta_k\}$ be a sequence of real numbers satisfying $\beta_{k+1}/\beta_k \to \infty$. Then $\{\sum_{k=1}^n f(\beta_k x)\}_{n\in\mathbf{N}}$ is recurrent a.e.*

We prove these results by the method of Ullich, Grubb and Moore, with the help of the mixing central limit theorem. We here introduce that notion. Let λ be the Lebesgue measure on $I = [\,0,1\,]$ and $\{g_n\}$ be a sequence of measurable functions on I. We say that $\{g_n\}$ obeys the mixing central limit theorem with limiting variance v if the probability measure $\frac{1}{\lambda(E)}\lambda(\{\, x \in E \mid g_n(x) \in \cdot \,\})$ on $\mathbf{R}$ converges weakly to the normal distribution $N_{0,v}$ with mean 0 and variance v for any $E \subset I$ with positive measure.

Study of the Riesz-Raikov sum has long history [7], [6], [8], and we [2] have proved the mixing central limit theorem holds for $\sum_{k=1}^n f(\theta^k x)/\sqrt{n}$, where $\theta > 1$. The limiting variance v is given by

$$v = \int_0^1 f^2(x)\,dx$$

if $\theta^r \notin \mathbf{Q}$ for all $r \in \mathbf{N}$, and

$$v = \int_0^1 f^2(x)\,dx + 2\sum_{k=1}^\infty \int_0^1 f(p^k x) f(q^k x)\,dx,$$

if $\theta = \sqrt[r]{p/q}$ where $r = \min\{k \in \mathbf{N} \mid \theta^k \in \mathbf{Q}\}$ and $p,\, q \in \mathbf{N}$. Here v is always non-negative, and is equal to 0 if and only if (2) holds.

As to the case of Theorem 2, we have the mixing central limit theorem with $v = \int_0^1 f^2(x)\,dx$ which was proved by Takahashi [9] by assuming that β_k are integers. We can easily drop the last condition in the same way as [2].

Lastly, we present another case when the mixing central limit theorem is proved. Let $\theta > 1$, real-valued functions $f_1, \dots, f_L$ on $\mathbf{R}$ satisfy (1), and $p_1, \dots, p_L$ be polynomials satisfying $p_m(\infty) = \infty$ and $(p_{m+1} - p_m)(\infty) = \infty$. Then $\sum_{k=1}^n \prod_{m=1}^L f_m(\theta^{p_m(k)}x)/\sqrt{n}$ obeys the mixing central limit theorem. Limiting variance is given as follows: If at lease one of the $p_m(k)$ is not linear, then

$$v = \prod_{m=1}^{L} \int_0^1 f_m^2(x)\,dx.$$

When all p_m are linear, i.e., $p_m(x) = a_m x + b_m$, if there exists m such that $\theta^{a_m r} \notin \mathbf{Q}$ for all $r \in \mathbf{N}$, then v is given as above. If $\theta^{a_m r} = p_m/q_m$ $(m = 1, \dots, L)$, then

$$v = \prod_{m=1}^{L} \int_0^1 f_m^2(x)\,dx + 2\sum_{k=1}^{\infty} \prod_{m=1}^{L} \int_0^1 f_m(p_m^k x) f_m(q_m^k x)\,dx.$$

By the help of this result, we can prove

Theorem 3. *If $v > 0$, then $\{\sum_{k=1}^n \prod_{m=1}^L f_m(\theta^{p_m(k)}x)\}_{n\in\mathbf{N}}$ is recurrent a.e.*

2. Proof of the Theorems

To verify our results, it is sufficient to prove the proposition below:

Proposition 4. *Let $L \in \mathbf{N}$, functions $f_1, \dots, f_L$ satisfy (1), and the sequences $\{\beta_{1,k}\}_{k\in\mathbf{N}}, \dots, \{\beta_{L,k}\}_{k\in\mathbf{N}}$ satisfy the Hadamard's gap condition: $\beta_{m,k+1}/\beta_{m,k} \ge q > 1$, $(k \in \mathbf{N},\, m = 1, \dots, L)$. Then the sequence $\{S_n = \sum_{k=1}^n \prod_{m=1}^L f_m(\beta_{m,k}x)\}_{k\in\mathbf{N}}$ is recurrent for a.e. x if $S_n/\sqrt{n}$ obeys the mixing central limit theorem with positive limiting variance.*

We use the lemma below proved by Ullich [10], [4].

Lemma 5. *Let $E_n,\, F_n \subset I$ $(n \in \mathbf{N})$. Assume that there exists $c > 0$ and $0 < \delta_n \downarrow 0$ such that, for all $x \in E_n$, there exists an interval J with $x \in J$, $\lambda(J) = \delta_n$ and $\lambda(F_n \cap J) \ge c\lambda(J)$. If $x \in E_n$ occurs infinitely often for almost every x, then $x \in F_n$ occurs infinitely often for almost every x.*

We follow the proof given by Grubb and Moore [4]. Take M large enough to satisfy both (1) and $|f_1(x)|, \dots, |f_L(x)| \le M$ for all x. Put $g_k(x) = \prod_{m=1}^{L} f_m(\beta_{m,k}x)$. We have $|g_k(x+h)-g_k(x)| \le M^L|h|^\alpha \sum_{l=1}^{L} \beta_{l,k}^\alpha$ by $|\xi_1 \dots \xi_m - \eta_1 \dots \eta_m| \le \sum_{l=1}^{L} |\xi_1 \dots \xi_{l-1}(\xi_l - \eta_l)\eta_{l+1} \dots \eta_m|$. By applying $\sum_{k=1}^{n} \beta_{l,k}^\alpha \le \beta_{l,n}^\alpha\bigl(1 + 1/q^\alpha + 1/q^{2\alpha} + \cdots\bigr) = \beta_{l,n}^\alpha/(1-1/q^\alpha)$, we have $|S_n(x+h) - S_n(x)| \le C|h|^\alpha \max_{l=1}^{L} \beta_{l,n}^\alpha$, where $C = LM^L/(1-1/q^\alpha)$.

Let us take small $\varepsilon > 0$ satisfying $c = (\varepsilon/C)^{1/\alpha} < 1/2$. Put $E_n = \{\, x \in I \mid S_n(x) \ge a, S_{n+1}(x) \le a \,\}$, $F_n = \{\, x \in I \mid S_n(x) \text{ or } S_{n+1}(x) \in (a-\varepsilon, a+\varepsilon) \}$, and $G_\pm = \{\, x \in I \mid \pm S_n(x) \ge \pm a \text{ f.e.} \,\}$. If $\lambda(G_+) > 0$ we have

$$\frac{\lambda(\{\, x \in G_+ \mid S_n(x) \ge a \})}{\lambda(G_+)} \le \frac{\lambda(\{\, x \in G_+ \mid S_n(x)/\sqrt{n} \ge -|a| \})}{\lambda(G_+)}$$

where the right hand side tends to $N_{0,v}(-|a|, \infty) < 1$ by the mixing central limit theorem, which contradicts with a definition of G_+. In the same way we have $\lambda(G_-) = 0$, and therefore we have proved $x \in E_n$ i.o. for a.e. x.

Let $x \in E_n$. Defining l_0 and δ_n by $\max_{l=1}^{L} \beta_{l,n+1} = \beta_{l_0,n+1} = 1/\delta_n$, we have $\max_{l=1}^{L} \beta_{l,n}^\alpha \le \max_{l=1}^{L} \beta_{l,n+1}^\alpha = 1/\delta_n^\alpha$. Let $J = (x - \delta_n/2, x + \delta_n/2)$.

Firstly, we assume that there exists an $x_0 \in J$ such that $S_n(x_0) = a$. If $|y - x_0| < c\delta_n$, we have $|S_n(y) - a| \le C|y-x_0|^\alpha/\delta_n^\alpha < \varepsilon$. Noting $c < 1/2$, we see that $(x_0 - c\delta_n, x_0)$ or $(x_0, x_0 + c\delta_n)$ is contained in $J \cap F_n$, and thereby $\lambda(J \cap F_n) \ge c\delta_n = c\lambda(J)$.

Secondly we assume that $S_n > a$ on J. Since J contains a period of $f_1(\beta_{l_0,n+1} \cdot\,)$, J contains its zero x_1. By $\prod_{m=1}^{L} f_m(\beta_{m,n+1}x_1) = 0$, we have $S_{n+1}(x_1) = S_n(x_1) > a$. Because of $S_{n+1}(x) \le a$, we have $x_2 \in J$ between x and x_1 such that $S_{n+1}(x_2) = a$. Therefore, if $|y - x_2| \le c\delta_n$, we have $|S_{n+1}(y) - a| \le C|y - x_2|^\alpha/\delta_n^\alpha < \varepsilon$, and thereby $(x_2 - c\delta_n, x_2)$ or $(x_2, x_2 + c\delta_n)$ is contained in $J \cap F_n$, and hence $\lambda(J \cap F_n) \ge c\delta_n = c\lambda(J)$.

We have verified the assumption of Lemma and proved $x \in F_n$ i.o. a.e. x.

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