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ON HARMONIC UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED RUSCHEWEYH DERIVATIVES OPERATOR

(submitted by M. A. Malakhaltsev)

Abstract. Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense preserving in the unit disk $\mathbf{U}$. Al-Shaqsi and Darus[7] introduced a generalized Ruscheweyh derivatives operator denoted by D^n_λ where $D^n_\lambda f(z) = z + \sum\limits_{k=2}^{\infty} [1 + \lambda(k-1)]C(n,k)a_k z^k$, where $C(n,k) = \binom{k+n-1}{n}$. The authors, using this operators, introduce the class $\mathcal{H}^n_\lambda$ of functions which are harmonic in $\mathbf{U}$. Coefficient bounds, distortion bounds and extreme points are obtained.

1. Introduction

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see [2]).

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Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbf{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad |b_1| < 1. \tag{1.1}$$

Observe that $\mathcal{S}_{\mathcal{H}}$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.
The class $\mathcal{T}$ is defined as the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of all functions $f = h + \overline{g}$ where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = -\sum_{n=1}^{\infty} |b_n| z^n. \tag{1.2}$$

In 1984 Clunie and Sheil-Small [2] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{S}_{\mathcal{H}}$ and its subclasses such that Silverman [3], Silverman and Silvia [4] and, Jahangiri [5] studied the harmonic univalent functions.
We denote by $\mathcal{H}_{\lambda}^{n}$ the class of all function of the form (1.1) that satisfy the condition

$$\Re(D_{\lambda}^{n} f(z))' > 0, \qquad z \in \mathbf{U}. \tag{1.3}$$

where $D_{\lambda}^{n} f(z) = D_{\lambda}^{n} h(z) + \overline{D_{\lambda}^{n} g(z)}$, and D_{λ}^{n} denotes the operator introduced by Al-Shaqsi and Darus[7] and is given by

$$D_{\lambda}^{n} f(z) = z + \sum_{k=2}^{\infty} \big[1 + \lambda(k-1)\big] C(n,k) a_k z^k, \qquad \lambda \geq 0, \tag{1.4}$$

where

$$C(n,k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1} (j+n)}{(k-1)!}, \qquad k \geq 2. \tag{1.5}$$

Note that when $\lambda = 0$, we get Ruscheweyh differential operator (see[1]). Also note that the class $\mathcal{H}_{\lambda}^{0} \equiv HP(\alpha)$ the class of harmonic univalent functions studied by Yalçin and Öztürk [6]. We further denote by $\mathcal{TH}_{\lambda}^{n}$ the subclass of $\mathcal{H}_{\lambda}^{n}$, where $\mathcal{TH}_{\lambda}^{n} = \mathcal{T} \cap \mathcal{H}_{\lambda}^{n}$.

2. Coefficients Bounds

Theorem 2.1. *Let $f = h + \overline{g}$ with h and g are given by (1.1). Let*

$$\sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)\big(|a_n|+|b_n|\big) \le 2, \tag{2.1}$$

where $a_1 = 1$ and $\lambda \ge 0$. Then f is harmonic univalent sense preserving in $\boldsymbol{U}$ and $f \in \mathcal{H}_\lambda^n$.

Proof. For $|z_1| \le |z_2| < 1$, we have by (2.1),

$$\begin{aligned}
&|f(z_1)-f(z_2)| \\
&\quad \ge |h(z_1)-h(z_2)| - |g(z_1)-g(z_2)| \\
&\quad = \left|(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)\right| - \left|\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)\right| \\
&\quad \ge |z_1-z_2|\left(1-|b_1|-\sum_{k=2}^{\infty} k|z_2|^{k-1}\right) \\
&\quad \ge |z_1-z_2|\left(1-|b_1|-|z_2|\sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)[|a_k|+|b_k|]\right) \\
&\quad \ge |z_1-z_2|(1-|b_1|)(1-|z_2|) > 0.
\end{aligned}$$

Consequently, f is univalent in $\mathbf{U}$. We note that f is sense preserving in $\mathbf{U}$. This is because

$$\begin{aligned}
|h'(z)| &\ge 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty} k|a_k| \\
&\ge 1-\sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|a_k||z| \\
&\ge \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \ge |g'(z)|.
\end{aligned}$$

Now we show that $f \in \mathcal{H}_\lambda^n$. Using the fact that $\Re w > 0$ if and only if $|1+w| \ge |1-w|$, it suffices to show that

$$\left|1 + (D_\lambda^n h(z))' + (\overline{D_\lambda^n h(z)})'\right| - \left|1 - (D_\lambda^n h(z))' - (\overline{D_\lambda^n h(z)})'\right| =$$

$$\left|2+\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)a_nz^{n-1}+\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)b_n\overline{z}^{n-1}\right|$$

$$-\left|-\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)a_nz^{n-1}-\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)b_n\overline{z}^{n-1}\right|$$

$$\geq 2-\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|a_n||z|^{n-1}$$

$$-\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|b_n||z|^{n-1}-\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|a_n||z|^{n-1}$$

$$-\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|b_n||z|^{n-1}$$

$$=2-2\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|a_n||z|^{n-1}$$

$$-2\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|b_n||z|^{n-1}\geq$$

$$2\left\{1-\left(\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|a_n|+\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)|b_n|\right)\right\}$$

$$\geq 0, \text{ by}(2.1).$$

The harmonic mappings

$$f(z)=z+\sum_{k=2}^{\infty}\frac{x_k}{k\big[1+\lambda(k-1)\big]C(n,k)}z^k$$

$$+\sum_{k=1}^{\infty}\frac{\overline{y}_k}{k\big[1+\lambda(k-1)\big]C(n,k)}\overline{z}^k$$

where $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.2) are in $\mathcal{H}_\lambda^n$ because

$$\sum_{k=1}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)\Big(|a_k|+|b_k|\Big)=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f=h+\overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f\in\mathcal{H}_\lambda^n$. We next show that the condition (2.1) is also necessary for functions in $\mathcal{TH}_{\lambda}^{n}$.

Theorem 2.2. *Let $f = h + \overline{g}$ with h and g are given by (1.2). Then $f \in \mathcal{TH}_{\lambda}^{n}$ if and only if*

$$\sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)\big(|b_k|+|b_k|\big) \le 2, \tag{2.2}$$

where $a_1 = 1$ and $\lambda \ge 0$.

Proof. We first suppose that $f \in \mathcal{TH}_{\lambda}^{n}$, then by (1.3) we have

$$\begin{aligned}\Re\big\{(D_{\lambda}^{n}h(z))' + \overline{(D_{\lambda}^{n}g(z))'}\big\} &= \Re\Big\{1 - \sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|a_k|z^{n-1} \\ &\qquad - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|b_k|\overline{z}^{n-1}\Big\} \\ &> 0.\end{aligned}$$

If we choose z to be real and let $z \to 1^{-}$, we get

$$1 - \sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|a_k| - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|b_k| \ge 0,$$

which is precisely the assertion (2.3) of Theorem 2.2.
Conversely, suppose that the inequality (2.3) holds true. Then we find from the definition (1.3) that

$$\begin{aligned}&\Re\big\{(D_{\lambda}^{n}h(z))' + \overline{(D_{\lambda}^{n}g(z))'}\big\} \\ &= \Re\Big\{1 - \sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|a_k|z^{n-1} \\ &\qquad - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)|b_k|\overline{z}^{n-1}\Big\} \\ &\ge 2 - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)\big(|b_k|+|b_k|\big)\overline{z}^{n-1} \\ &> 2 - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)\big(|b_k|+|b_k|\big) \ge 0.\end{aligned}$$

provided that the inequality (2.3) is satisfied.

3. Distortion Bounds and Extreme Points.

In this section, we shall obtain distortion bounds for functions in $\mathcal{TH}_\lambda^n$ and also provide extreme points for the class $\mathcal{TH}_\lambda^n$.

Theorem 3.1. *If $f \in \mathcal{TH}_\lambda^n$, for $\lambda \geq 0$ and $|z| = r > 1$, then*

$$|f(z)| \leq (1+b_1)r + (1-b_1)r + \frac{1-|b_1|}{2(1+\lambda)(n+1)}r^2,$$

and

$$|f(z)| \geq (1-b_1)r - (1-b_1)r + \frac{1-|b_1|}{2(1+\lambda)(n+1)}r^2.$$

Proof. We only prove the second inequality. The argument for first inequality is similar and will be omitted. Let $f \in \mathcal{TH}_\lambda^n$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\geq (1-b_1)r - \sum_{k=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\geq (1-b_1)r - \sum_{k=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&= (1-b_1)r - \frac{1}{2(1+\lambda)(n+1)}\sum_{k=2}^{\infty}2(1+\lambda)(n+1)(|a_n|+|b_n|)r^2 \\
&\geq (1-b_1)r - \frac{1}{2(1+\lambda)(n+1)}\sum_{k=2}^{\infty}k\big[1+\lambda(k-1)\big]C(n,k)(|a_n|+|b_n|)r^2 \\
&\geq (1-b_1)r - \frac{1}{2(1+\lambda)(n+1)}[1-|b_1|]r^2.
\end{aligned}$$

The bounds given in Theorem 3.1 for the functions $f = h + \overline{g}$ of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

$$f(z) = z + |b_1|\overline{z} - \frac{1-|b_1|}{2(1+\lambda)(n+1)}\overline{z}^2$$

and

$$f(z) = (1-|b_1|)z - \frac{1-|b_1|}{2(1+\lambda)(n+1)}z^2$$

for $|b_1| < 1$ show that the bounds given Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

Corollary 3.2. *If $f \in \mathcal{TH}_\lambda^n$, then*

$$\left\{ w : |w| < \frac{1}{2(1+\lambda)(n+1)} [(1-|b_1|)(2(1+\lambda)(n+1)-1)] \right\} \subset f(\boldsymbol{U}).$$

Theorem 3.3. *$f \in \mathcal{TH}_\lambda^n$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty} (\gamma_k h_k + \mu_k g_k) \tag{3.1}$$

where $z \in \boldsymbol{U}$,

$$h_1(z) = z, \ h_k(z) = z - \frac{1}{k\big[1+\lambda(k-1)\big]C(n,k)} z^k, \ (k = 2, 3, ...),$$

$$g_k(z) = z - \frac{1}{k\big[1+\lambda(k-1)\big]C(n,k)} \overline{z}^k, \ (n = 1, 2, ...),$$

$$\sum_{k=1}^{\infty} (\gamma_k + \mu_k) = 1, \ \gamma_k \geq 0 \ \textit{and} \ \mu_k \geq 0.$$

In particular, the extreme points of $\mathcal{TH}_\lambda^n$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Note that for f we may write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (\gamma_k h_k + \mu_k g_k) \\ &= \sum_{k=1}^{\infty} (\gamma_k + \mu_k) z - \sum_{k=2}^{\infty} \frac{1}{k\big[1+\lambda(k-1)\big]C(n,k)} \gamma_k z^k \\ &\quad - \sum_{k=1}^{\infty} \frac{1}{k\big[1+\lambda(k-1)\big]C(n,k)} \mu_k \overline{z}^k. \end{aligned}$$

Now the first part of the proof is complete, since by Theorem 2.2

$$\begin{aligned} &\sum_{k=2}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k) \frac{\gamma_k}{k\big[1+\lambda(k-1)\big]C(n,k)} \\ &\quad - \sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k) \frac{\mu_k}{k\big[1+\lambda(k-1)\big]C(n,k)} \\ &= \sum_{k=1}^{\infty} (\gamma_k + \mu_k) - \gamma_1 = 1 - \gamma_1 \leq 1. \end{aligned}$$

Conversely, suppose that $f \in \mathcal{TH}_{\lambda}^{n}$. Then

$$\sum_{k=1}^{\infty} k\big[1+\lambda(k-1)\big]C(n,k)(|a_k|+|b_k|) \leq 2.$$

Setting

$$\gamma_k = k\big[1+\lambda(k-1)\big]C(n,k)|a_k|, \quad 0 \leq \gamma_k \leq 1,\ (k=2,3,...),$$
$$\mu_k = k\big[1+\lambda(k-1)\big]C(n,k)|b_k|, \quad 0 \leq \mu_k \leq 1,\ (k=1,2,3,...),$$

and $\mu_1 = 1-\gamma_1-\sum\limits_{k=2}^{\infty}(\gamma_k+\mu_k)$ we obtain

$f(z) = \sum\limits_{k=1}^{\infty}(\gamma_k h_k + \mu_k g_k)$ as required.

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