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S-extremal strongly modular lattices

par Gabriele NEBE et Kristina SCHINDELAR

Résumé. Un réseau fortement modulaire est dit s-extrémal, s'il maximise le minimum du réseau et son ombre simultanément. La dimension des réseaux s-extrémaux dont le minimum est pair peut être bornée par la théorie des formes modulaires. En particulier de tels réseaux sont extrémaux.

Abstract. S-extremal strongly modular lattices maximize the minimum of the lattice and its shadow simultaneously. They are a direct generalization of the s-extremal unimodular lattices defined in [6]. If the minimum of the lattice is even, then the dimension of an s-extremal lattices can be bounded by the theory of modular forms. This shows that such lattices are also extremal and that there are only finitely many s-extremal strongly modular lattices of even minimum.

1. Introduction.

Strongly modular lattices have been defined in [11] to generalize the notion of unimodular lattices. For square-free $N \in \mathbb{N}$ a lattice $L \subset (\mathbb{R}^n, (.,.))$ in Euclidean space is called *strongly N-modular*, if L is integral, i.e. contained in its dual lattice

$$L^* = \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \ \forall \ \ell \in L\}$$

and isometric to its rescaled partial dual lattices $\sqrt{d}(L^* \cap \frac{1}{d}L)$ for all $d \mid N$. The simplest strongly modular lattice is

$$C_N := \perp_{d|N} \sqrt{d}\mathbb{Z}$$

of dimension $\sigma_0(N)$, the number of divisors of N. For

$$N \in \mathcal{L} = \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\}$$

which is the set of square-free numbers such that $\sigma_1(N) = \sum_{d|N} d$ divides 24, Theorems 1 and 2 in [13] bound the minimum $\min(L) := \min\{(\ell, \ell) \mid$

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$0 \neq \ell \in L\}$ of a strongly N-modular lattice that is rational equivalent to C_N^k by

$$\min(L) \leq 2 + 2\lfloor \frac{k}{s(N)} \rfloor, \text{ where } s(N) = \frac{24}{\sigma_1(N)}. \tag{1.1}$$

For $N \in \{1,3,5,7,11\}$ there is one exception to this bound: $k = s(N) - 1$ and $L = S^{(N)}$ of minimum 3 (see [13, Table 1]). Lattices achieving this bound are called *extremal.*

For an odd strongly N-modular lattice L let

$$S(L) = L_0^* \setminus L^*$$

denote the shadow of L, where $L_0 = \{\ell \in L \mid (\ell,\ell) \in 2\mathbb{Z}\}$ is the even sublattice of L. For even strongly N-modular lattices L let $S(L) := L^*$. Then the *shadow-minimum* of an N-modular lattice is defined as

$$\operatorname{smin}(L) := \min\{N(x,x) \mid x \in S(L)\}.$$

In particular $\operatorname{smin}(L) = 0$ for even lattices L. In this paper we show that for all $N \in \mathcal{L}$ and for all strongly N-modular lattices L that are rational equivalent to C_N^k

$$\begin{array}{ll} 2\min(L) + \operatorname{smin}(L) \leq k\frac{\sigma_1(N)}{4} + 2 & \text{if } N \text{ is odd and} \\ \min(L) + \operatorname{smin}(L) \leq k\frac{\sigma_1(N/2)}{2} + 1 & \text{if } N \text{ is even} \end{array}$$

with the exceptions $L = S^{(N)}$, $k = s(N) - 1$ ($N \neq 23, 15$ odd) where the bound has to be increased by 2 and $L = O^{(N)}$, $k = s(N)$ and N even, where the bound has to be increased by 1 (see [13, Table 1] for the definition of the lattices $S^{(N)}, O^{(N)}$ and also $E^{(N)}$). Lattices achieving this bound are called *s-extremal.* The theory of modular forms allows us to bound the dimension $\sigma_0(N)k$ of an s-extremal lattice of even minimum μ by

$$2k < \mu s(N).$$

In particular s-extremal lattices of even minimum are automatically extremal and hence by [12] there are only finitely many strongly N-modular s-extremal lattices of even minimum. This is also proven in Section 3, where explicit bounds on the dimension of such s-extremal lattices and some classifications are obtained. It would be interesting to have a similar bound for odd minimum $\mu \geq 3$. Of course for $\mu = 1$, the lattices C_N^k are s-extremal strongly N-modular lattices of minimum 1 for arbitrary $k \in \mathbb{N}$ (see [9]), but already for $\mu = 3$ there are only finitely many s-extremal unimodular lattices of minimum 3 (see [10]). The s-extremal strongly N-modular lattices of minimum $\mu = 2$ are classified in [9] and some s-extremal lattices of minimum 3 are constructed in [15]. For all calculations we used the computer algebra system MAGMA [2].

2. S-extremal lattices.

For a subset $S \subset \mathbb{R}^n$, which is a finite union of cosets of an integral lattice we put its *theta series*

$$\Theta_S(z) := \sum_{v\in S} q^{(v,v)}, \quad q = \exp(\pi i z).$$

The theta series of strongly N-modular lattices are modular forms for a certain discrete subgroup Γ_N of $SL_2(\mathbb{R})$ (see [13]). Fix $N \in \mathcal{L}$ and put

$$g_1^{(N)}(z) := \Theta_{C_N}(z) = \prod_{d|N} \Theta_{\mathbb{Z}}(dz) = \prod_{d|N} \prod_{j=1}^{\infty} (1-q^{2dj})(1+q^{d(2j-1)})^2$$

(see [4, Section 4.4]). Let η be the Dedekind eta-function

$$\eta(z) := q^{\frac{1}{12}} \prod_{j=1}^{\infty}(1-q^{2j}) \text{ and put } \eta^{(N)}(z) := \prod_{d|N} \eta(dz).$$

If N is odd define

$$g_2^{(N)}(z) := \Big(\frac{\eta^{(N)}(z/2)\eta^{(N)}(2z)}{\eta^{(N)}(z)^2}\Big)^{s(N)}$$

and if N is even then

$$g_2^{(N)}(z) := \Big(\frac{\eta^{(N/2)}(z/2)\eta^{(N/2)}(4z)}{\eta^{(N/2)}(z)\eta^{(N/2)}(2z)}\Big)^{s(N)}.$$

The meromorphic function $g_2^{(N)}$ generates the field of modular functions of Γ_N. It is a power series in q starting with

$$g_2^{(N)}(z) = q - s(N)q^2 + \dots .$$

Using the product expansion of the η-function we find that

$$q^{-1} g_2^{(N)}(z) = \prod_{d|N} \prod_{j=1}^{\infty} (1+q^{d(2j-1)})^{-s(N)}.$$

For even N one has to note that

$$\begin{aligned} q^{-1} g_2^{(N)}(z) &= \prod_{d|\frac{N}{2}} \prod_{j=1}^{\infty} \Big(\frac{1+q^{4dj}}{1+q^{dj}}\Big)^{s(N)} \\ &= \prod_{d|\frac{N}{2}} \prod_{j=1}^{\infty} (1+q^{2d(2j-1)})^{-s(N)} (1+q^{d(2j-1)})^{-s(N)}. \end{aligned}$$

By [13, Theorem 9, Corollary 3] the theta series of a strongly N-modular lattice L that is rational equivalent to C_N^k is of the form

$$\Theta_L(z) = g_1^{(N)}(z)^k \sum_{i=0}^{b} c_i g_2^{(N)}(z)^i \tag{2.1}$$

for $c_i \in \mathbb{R}$ and some explicit b depending on k and N. The theta series of the rescaled shadow $S := \sqrt{N} S(L)$ of L is

$$\Theta_S(z) = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i \tag{2.2}$$

where $s_1^{(N)}$ and $s_2^{(N)}$ are the corresponding "shadows" of $g_1^{(N)}$ and $g_2^{(N)}$ as defined in [13] (see also [9]).

If N is odd, then

$$s_1^{(N)} = 2^{\sigma_0(N)} q^{\sigma_1(N)/4}(1 + q^2 + \ldots)$$

and

$$s_2^{(N)} = 2^{-s(N)\sigma_0(N)/2}(-q^{-2} + s(N) + \ldots).$$

If N is even, then

$$s_1^{(N)} = 2^{\sigma_0(N)/2} q^{\sigma_1(\frac{N}{2})/2}(1 + 2q + \ldots),$$
$$s_2^{(N)} = 2^{-s(N)\sigma_0(\frac{N}{2})/2}(-q^{-1} + s(N) + \ldots).$$

Theorem 2.1. *Let $N \in \mathcal{L}$ be odd and let L be a strongly N-modular lattice in the genus of C_N^k. Let $\sigma := \operatorname{smin}(L)$ and let $\mu := \min(L)$. Then*

$$\sigma + 2\mu \le k\frac{\sigma_1(N)}{4} + 2$$

unless $k = s(N) - 1$ and $\mu = 3$. In the latter case the lattice $S^{(N)}$ is the only exception (with $\min(S^{(N)}) = 3$ and $\operatorname{smin}(S^{(N)}) = 4 - \sigma_1(N)/4$).

Proof. The proof is a straightforward generalization of the one given in [6]. We always assume that $L \neq S^{(N)}$ and put $g_1 := g_1^{(N)}$ and $g_2 := g_2^{(N)}$. Let $m := \mu - 1$ and assume that $\sigma + 2\mu \ge k\frac{\sigma_1(N)}{4} + 2$. Then from the expansion of

$$\Theta_S = \sum_{j=\sigma}^{\infty} b_j q^j = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i$$

in formula (2.2) above we see that $c_i = 0$ for $i > m$ and (2.1) determines the remaining coefficients $c_0 = 1, c_1, \ldots, c_m$ uniquely from the fact that

$$\Theta_L = 1 + \sum_{j=\mu}^{\infty} a_j q^j \equiv 1 \pmod{q^{m+1}}.$$

The number of vectors of norm $k\frac{\sigma_1(N)}{4}+2-2\mu$ in $S=\sqrt{N}S(L)$ is

$$c_m(-1)^m 2^{-m\sigma_0(N)s(N)/2+k\sigma_0(N)}$$

and nonzero, iff $c_m \neq 0$. The expansion of g_1^{-k} in a power series in g_2 is given by

$$g_1^{-k}=\sum_{i=0}^{m} c_i g_2^i - a_{m+1}q^{m+1}g_1^{-k}+\star q^{m+2}+\ldots=\sum_{i=0}^{\infty}\tilde{c}_i g_2^i \tag{2.3}$$

with $\tilde{c}_i=c_i$ $(i=0,\ldots,m)$ and $\tilde{c}_{m+1}=-a_{m+1}$. Hence Bürmann-Lagrange (see for instance [16]) yields that

$$c_m=\frac{1}{m!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left(\frac{\partial}{\partial q}(g_1^{-k})(qg_2^{-1})^m\right)_{q=0}=\frac{-k}{m}(\text{ coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1)$$

with $f_1=(q^{-1}g_2)^m g_1^k$. Using the product expansion of g_1 and g_2 above we get

$$f_1=\prod_{d|N}\prod_{j=1}^{\infty}(1-q^{2dj})^k(1+q^{d(2j-1)})^{2k-s(N)m}.$$

Since

$$g_1'/g_1=\sum_{d|N}\frac{\frac{\partial}{\partial q}\theta_3(dz)}{\theta_3(dz)}$$

is alternating as a sum of alternating power series, the series $P:=g_1'/g_1/f_1$ is alternating, if $2k-s(N)m\geq 0$. In this case all coefficients of P are nonzero, since all even powers of q occur in $(1-q^2)^{-1}$ and g_1'/g_1 has a non-zero coefficient at q^1. Otherwise write

$$P=g_1'\prod_{d|N}\prod_{j=1}^{\infty}\frac{(1+q^{d(2j-1)})^{s(N)m-2k-2}}{(1-q^{2dj})^{k+1}}.$$

If $2k-s(N)m<-2$ then P is a positive power series in which all q-powers occur. Hence $c_m<0$ in this case. If the minimum μ is odd then this implies that $b_\sigma<0$ and hence the nonexistence of an s-extremal lattice of odd minimum for $s(N)m-2>2k$. Assume now that $2k-s(N)m=-2$, i.e. $k=s(N)m/2-1$. By the bound in [13] one has

$$m+1\leq 2\lfloor\frac{k}{s(N)}\rfloor+2=2\lfloor\frac{m}{2}-\frac{1}{s(N)}\rfloor+2.$$

This is only possible if m is odd. Since g_1' has a non-zero constant term, P contains all even powers of q. In particular the coefficient of q^{m-1} is positive. The last case is $2k - s(N)m = -1$. Then clearly m and $s(N)$ are odd and $P = GH^{(m-1)/2}$ where

$$G = g_1' \prod_{d|N} \prod_{j=1}^{\infty} (1 + q^{d(2j-1)})^{-1} (1 - q^{2dj})^{-(s(N)+1)/2}$$

and

$$H = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2dj})^{-s(N)}.$$

If m is odd then the coefficient of P at q^{m-1} is

$$\int_{1+iy_0}^{-1+iy_0} e^{-(m-1)\pi i z} G(e^{\pi i z}) H(e^{\pi i z})^{(m-1)/2} dz$$

which may be estimated by the saddle point method as illustrated in [8, Lemma 1]. In particular this coefficient grows like a constant times

$$\frac{c^{(m-1)/2}}{m^{1/2}}$$

where $c = F(y_0)$, $F(y) = e^{2\pi y} H(e^{-2\pi y})$ and y_0 is the first positive zero of F'. Since $c > 0$ and also $F''(y_0) > 0$ and the coefficient of P at q^{m-1} is positive for the first few values of m (we checked 10000 values), this proves that $b_\sigma > 0$ also in this case. ☐

To treat the even $N \in \mathcal{L}$, we need two easy (probably well known) observations:

Lemma 2.1. *Let*

$$f(q) := \prod_{j=1}^{\infty} (1 + q^{2j-1})(1 + q^{2(2j-1)}).$$

Then the q-series expansion of $1/f$ is alternating with non zero coefficients at q^a for $a \neq 2$.

Proof.

$$1/f = \prod_{j=1}^{\infty} (1 + q^{2j-1} + q^{2(2j-1)} + q^{3(2j-1)})^{-1} = \prod_{j=1}^{\infty} \sum_{\ell=0}^{\infty} q^{4\ell(2j-1)} - q^{(4\ell+1)(2j-1)}$$

is alternating as a product of alternating series. The coefficient of q^a is non-zero, if and only if a is a sum of numbers of the form $4\ell(2j-1)$ and $(4\ell+1)(2j-1)$ with distinct ℓ. One obtains 0 and 1 with $\ell = 0$ and $j = 1$ and $3 = 1(2 \cdot 2 - 1)$ and $6 = 1 + 5$. Since one may add arbitrary multiples of 4, this shows that the coefficients are all non-zero except for the case that $a = 2$. ☐

Lemma 2.2. *Let* $g_1 := g_1^{(N)}$ *for even* N *such that* $N/2$ *is odd and denote by* g_1' *the derivative of* g_1 *with respect to* q. *Then* $\frac{g_1'}{g_1}$ *is an alternating series with non-zero coefficients for all* q^a *with* $a \not\equiv 1 \pmod 4$. *The coefficients for* q^a *with* $a \equiv 1 \pmod 4$ *are zero.*

Proof. Using the product expansion

$$g_1 = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2jd})(1 + q^{(2j-1)d})^2$$

we calculate

$$\begin{aligned} g_1'/g_1 &= \sum_{d|\frac{N}{2}} \sum_{j=1}^{\infty} \frac{2(2j-1)dq^{d(2j-1)-1}}{1 - q^{d(2j-1)}} - \frac{2djq^{2dj-1}}{1-q^{2dj}} - \frac{4djq^{4dj-1}}{1-q^{4dj}} \\ &\qquad + \frac{2(4j-2)dq^{d(4j-2)-1}}{1-q^{d(4j-2)}} \\ &= \sum_{d|\frac{N}{2}} \sum_{j=1}^{\infty} \frac{(4j-2)dq^{(2j-1)d-1}}{1+q^{(2j-1)d}} - \frac{8djq^{4dj-1}}{1-q^{4dj}} \\ &\qquad + \frac{(4j-2)d(q^{(4j-2)d-1} - 3q^{(8j-4)d-1}}{1-q^{(8j-4)d}} \\ &= \sum_{d|\frac{N}{2}} \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} -8jdq^{4jd\ell-1} - 3(4j-2)dq^{(8j-4)d\ell-1} \\ &\qquad + (4j-2)dq^{(2j-1)d(4\ell-2)-1} - (-1)^{\ell}(4j-2)dq^{(2j-1)d\ell-1} \end{aligned}$$

Hence the coefficient of q^a is positive if a is even and negative if $a \equiv -1 \pmod 4$. The only cancellation that occurs is for $a \equiv 1 \pmod 4$. In this case the coefficient of q^a is zero. □

Theorem 2.2. *Let* $N \in \mathcal{L}$ *be even and let* L *be a strongly* N*-modular lattice in the genus of* C_N^k. *Let* $\sigma := \mathrm{smin}(L)$ *and let* $\mu := \min(L)$. *Then*

$$\sigma + \mu \le k\frac{\sigma_1(N/2)}{2} + 1$$

unless $k = s(N)$ *and* $\mu = 3$ *where this bound has to be increased by 1. In these cases* L *is the unique lattice* $L = O^{(N)}$ *(from* [13, Table 1]*) of minimum 3 described in* [9, Theorem 3].

Proof. As in the proof of Theorem 2.1 let $g_1 := g_1^{(N)}$ and $g_2 := g_2^{(N)}$, $m := \mu - 1$ and assume that $\sigma + \mu \ge k\frac{\sigma_1(N/2)}{2} + 1$. Again all coefficients c_i in (2.2) and (2.1) are uniquely determined by the conditions that $\mathrm{smin}(L) \ge k\frac{\sigma_1(N/2)}{4} - m$ and $\Theta_L \equiv 1 \pmod{q^{m+1}}$. The number of vectors of norm

$k\frac{\sigma_1(N/2)}{2} - m$ in $S = \sqrt{N}S(L)$ is $c_m(-1)^m 2^{\sigma_0(N)k/2-ms(N)}$. As in the proof of Theorem 2.1 the formula of Bürmann-Lagrange yields that

$$c_m = \frac{-k}{m}(\text{ coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1)$$

with f_1 as in the proof of Theorem 2.1. We have

$$f_1 = \prod_{d|\frac{N}{2}} f(dq)^{2k-s(N)m} \prod_{j=1}^{\infty}(1-q^{2dj})^k(1-q^{4dj})^k$$

where f is as in Lemma 2.1. If $2k - s(N)m > 0$ then $1/f_1$ is alternating by Lemma 2.1 and $\frac{g_1'}{g_1}$ is alternating (with a non-zero coefficient at q^3) by Lemma 2.2 and we can argue as in the proof of Theorem 2.1. Since $k > 0$ all even coefficients occur in the product

$$\prod_{j=1}^{\infty}(1-q^{2j})^{-k}$$

hence all coefficients in $(g_1'/g_1)/f_1$ are non-zero. If $2k-s(N)m = 0$ similarly the only zero coefficient in $(g_1'/g_1)/f_1$ is at q^1 yielding the exception stated in the Theorem. Now assume that $2k - s(N)m < 0$ and write

$$P = (g_1'/g_1)/f_1 = g_1' \prod_{d|\frac{N}{2}} \frac{f(dq)^{s(N)m-2k-2}}{\prod_{j=1}^{\infty}((1-q^{2dj})(1-q^{4dj}))^{k+1}}.$$

If $2k - s(N)m < -2$ then P is a positive power series in which all q-powers occur and hence $c_m < 0$. If the minimum μ is odd then this implies that $b_\sigma < 0$ and hence the nonexistence of an s-extremal lattice of odd minimum for $s(N)m - 2 > 2k$. Assume now that $2k - s(N)m = -2$, i.e. $k = s(N)m/2-1$. Then again m is odd and since g_1' has a non-zero constant term P contains all even powers of q. In particular the coefficient of q^{m-1} is positive. The last case is $2k - s(N)m = -1$ and dealt with as in the proof of Theorem 2.1. □

From the proof of Theorem 2.1 and 2.2 we obtain the following bound on the minimum of an s-extremal lattice which is sometimes a slight improvement of the bound (1.1).

Corollary 2.1. *Let L be an s-extremal strongly N-modular lattice in the genus of C_N^k with odd minimum $\mu := \min(L)$. Then*

$$\mu < \frac{2k+2}{s(N)} + 1.$$

3. S-extremal lattices of even minimum.

In this section we use the methods of [8] to show that there are only finitely many s-extremal lattices of even minimum. The first result generalizes the bound on the dimension of an s-extremal lattice of even minimum that is obtained in [6] for unimodular lattices. In particular such s-extremal lattices are automatically extremal. Now [12, Theorem 5.2] shows that there are only finitely many extremal strongly N-modular lattices which also implies that there are only finitely many such s-extremal lattices with even minimum. To get a good upper bound on the maximal dimension of an s-extremal strongly N-modular lattice, we show that the second (resp. third) coefficient in the shadow theta series becomes eventually negative.

Theorem 3.1. *Let $N \in \mathcal{L}$ and let L be an s-extremal strongly N-modular lattice in the genus of C_N^k. Assume that $\mu := \min(L)$ is even. Then*

$$s(N)(\mu - 2) \leq 2k < \mu s(N).$$

Proof. The lower bound follows from (1.1). As in the proof of Theorem 2.1 we obtain the number a_μ of minimal vectors of L as

$$a_\mu = \frac{k}{\mu - 1}(\text{ coeff. of } q^{\mu-1} \text{ in } (g_1'/g_1)/f_2)$$

with

$$f_2 = (q^{-1}g_2)^\mu g_1^k.$$

If N is odd, then

$$f_2 = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2dj})^k (1 + q^{d(2j-1)})^{2k - s(N)\mu}$$

and for even N we obtain

$$f_2 = \prod_{d|\frac{N}{2}} f(dq)^{2k - s(N)\mu} \prod_{j=1}^{\infty} (1 - q^{2dj})^k (1 + q^{4dj})^k$$

where f is as in Lemma 2.1. If $2k - s(N)\mu \geq 0$ then in both cases $(g_1'/g_1)/f_2$ is an alternating series and since $\mu - 1$ is odd the coefficient of $q^{\mu-1}$ in this series is negative. Therefore a_μ is negative which is a contradiction. □

We now proceed as in [8] and express the first coefficients of the shadow theta series of an s-extremal N-modular lattice.

Lemma 3.1. *Let $N \in \mathcal{L}$, $s_1 := s_1^{(N)}$ and $s_2 := s_2^{(N)}$. Then $s_1^k \sum_{i=0}^m c_i s_2^i$ starts with $(-1)^m 2^{\sigma_0(N)(k - ms(N)/2)} q^{k\sigma_1(N)/4 - 2m}$ times*

$$c_m - (2^{s(N)\sigma_0(N)/2} c_{m-1} + (s(N)m - k)c_m)q^2$$

if N is odd, and with $(-1)^m 2^{\sigma_0(N)k/2-ms(N)\sigma_0(N)/4} q^{k\sigma_1(N/2)/2-m}$ *times*

$$\begin{aligned} & c_m - (2^{s(N)\sigma_0(N)/4} c_{m-1} + (s(N)m-2k)c_m)q \\ & + \Big(2^{s(N)\sigma_0(N)/2} c_{m-2} + 2^{s(N)\sigma_0(N)/4}(s(N)(m-1)-2k)c_{m-1} \\ & + (s(N)^2 \tfrac{m(m-1)}{2} - 2kms(N) + 2k(k-1) + 2^{s(N)\sigma_0(N)/4} \tfrac{m(s(N)+1)}{4})c_m\Big) q^2 \end{aligned}$$

if N is even.

Proof. If N is odd then

$$\begin{aligned} s_1 &= 2^{\sigma_0(N)} q^{\sigma_1(N)/4}(1+q^2) + \dots \\ s_2 &= 2^{-s(N)\sigma_0(N)/2}(-q^{-2} + s(N)) + \dots \end{aligned}$$

and for even N

$$\begin{aligned} s_1 &= 2^{\sigma_0(N)/2} q^{\sigma_1(N/2)/2}(1+2q+0q^2+)\dots \\ s_2 &= 2^{-s(N)\sigma_0(N)/4}(-q^{-1} + s(N)) - \tfrac{s(N)+1}{4} q + \dots \end{aligned}$$

Explicit calculations prove the lemma. □

We now want to use [8, Lemma 1] to show that the coefficients c_m and c_{m-1} determined in the proof of Theorem 2.1 for the theta series of an s-extremal lattice satisfy $(-1)^j c_j > 0$ and c_m/c_{m-1} is bounded.

If L is an s-extremal lattice of even minimum $\mu = m+1$ in the genus of C_N^k, then Theorem 3.1 yields that

$$k = \frac{s(N)}{2}(m-1) + b \text{ for some } 0 \le b < s(N).$$

Let

$$\psi := \psi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N} (1 - q^{2jd}) \text{ and } \varphi := \varphi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N} (1 + q^{(2j-1)d}).$$

Then

$$\begin{aligned} c_{m-\ell} &= \tfrac{-k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } g_1' \psi^{-k-1} \varphi^{s(N)(m-\ell)-2(k+1)} \\ &= \tfrac{-k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } G_\ell^{(b)} H^{m-\ell-1} \end{aligned}$$

where

$$G_\ell^{(b)} = g_1' \psi^{-b-1-\ell s(N)/2} \varphi^{-2b-2+(1-\ell)s(N)} = G_\ell^{(0)} (\psi^{-1}\phi^{-2})^b$$

and

$$H = \psi^{-s(N)/2} = 1 + \frac{s(N)}{2} q^2 + \dots.$$

In particular the first two coefficients of H are positive and the remaining coefficients are nonnegative. Since also odd powers of q arise in $G_\ell^{(b)}$ the coefficient $\beta_{m-\ell-1}$ of $q^{m-\ell-1}$ in $G_\ell^{(b)} H^{m-\ell-1}$ is by Cauchy's formula

$$\beta_{m-\ell-1} = \frac{1}{2} \int_{-1+iy}^{1+iy} e^{-\pi i(m-\ell-1)z} G_\ell^{(b)}(e^{\pi iz}) H^{m-\ell-1}(e^{\pi iz}) dz$$

for arbitrary $y > 0$.

Put $F(y) := e^{\pi y} H(e^{-\pi y})$ and let y_0 be the first positive zero of F'. Then we check that $d_1 := F(y_0) > 0$ and $d_2 := F''(y_0)/F(y_0) > 0$. Now H has two saddle points in $[-1 + iy_0, 1 + iy_0]$ namely at $\pm 1 + iy_0$ and iy_0. By the saddle point method (see [1, (5.7.2)]) we obtain

$$\begin{aligned}\beta_{m-\ell-1} \sim d_1^{m-\ell-1}(G_\ell^{(b)}(e^{-\pi y_0}) + (-1)^{m-\ell-1} G_\ell^{(b)}(-e^{-\pi y_0}))\\ \times (2\pi(m-\ell-1)d_2)^{-1/2}\end{aligned}$$

as m tends to infinity. In particular

$$c_m \sim d_1 \frac{G_0^{(b)}(e^{-\pi y_0}) + (-1)^{m-1} G_0^{(b)}(-e^{-\pi y_0})}{G_1^{(b)}(e^{-\pi y_0}) + (-1)^m G_1^{(b)}(-e^{-\pi y_0})} c_{m-1}.$$

Lemma 3.2. *For $N \in \mathcal{L}$ and $b \in \{0, \ldots, s(N)-1\}$ let $k := \frac{s(N)}{2}(m-1)+b = js(N) + b$, $G_\ell^{(b)}$, d_1, d_2, y_0 be as above where $m = 2j+1$ is odd. Then c_{2j+1}/c_{2j} tends to*

$$Q(N,b) := d_1 \frac{G_0^{(b)}(e^{-\pi y_0}) + G_0^{(b)}(-e^{-\pi y_0})}{G_1^{(b)}(e^{-\pi y_0}) - G_1^{(b)}(-e^{-\pi y_0})} \in \mathbb{R}_{<0}$$

if j goes to infinity.

By Lemma 3.1 the second coefficient $b_{\sigma+2}$ in the shadow theta series of a putative s-extremal strongly N-modular lattice of even minimum $\mu = m+1$ in the genus of C_N^k ($k = \frac{s(N)}{2}(m-1) + b$ as above) is a positive multiple of

$$\begin{aligned}2^{s(N)\frac{\sigma_0(N)}{2}} c_{m-1} + (s(N)m - k)c_m\\ \sim (2^{s(N)\frac{\sigma_0(N)}{2}} + Q(N,b)\frac{s(N)(m+1)-2b}{2})c_{m-1}\end{aligned}$$

when m tends to infinity. In particular this coefficient is expected to be negative if

$$\mu = m+1 > B(N,b) := \frac{2}{s(N)}(b + \frac{2^{s(N)\sigma_0(N)/2}}{-Q(N,b)}).$$

Since all these are asymptotic values, the actual value $\mu_-(N,b)$ of the first even minimum μ where $b_{\sigma+2}$ becomes negative may be different. In all cases, the second coefficient of the relevant shadow theta series seems to remain negative for even minimum $\mu \geq \mu_-(N,b)$.

For odd $N \in \mathcal{L}$ the values of $B(N,b)$ and $\mu_-(N,b)$ are given in the following tables:

$N=1$	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$	$b=8$
Q(1,b)	-380	-113	-43.8	-18.4	-8	-3.53	-1.57	-0.71	-0.33
B(1,b)	0.9	3.1	7.96	18.8	43	97.1	217.4	480.4	1036.6
$\mu_-(1,b)$	6	6	12	20	44	96	216	478	1032
$k_-(1,b)$	48	49	122	219	508	1133	2574	5719	12368

$N=1$	$b=9$	$b=10$	$b=11$	$b=12$	$b=13$	$b=14$	$b=15$
Q(1,b)	-0.16	-0.08	-0.05	-0.04	-0.03	-0.027	-0.026
B(1,b)	2131.3	4012.4	6597.4	9240.4	11239.4	12433.6	13049.1

$N=1$	$b=16$	$b=17$	$b=18$	$b=19$	$b=20$	$b=21$	$b=22$	$b=23$
Q(1,b)	-0.026	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025	-0.025
B(1,b)	13342	13477	13538	13565	13577	13582	13585	13586

$N=3$	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
Q(3,b)	-15.6	-2	-0.45	-0.2	-0.16	-0.15
B(3,b)	1.36	11	47.6	107.13	137.07	144.34
$\mu_-(3,b)$	6	12	44	100	126	130
$k_-(3,b)$	12	31	128	297	376	389

$N=5$	$b=0$	$b=1$	$b=2$	$b=3$	$N=7$	$b=0$	$b=1$	$b=2$
Q(5,b)	-5	-0.73	-0.31	-0.25	Q(7,b)	-2.88	-0.51	-0.32
B(5,b)	1.6	11	27	33.5	B(7,b)	1.85	11	17.8
$\mu_-(5,b)$	6	12	22	24	$\mu_-(7,b)$	6	10	12
$k_-(5,b)$	8	21	42	47	$k_-(7,b)$	6	13	17

$N=11$	$b=0$	$b=1$	$N=15$	$b=0$	$N=23$	$b=0$
Q(11,b)	-1.72	-0.45	Q(15,b)	-2.03	Q(23,b)	-1.08
B(11,b)	2.33	9.8	B(15,b)	3.93	B(23,b)	3.69
$\mu_-(11,b)$	6	6	$\mu_-(15,b)$	6	$\mu_-(23,b)$	6
$k_-(11,b)$	4	5	$k_-(15,b)$	2	$k_-(23,b)$	2

For even $N \in \mathcal{L}$ the situation is slightly different. Again $k = b + \frac{s(N)}{2}(m-1)$ for some $0 \leq b < s(N)$. From Lemma 3.1 the second coefficient $b_{\sigma+1}$ in the s-extremal shadow theta series is a nonzero multiple of $2^{s(N)\sigma_0(N)/4}c_{m-1} + (s(N)-2b)c_m$ and in particular its sign is asymptotically independent of m. Therefore we need to consider the third coefficient $b_{\sigma+2}$, which is by Lemma 3.1 for odd m a positive multiple of

$$\begin{aligned}- a^2c_{m-2} &+ a(2k - s(m-1))c_{m-1} \\ &+ (2kms - s^2\frac{m(m-1)}{2} - 2k(k-1) - am\frac{s+1}{4})c_m\end{aligned}$$

where for short $a := 2^{s\sigma_0(N)/4}$ and $s := s(N)$. For $k = \frac{s(N)}{2}(m-1) + b$ this becomes

$$-a^2c_{m-2} + 2abc_{m-1} + (m(2b(b-1-s) - a\frac{s+1}{4} + s\frac{s+2}{2}) + \frac{2s+s^2}{2})c_m.$$

Since the quotients c_{m-1}/c_{m-2} and c_m/c_{m-2} are bounded, there is an explicit asymptotic bound $B(N,b)$ for $\mu = m+1$ after which this coefficient should become negative. Again, the true values $\mu_-(N,b)$ differ and the results are displayed in the following table.

$N=2$	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$
B(2,b)	-4.9	10	52.5	170.1	382.6	575.9	677.7	725.7
$\mu_-(2,b)$	16	22	54	166	374	564	666	716
$k_-(2,b)$	56	81	210	659	1492	2253	2662	2863

$N=6$	$b=0$	$b=1$	$N=14$	$b=0$
B(6,b)	1	33.58	B(14,b)	2
$\mu_-(6,b)$	10	28	$\mu_-(14,b)$	10
$k_-(6,b)$	8	27	$k_-(14,b)$	4

3.1. Explicit classifications.

In this section we classify the s-extremal strongly N-modular lattices $L_N(\mu,k)$ rational equivalent to C_N^k for certain N and even minimum μ. For $N \in \{11, 14, 15, 23\}$ a complete classification is obtained. For convenience we denote the uniquely determined modular form that should be the theta series of $L_N(\mu,k)$ by $\theta_N(\mu,k)$ and its shadow by $\sigma_N(\mu,k)$.

Important examples are the unique extremal even strongly N-modular lattices $E^{(N)}$ of minimum 4 and with $k = s(N)$ from [13, Table 1]. For odd

N, these lattices are s-extremal since $2\mu + \sigma = 8 = s(N)\sigma_1(N)/4 + 2$ and hence $E^{(N)} = L_N(4, s(N))$.

Theorem 3.1 suggests to write $k = \frac{s(N)(\mu-2)}{2} + b$ for some $0 \leq b \leq s(N) - 1$ and we will organize the classification according to the possible b. Note that for every b the maximal minimum μ is bounded by $\mu_-(N, b)$ above.

If $N = 14, 15$ or 23, then $s(N) = 1$ and hence Theorem 3.1 implies that $k = \frac{\mu-2}{2}$. For $N = 15, 23$ the only possibility is $k = 1$ and $\mu = 4$ and $L_N(4, 1) = E^{(N)}$. The second coefficient of $\sigma_{14}(4, 1)$ and $\sigma_{14}(8, 3)$ is negative, hence the only s-extremal strongly 14-modular lattice with even minimum is $L_{14}(6, 2)$ of minimum 6. The series $\sigma_{14}(6, 2)$ starts with $8q^3 + 8q^5 + 16q^6 + \ldots$. Therefore the even neighbour of $L_{14}(6, 2)$ in the sense of [13, Theorem 8] is the unique even extremal strongly 14-modular lattice of dimension 8 (see [14, p. 160]). Constructing all odd 2-neighbours of this lattice, it turns out that there is a unique such lattice $L_{14}(6, 2)$. Note that $L_{14}(6, 2)$ is an odd extremal strongly modular lattice in a jump dimension and hence the first counterexample to conjecture (3) in the Remark after [13, Theorem 2].

For $N = 11$ and $b = 0$ the only possibility is $\mu = 4$ and $k = 2 = s(N)$ whence $L_{11}(4, 2) = E^{(11)}$. If $b = 1$ then either $\mu = 2$ and $L_{11}(2, 1) = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$ or $\mu = 4$. An explicit enumeration of the genus of C_{11}^3 with the Kneser neighbouring method [7] shows that there is a unique lattice $L_{11}(4, 3)$.

Now let $N = 7$. For $b = 0$ again the only possibility is $k = s(N)$ and $L_7(4, 3) = E^{(7)}$. For $b = 1$ and $b = 2$ one obtains unique lattices $L_7(2, 1)$ (with Grammatrix $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$) $L_7(4, 4)$ and $L_7(4, 5)$. There is no contradiction for the existence of lattices $L_7(6, 7)$, $L_7(6, 8)$, $L_7(8, 10)$, $L_7(8, 11)$, though a complete classification of the relevant genera seems to be difficult. For the lattice $L_7(6, 8)$ we tried the following: Both even neighbours of such a lattice are extremal even 7-modular lattices. Starting from the extremal 7-modular lattice constructed from the structure over $\mathbb{Z}[\sqrt{2}]$ of the Barnes-Wall lattice as described in [14], we calculated the part of the Kneser 2-neighbouring graph consisting only of even lattices of minimum 6 and therewith found 126 such even lattices 120 of which are 7-modular. None of the edges between such lattices gave rise to an s-extremal lattice. The lattice $L_7(10, 14)$ does not exist because $\theta_7(10, 14)$ has a negative coefficient at q^{13}.

Now let $N := 6$. For $k = \mu - 2$ the second coefficient in the shadow theta series is negative, hence there are no lattices $L_6(\mu, \mu - 2)$ of even minimum μ. For $k = \mu - 1 < 27$ the modular forms $\theta_6(\mu, \mu - 1)$ and $\sigma_6(\mu, \mu - 1)$ seem to have nonnegative integral coefficients. The lattice $L_6(2, 1)$ is unique and already given in [9]. For $\mu = 4$ the even neighbour of any lattice $L_6(4, 3)$ (as defined in [13, Theorem 8]) is one of the five even extremal strongly

6-modular lattices given in [14]. Constructing all odd 2-neighbours of these lattices we find a unique lattice $L_6(4,3)$ as displayed below.

For $N=5$ the lattice $L_5(4,4)=E^{(5)}$ is is the only s-extremal lattice of even minimum μ for $k=2(\mu-2)$, because $\mu_-(5,0)=6$. For $k=2(\mu-2)+1$ the shadow series $\sigma_5(2,1)$, $\sigma_5(4,5)$ and $\sigma_5(6,9)$ have non integral respectively odd coefficients so the only lattices that might exist here are $L_5(8,13)$ and $L_5(10,17)$. The s-extremal lattice $L_5(2,2)=\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ is unique. The theta series $\theta_5(2,3)$ starts with $1+20q^3+\ldots$, hence $L_5(2,3)=S^{(5)}$ has minimum 3. The genus of C_5^6 contains 1161 isometry classes, 3 of which represent s-extremal lattices of minimum 4 and whose Grammatrices $L_5(4,6)_{a,b,c}$ are displayed below. For $k=7$ a complete classification of the genus of C_5^k seems to be out of range. A search for lattices in this genus that have minimum 4 constructs the example $L_5(4,7)_a$ displayed below of which we do not know whether it is unique. For the remaining even minima $\mu<\mu_-(5,b)$ we do not find a contradiction against the existence of such s-extremal lattices.

For $N=3$ and $b=0$ again $E^{(3)}=L_3(4,6)$ is the unique s-extremal lattice. For $k=3(\mu-2)+1$, the theta series $\theta_3(8,19)$ and $\theta_3(10,25)$ as well as their shadows seem to have integral non-negative coefficients, whereas $\sigma_3(4,7)$ and $\sigma_3(6,13)$ have non-integral coefficients. The remaining theta-series and their shadows again seem to have integral non-negative coefficients. The lattices of minimum 2 are already classified in [9]. In all cases $L_3(2,b)$ $(2\le b\le 5)$ is unique but $L_3(2,5)=S^{(3)}$ has minimum 3.

Now let $N:=2$. For $b=0$ and $b=1$ the second coefficient in $\sigma_2(\mu,4(\mu-2)+b)$ is always negative, proving the non-existence of such s-extremal lattices. The lattices of minimum 2 are already classified in [9]. There is a unique lattice $L_2(2,2)\cong D_4$, no lattice $L_2(2,3)$ since the first coefficient of $\sigma_2(2,3)$ is 3, unique lattices $L_2(2,b)$ for $b=4,5$ and 7 and two such lattices $L_2(2,6)$.

For $N=1$ we also refer to the paper [6] for the known classifications. Again for $b=0$, the Leech lattice $L_1(4,24)=E^{(1)}$ is the unique s-extremal lattice. For $\mu=2$, these lattices are already classified in [5]. The possibilities for $b=k$ are $8,12,14\le b\le 22$. For $\mu=4$, the possibilities are either $b=0$ and $k=24$ or $8\le b\le 23$ whence $32\le k\le 47$ since the other shadow series have non-integral coefficients. The lattices $L_1(4,32)$ are classified in [3]. For $\mu=6$ no such lattices are known. The first possible dimension is 56, since the other shadow series have non-integral coefficients.

Since for odd N the value $\mu_-(N,0)=6$ and the s-extremal lattices of minimum 4 with $k=s(N)$ are even and hence isometric to $E^{(N)}$ we obtain the following theorem.

Theorem 3.2. *Let L be an extremal and s-extremal lattice rational equivalent to C_N^k for some $N \in \mathcal{L}$ such that k is a multiple of $s(N)$. Then $\mu := \min(L)$ is even and $k = s(N)(\mu - 2)/2$ and either $\mu = 4$, N is odd and $L = E^{(N)}$ or $\mu = 6$, $N = 14$ and $L = L_{14}(6, 2)$.*

For $N \in \{11, 14, 15, 23\}$ the complete classification of s-extremal strongly N-modular lattices in the genus of C_N^k is as follows:

N	23	15	14	11	11	11
min	4	4	6	2	4	4
k	1	1	2	1	2	3
lattice	$E^{(23)}$	$E^{(15)}$	$E^{(14)}$	$L_{11}(2,1)$	$E^{(11)}$	$L_{11}(4,3)$

For the remaining $N \in \mathcal{L}$, the results are summarized in the following tables. The last line, labelled with # displays the number of lattices, where we display $-$ if there is no such lattice, ? if we do not know such a lattice, + if there is a lattice, but the lattices are not classified. We always write $k = \ell s(N) + b$ with $0 \leq b \leq s(N) - 1$ such that $\mu = \min(L) = 2\ell + 2$ by Theorem 3.1 and $\dim(L) = k\sigma_0(N)$.

$N = 7,\ s(N) = 3,\ k = \ell s(N) + b$

b	0		1					2				
ℓ	1	≥ 2	0	1	2	3	≥ 4	0	1	2	3	≥ 4
min	4	≥ 6	2	4	6	8	≥ 10	3	4	6	8	≥ 10
#	1	-	1	1	?	?	-	1	1	?	?	-

$N = 6,\ s(N) = 2,\ k = \ell s(N) + b$

b	0	1			
ℓ	≥ 1	0	1	$2 \leq \ell \leq 12$	≥ 13
min	≥ 4	2	4	$6 \leq \mu \leq 26$	≥ 28
#	-	1	1	?	-

$N = 5,\ s(N) = 4,\ k = \ell s(N) + b$

b	0		1					
ℓ	1	≥ 2	0	1	2	3	4	≥ 5
min	4	≥ 6	2	4	6	8	10	≥ 12
#	1	-	-	-	-	?	?	-

b	2				3			
ℓ	0	1	$2 \leq \ell \leq 9$	≥ 10	0	1	$2 \leq \ell \leq 10$	≥ 11
min	2	4	$6 \leq \mu \leq 20$	≥ 22	3	4	$6 \leq \mu \leq 22$	≥ 24
#	1	3	?	-	1	+	?	-

$$N=3,\ s(N)=6,\ k=\ell s(N)+b$$

b	0		1					2			3	
ℓ	1	≥ 2	1	2	3	4	≥ 5	0	$1 \leq \ell \leq 20$	≥ 21	0	$1 \leq \ell \leq 48$
min	4	≥ 6	4	6	8	10	≥ 12	2	$4 \leq \ell \leq 42$	≥ 44	2	$4 \leq \mu \leq 98$
#	1	-	-	-	?	?	-	1	?	-	1	?

b	3	4			5		
ℓ	≥ 49	0	$1 \leq \ell \leq 61$	≥ 62	0	$1 \leq \ell \leq 63$	≥ 64
min	≥ 100	2	$4 \leq \mu \leq 124$	≥ 126	3	$4 \leq \mu \leq 128$	≥ 130
#	-	1	?	-	1	?	-

$$N=2,\ s(N)=8,\ k=\ell s(N)+b$$

b	0	1	2			3		
ℓ	≥ 1	≥ 1	0	$1 \leq \ell \leq 25$	≥ 26	0	$1 \leq \ell \leq 81$	≥ 82
min	≥ 4	≥ 4	2	$4 \leq \mu \leq 52$	≥ 54	2	$4 \leq \mu \leq 164$	≥ 166
#	-	-	1	?	-	-	?	-

b	4			5		
ℓ	0	$1 \leq \ell \leq 185$	≥ 186	0	$1 \leq \ell \leq 280$	≥ 281
min	2	$4 \leq \mu \leq 372$	≥ 374	2	$4 \leq \mu \leq 562$	≥ 564
#	1	?	-	1	?	-

b	6			7		
ℓ	0	$1 \leq \ell \leq 331$	≥ 332	0	$1 \leq \ell \leq 356$	≥ 357
min	2	$4 \leq \mu \leq 664$	≥ 666	2	$4 \leq \mu \leq 714$	≥ 716
#	2	?	-	1	?	-

Grammatrices of the new s-extremal lattices:

$$L_{14}(6,2)=\begin{pmatrix} 6 & 3 & 0 & 2 & -3 & 3 & -1 & -2 \\ 3 & 6 & 3 & 2 & -3 & 3 & -3 & -2 \\ 0 & 3 & 6 & 0 & -3 & 2 & -2 & -3 \\ 2 & 2 & 0 & 6 & -2 & -1 & 1 & -3 \\ -3 & -3 & -3 & -2 & 6 & -3 & 3 & 3 \\ 3 & 3 & 2 & -1 & -3 & 7 & -4 & -2 \\ -1 & -3 & -2 & 1 & 3 & -4 & 7 & -1 \\ -2 & -2 & -3 & -3 & 3 & -2 & -1 & 7 \end{pmatrix},\quad L_{11}(4,3)=\begin{pmatrix} 4 & 0 & 0 & 2 & -2 & -1 \\ 0 & 4 & 0 & -1 & 2 & 2 \\ 0 & 0 & 4 & -2 & -1 & -2 \\ 2 & -1 & -2 & 5 & -1 & 0 \\ -2 & 2 & -1 & -1 & 5 & 2 \\ -1 & 2 & -2 & 0 & 2 & 5 \end{pmatrix}$$

$$L_7(4,4)=\begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 2 & -2 & -1 \\ 0 & 4 & 0 & 2 & 2 & -1 & -2 & 2 \\ 0 & 0 & 4 & -1 & 2 & 2 & 1 & 2 \\ 2 & 2 & -1 & 5 & 2 & 1 & -3 & 1 \\ 2 & 2 & 2 & 2 & 5 & 2 & -1 & 2 \\ 2 & -1 & 2 & 1 & 2 & 5 & 1 & 1 \\ -2 & -2 & 1 & -3 & -1 & 1 & 5 & 1 \\ -1 & 2 & 2 & 1 & 2 & 1 & 1 & 5 \end{pmatrix},\quad L_7(4,5)=\begin{pmatrix} 4 & 0 & 0 & -2 & 1 & 1 & 1 & 2 & 1 & -1 \\ 0 & 4 & 0 & -2 & 1 & -2 & -1 & 2 & 2 & 2 \\ 0 & 0 & 4 & -2 & -2 & 0 & 2 & 2 & 2 & -1 \\ -2 & -2 & -2 & 5 & 0 & 0 & -1 & -3 & -2 & 1 \\ 1 & 1 & -2 & 0 & 5 & -2 & -1 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 & -2 & 5 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & -1 & -1 & -1 & 5 & 1 & -1 & -3 \\ 2 & 2 & 2 & -3 & 0 & 0 & 1 & 5 & 3 & -1 \\ 1 & 2 & 2 & -2 & -2 & 1 & -1 & 3 & 6 & 2 \\ -1 & 2 & -1 & 1 & -1 & 0 & -3 & -1 & 2 & 6 \end{pmatrix}$$

$$L_6(4,3) = \begin{pmatrix} 4 & 1 & -2 & 1 & 0 & 1 & 1 & 1 & -2 & 2 & 0 & -1 \\ 1 & 4 & -2 & 0 & 1 & -1 & 1 & 1 & 0 & 2 & 0 & 1 \\ -2 & -2 & 4 & -1 & -1 & 0 & -1 & 0 & 1 & -2 & -1 & -1 \\ 1 & 0 & -1 & 4 & -1 & 2 & -1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 4 & -2 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 2 & -2 & 4 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 4 & 2 & -2 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 & 2 & 4 & -2 & 0 & -2 & 0 \\ -2 & 0 & 1 & 1 & 1 & -1 & -2 & -2 & 5 & -1 & 1 & 0 \\ 2 & 2 & -2 & -1 & 1 & 0 & 0 & 0 & -1 & 5 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & -2 & 1 & 0 & 4 & 2 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & 2 & 5 \end{pmatrix}, \quad L_5(4,6)_a = \begin{pmatrix} 4 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -2 & -1 \\ -1 & 4 & 0 & 0 & -2 & -2 & 0 & 2 & -2 & 0 & 1 & -1 \\ -1 & 0 & 4 & 0 & 0 & 0 & 2 & 1 & -2 & -2 & 1 & 1 \\ 1 & 0 & 0 & 4 & -1 & 1 & 1 & 2 & -1 & -2 & -2 & 2 \\ 1 & -2 & 0 & -1 & 4 & 1 & 0 & -2 & 2 & 2 & 1 & -1 \\ 0 & -2 & 0 & 1 & 1 & 4 & 0 & 0 & 0 & -1 & -1 & 2 \\ -1 & 0 & 2 & 1 & 0 & 0 & 4 & 1 & 0 & -2 & -1 & 1 \\ -1 & 2 & 1 & 2 & -2 & 0 & 1 & 5 & -3 & -1 & -1 & 0 \\ 0 & -2 & -2 & -1 & 2 & 0 & 0 & -3 & 5 & 2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & -1 & -2 & -1 & 2 & 5 & 2 & -3 \\ -2 & 1 & 1 & -2 & 1 & -1 & -1 & -1 & 0 & 2 & 5 & 0 \\ -1 & -1 & 1 & 2 & -1 & 2 & 1 & 0 & 0 & -3 & 0 & 5 \end{pmatrix}$$

$$L_5(4,6)_b = \begin{pmatrix} 4 & 1 & 1 & 0 & 2 & 0 & -1 & 0 & 1 & -1 & -1 & 2 \\ 1 & 4 & -1 & 2 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 2 \\ 1 & -1 & 4 & 0 & 1 & -2 & -2 & 2 & 2 & 1 & 1 & -1 \\ 0 & 2 & 0 & 4 & 0 & -1 & -1 & 2 & 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & 0 & 4 & 0 & -1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 & 4 & 1 & -1 & -2 & 1 & -1 & 0 \\ -1 & -1 & -2 & -1 & -1 & 1 & 4 & -2 & -1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 2 & 2 & -1 & -2 & 5 & 2 & 1 & 3 & 0 \\ 1 & 0 & 2 & 2 & 1 & -2 & -1 & 2 & 5 & -1 & 2 & 2 \\ -1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & -1 & 5 & 2 & -2 \\ -1 & 0 & 1 & 2 & 1 & -1 & -1 & 3 & 2 & 2 & 5 & -1 \\ 2 & 2 & -1 & 2 & 1 & 0 & 0 & 0 & 2 & -2 & -1 & 5 \end{pmatrix}, \quad L_5(4,6)_c = \begin{pmatrix} 4 & -2 & 0 & 0 & 1 & 2 & 0 & 2 & -2 & 2 & 2 & 2 \\ -2 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 4 & -1 & 0 & 1 & -1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 0 & -1 & 4 & 0 & 2 & -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & 1 & 2 & 0 & 5 & 0 & 1 & 0 & 2 \\ -2 & 0 & 2 & 2 & -1 & -2 & 0 & 0 & 5 & -1 & 1 & -3 \\ 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & -1 & 5 & 0 & 2 \\ 2 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 & -1 \\ 2 & 0 & 0 & -1 & 1 & 2 & 0 & 2 & -3 & 2 & -1 & 5 \end{pmatrix}$$

$$L_5(4,7)_a = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -1 \\ 0 & 4 & 0 & -1 & -2 & 0 & -2 & 2 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & 5 & 2 & 0 & -1 & 1 & 2 & -1 & 0 & 2 & 0 & -1 \\ 0 & -2 & -2 & 2 & 5 & -2 & 2 & 0 & 0 & -2 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 & 5 & -1 & 1 & -1 & 2 & 2 & 0 & -1 & -2 \\ 0 & -2 & -2 & -1 & 2 & -1 & 5 & -2 & -1 & -2 & 0 & -1 & -1 & 2 \\ -2 & 2 & 0 & 1 & 0 & 1 & -2 & 5 & 1 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & -1 & 1 & 5 & -2 & -2 & 1 & -1 & -1 \\ -2 & 0 & 1 & -1 & -2 & 2 & -2 & 1 & -2 & 5 & 2 & 0 & 1 & -1 \\ 0 & -2 & 0 & 0 & 0 & 2 & 0 & -2 & -2 & 2 & 5 & 0 & -2 & -2 \\ -2 & 0 & 0 & 2 & 2 & 0 & -1 & 2 & 1 & 0 & 0 & 5 & -2 & 0 \\ 0 & 2 & -1 & 0 & -1 & -1 & -1 & 1 & -1 & 1 & -2 & -2 & 5 & 1 \\ -1 & 0 & -2 & -1 & 2 & -2 & 2 & 0 & -1 & -1 & -2 & 0 & 1 & 5 \end{pmatrix}$$

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Gabriele Nebe
Lehrstuhl D für Mathematik
RWTH Aachen
52056 Aachen, Germany
E-mail : `nebe@math.rwth-aachen.de`
URL: `http://www.math.rwth-aachen.de/~Gabriele.Nebe/`

Kristina Schindelar
Lehrstuhl D für Mathematik
RWTH Aachen
52056 Aachen, Germany
E-mail : `schindelar@math.rwth-aachen.de`