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On the generalized principal ideal theorem of complex multiplication

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Dedicated to Michael Pohst on his 60th birthday

Résumé. Dans le p^n-ième corps cyclotomique $\mathbb{Q}_{p^n}$, p un nombre premier, $n \in \mathbb{N}$, le premier p est totalement ramifié, l'idéal au dessus de p dans $\mathbb{Q}_{p^n}$ étant engendré par $\omega_n = \zeta_{p^n} - 1$ avec une racine primitive p^n-ième de l'unité $\zeta_{p^n} = e^{\frac{2\pi i}{p^n}}$. De plus ces nombres constituent un ensemble qui vérifie la relation de norme $\mathbf{N}_{\mathbb{Q}_{p^{n+1}}/\mathbb{Q}_{p^n}}(\omega_{n+1}) = \omega_n$. Le but de cet article est d'établir un résultat analogue pour les corps de classes de rayon $K_{\mathfrak{p}^n}$ de conducteur $\mathfrak{p}^n$ d'un corps quadratique imaginaire K, où $\mathfrak{p}^n$ est une puissance d'un idéal premier dans K. Un tel résultat est obtenu en remplaçant la fonction exponentielle par une fonction elliptique convenable.

Abstract. In the p^n-th cyclotomic field $\mathbb{Q}_{p^n}$, p a prime number, $n \in \mathbb{N}$, the prime p is totally ramified and the only ideal above p is generated by $\omega_n = \zeta_{p^n} - 1$, with the primitive p^n-th root of unity $\zeta_{p^n} = e^{\frac{2\pi i}{p^n}}$. Moreover these numbers represent a norm coherent set, i.e. $\mathbf{N}_{\mathbb{Q}_{p^{n+1}}/\mathbb{Q}_{p^n}}(\omega_{n+1}) = \omega_n$. It is the aim of this article to establish a similar result for the ray class field $K_{\mathfrak{p}^n}$ of conductor $\mathfrak{p}^n$ over an imaginary quadratic number field K where $\mathfrak{p}^n$ is the power of a prime ideal in K. Therefore the exponential function has to be replaced by a suitable elliptic function.

1. Introduction and results

Let K be an imaginary quadratic number field, $\mathfrak{f}$ an integral ideal in K and $K_{\mathfrak{f}}$ the ray class field modulo $\mathfrak{f}$ over K. In particular $K_{(1)}$ is the Hilbert class field of K. The generalized Principle Ideal Theorem [Sch2], [Sch3] [1] says that for any power of a prime ideal $\mathfrak{p}^n$ there is an element $\pi_n \in K_{\mathfrak{p}^n}$

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[1]In [Sch3] the following has to be corrected:
1) The prime ideal $\mathfrak{q}$ in the definition of $H_{\mathfrak{q}}(z)$ must have the additional property $\gcd(\mathfrak{q}, \overline{\mathfrak{q}}) = 1$,
2) $H_{\mathfrak{q}}(1)$ has to be replaced by $H_{\mathfrak{q}}(\omega)$ with $\omega \equiv 1 \bmod \mathfrak{q}$, $\omega \equiv 0 \bmod \overline{\mathfrak{q}}$.

associated to $\mathfrak{p}^{\frac{1}{[K_{\mathfrak{p}^n}:K_{(1)}]}}$:

$$\pi_n \sim \mathfrak{p}^{\frac{1}{[K_{\mathfrak{p}^n}:K_{(1)}]}}.$$

The element π_n can be viewed as the elliptic analogue of the cyclotomic unit

$$\omega_n = e^{\frac{2\pi i}{p^n}} - 1$$

for the power p^n of a prime p. As an element of the p^n-th cyclotomic field $\mathbb{Q}_{p^n}$ the element ω_n has the factorisation

$$\omega_n \sim (p)^{\frac{1}{[\mathbb{Q}_{p^n}:\mathbb{Q}]}}.$$

Moreover ω_n has the following nice properties that can easily be verified:

- $\omega_n = e_n(1)$ with the p^n periodic function $e_n(z) = 1 - e^{\frac{2\pi i}{p^n} z}$.
- Let $\mathbb{C}_{p^n\mathbb{Z}}$ denote the field of p^n periodic meromorphic functions on $\mathbb{C}$, then we have the norm relation for $n \geq 0$
$$e_n(z) = N_{\mathbb{C}_{p^{n+1}\mathbb{Z}}/\mathbb{C}_{p^n\mathbb{Z}}}(e_{n+1}(z)) = \prod_{\xi \in p^n\mathbb{Z} \bmod p^{n+1}\mathbb{Z}} e_{n+1}(z+\xi).$$
- For $z = 1$ the last relation becomes a norm relation between number fields, if $n \geq 1$:
$$\omega_n = N_{\mathbb{Q}_{p^{n+1}}/\mathbb{Q}_{p^n}}(\omega_{n+1}) = \prod_{\xi \in p^n\mathbb{Z} \bmod p^{n+1}\mathbb{Z}} e_{n+1}(1+\xi)$$
- and
$$\left.\frac{e_0(z)}{e_1(z-1)}\right|_{z=1} = p.$$

It is the aim of this article to give a construction of π_n having the same nice properties. For a complex lattice Γ we therefore consider the Klein normalization of the Weierstrass σ-function

$$\varphi(z|\Gamma) = e^{-\frac{zz^*}{2}}\sigma(z|\Gamma)\sqrt[12]{\Delta(\Gamma)},$$

where $\Delta(\Gamma)$ is the discriminant of the theory of elliptic functions. Herein z^* is defined for a complex number z by

$$z^* = z_1\omega_1^* + z_2\omega_2^*,$$

with the real coordinates z_1, z_2 from the representation $z = z_1\omega_1 + z_2\omega_2$ by a basis ω_1, ω_2 of Γ and the quasiperiods $\omega_i^* = 2\zeta(\frac{\omega_i}{2}|\Gamma)$ of the Weierstrass ζ-function. The first factor $e^{-\frac{zz^*}{2}}\sigma(z|\Gamma)$ in the definition of $\varphi(z|\Gamma)$ is clearly independent of the choice of basis ω_1, ω_2 of Γ. To fix the 12-th root $\sqrt[12]{\Delta(\Gamma)}$ we use the identity

$$\Delta(\Gamma) = \left(\frac{2\pi i}{\omega_2}\right)^{12} \eta\left(\frac{\omega_1}{\omega_2}\right)^{24}$$

for a basis of Γ oriented by $\Im\left(\frac{\omega_1}{\omega_2}\right) > 0$ and set

$$\sqrt[12]{\Delta(\Gamma)} = \left(\frac{2\pi i}{\omega_2}\right)\eta\left(\frac{\omega_1}{\omega_2}\right)^2.$$

So the value $\varphi(z|\Gamma)$ is only well defined up to a 12-th root of unity depending on the basis chosen for its definition. However products where all the $\sqrt[12]{\Delta(\Gamma)}$-factors cancel out are independent of the choice of basis choosing the same basis for each factor.

We fix an arbitrary prime ideal $\mathfrak{p}$ in K and an integral auxiliary ideal $\mathfrak{q} \nmid 2$ that is prime to $\mathfrak{p}$ and satisfies

$$\gcd(\mathfrak{q}, \overline{\mathfrak{q}}) = 1.$$

For $n \in \mathbb{N}$ we define

$$E_n(z) := \frac{\varphi\left(z - \gamma_n \middle|\, \mathfrak{q}\mathfrak{p}^n\right)\varphi\left(z + \gamma_n \middle|\, \mathfrak{q}\mathfrak{p}^n\right)}{\varphi^2\left(z \middle|\, \mathfrak{q}\mathfrak{p}^n\right)}$$

with a solution γ_n of the congruences

$$\gamma_n \equiv 0 \bmod \mathfrak{p}^n,$$
$$\gamma_n \equiv 1 \bmod \mathfrak{q},$$
$$\gamma_n \equiv 0 \bmod \overline{\mathfrak{q}}.$$

Note that $E_n(z)$ is well defined because all Δ-factors are canceling out if we choose the same basis of $\mathfrak{q}\mathfrak{p}^n$ for every φ-value. Using the identity $\wp(u) - \wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}$, we can express E_n by the Weierstrass $\wp$-function:

$$E_n(z) = -\varphi^2\left(\gamma_n \middle|\, \mathfrak{q}\mathfrak{p}^n\right)\left(\frac{\wp\left(z \middle|\, \mathfrak{q}\mathfrak{p}^n\right)}{\sqrt[6]{\Delta(\mathfrak{q}\mathfrak{p}^n)}} - \frac{\wp\left(\gamma_n \middle|\, \mathfrak{q}\mathfrak{p}^n\right)}{\sqrt[6]{\Delta(\mathfrak{q}\mathfrak{p}^n)}}\right)$$

and we can conclude that E_n is elliptic with respect to the lattice $\mathfrak{q}\mathfrak{p}^n$. Moreover E_n satisfies the following norm relation:

Theorem 1. *Let $\mathbb{C}_{\mathfrak{q}\mathfrak{p}^n}$ denote the field of elliptic functions with respect to $\mathfrak{q}\mathfrak{p}^n$, $n \geq 0$. Then $\mathbb{C}_{\mathfrak{q}\mathfrak{p}^{n+1}}/\mathbb{C}_{\mathfrak{q}\mathfrak{p}^n}$ is a Galois extension, the Galois group consisting of all substitutions*

$$g(z) \mapsto g(z+\xi), \qquad \xi \in \mathfrak{q}\mathfrak{p}^n \bmod \mathfrak{q}\mathfrak{p}^{n+1}$$

for $g \in \mathbb{C}_{\mathfrak{q}\mathfrak{p}^{n+1}}$ and we have the norm relation

$$E_n(z) = \mathbf{N}_{\mathbb{C}_{\mathfrak{q}\mathfrak{p}^{n+1}}/\mathbb{C}_{\mathfrak{q}\mathfrak{p}^n}}\left(E_{n+1}(z)\right) = \prod_{\xi \in \mathfrak{q}\mathfrak{p}^n \bmod \mathfrak{q}\mathfrak{p}^{n+1}} E_{n+1}(z+\xi).$$

For the singular values $E_n(1)$ we obtain:

Theorem 2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be as above and let Φ denote the Euler function in K. Then*

(1) $E_n(1) \in K_{\mathfrak{q}\mathfrak{p}^n}$ *for* $n \geq 0$,

(2) $E_n(1) \sim \mathfrak{p}^{\frac{1}{\Phi(\mathfrak{p}^n)}}$ *for* $n \geq 1$,

(3) $E_n(1) = \mathbf{N}_{K_{\mathfrak{q}\mathfrak{p}^{n+1}}/K_{\mathfrak{q}\mathfrak{p}^n}}(E_{n+1}(1)) = \prod_{\xi \in \mathfrak{q}\mathfrak{p}^n \bmod \mathfrak{q}\mathfrak{p}^{n+1}} E_{n+1}(1+\xi)$ *for* $n \geq 1$,

(4) $\left.\frac{E_0(z)}{E_1(z-1+\gamma_1)}\right|_{z=1} = \mathbf{N}_{K_{\mathfrak{q}\mathfrak{p}}/K_{\mathfrak{q}}}(E_1(1)) = \frac{\varphi(1+\gamma_1|\mathfrak{q})\varphi(\gamma_1|\mathfrak{q}\mathfrak{p})^2}{\varphi(2\gamma_1|\mathfrak{q}\mathfrak{p})\varphi(1|\mathfrak{q})^2}\sqrt[12]{\frac{\Delta(\mathfrak{q})}{\Delta(\mathfrak{q}\mathfrak{p})}} \sim \mathfrak{p}$.

To obtain the analogous result for the extension $K_{\mathfrak{p}^{n+1}}/K_{\mathfrak{p}^n}$ that we were aiming at, we have to get rid of the auxiliary ideal $\mathfrak{q}$. Therefore we need the following (well known)

Lemma. *For any integral ideal $\mathfrak{a}$ in K*

$$\gcd\{N(\mathfrak{q})-1 \mid \mathfrak{q} \text{ prime ideal in } K,\ \mathfrak{q} \nmid 2\overline{\mathfrak{q}}\ \mathfrak{a}\} = w_K,$$

where w_K denotes the number of roots of unity in K.

So we can choose finitely many prime ideals $\mathfrak{q}_i$, $i = 1, ..., s$ of degree 1 that are prime to $N(\mathfrak{p})$ and integers $x_i \in \mathbb{Z}$ so that

$$x_1(N(\mathfrak{q}_1)-1) + ... + x_s(N(\mathfrak{q}_s)-1) = w_K.$$

For each $\mathfrak{q}_i$ we define a set of functions $E_{n,i}(z)$ as above with parameters $\gamma_{n,i}$. Taking relative norms we obtain

$$\mathbf{N}_{K_{\mathfrak{q}_i\mathfrak{p}^n}/K_{\mathfrak{p}^n}}(E_{n,i}(1)) \sim \mathfrak{p}^{\frac{N(\mathfrak{q}_i)-1}{\Phi(\mathfrak{p}^n)}}$$

for $n \geq 1$. Hence

$$\pi_n := \prod_{i=1}^{s}\left(\mathbf{N}_{K_{\mathfrak{q}_i\mathfrak{p}^n}/K_{\mathfrak{p}^n}}(E_{n,i}(1))\right)^{x_i}$$

is an element in $K_{\mathfrak{p}^n}$ having the factorisation

$$\pi_n \sim \mathfrak{p}^{\frac{w_K}{\Phi(\mathfrak{p}^n)}}.$$

This is what we were aiming at because

$$[K_{\mathfrak{p}^n} : K_{(1)}] = \frac{w(\mathfrak{p}^n)}{w_K}\Phi(\mathfrak{p}^n),$$

where $w(\mathfrak{p}^n)$ denotes the number of roots of unity in K that are congruent to 1 mod $\mathfrak{p}^n$. This implies

$$\pi_n \sim \mathfrak{p}^{\frac{w(\mathfrak{p}^n)}{[K_{\mathfrak{p}^n}:K_{(1)}]}}$$

where

$$w(\mathfrak{p}^n) = 1$$

except for the cases

(i) $\mathfrak{p} \mid 2$, $n \leq 2$, where $w(\mathfrak{p}^n) \in \{1,2\}$, if $d_K \neq -4$ and $w(\mathfrak{p}^n) \in \{1,2,4\}$ if $d_K = -4$;

(ii) $\mathfrak{p} \mid 3, n = 1, d_K = -3$, where $w(\mathfrak{p}^n) = 2$.

Moreover we will show now that this element can be written analogously to the cyclotomic case. We therefore observe that by reciprocity law the conjugates of the singular values $E_{n,i}(1)$ over $K_{\mathfrak{p}^n}$ are given by

$$E_{n,i}(1)^{\sigma(\lambda)} = \frac{\varphi\left(\lambda - \gamma_{n,i}\lambda \middle| \, \mathfrak{q}_i\mathfrak{p}^n\right)\varphi\left(\lambda + \gamma_{n,i}\lambda \middle| \, \mathfrak{q}_i\mathfrak{p}^n\right)}{\varphi^2\left(\lambda \middle| \, \mathfrak{q}_i\mathfrak{p}^n\right)},$$

where $\sigma(\lambda)$ denotes the Frobenius automorphism of $K_{\mathfrak{q}\mathfrak{p}^n}/K_{\mathfrak{p}^n}$ of the ideal $(\lambda), \lambda \equiv 1 \bmod \mathfrak{p}^n$. So we define the function

$$E_n^*(z) := \prod_{i=1}^{s} \prod_{j=1}^{N(\mathfrak{q}_i)-1} \left(\frac{\varphi\Big(z+(\lambda_{i,j}^{(n)}-1)-\gamma_{n,i}\lambda_{i,j}^{(n)}\Big|\mathfrak{q}_i\mathfrak{p}^n\Big)\varphi\Big(z+(\lambda_{i,j}^{(n)}-1)+\gamma_{n,i}\lambda_{i,j}^{(n)}\Big|\mathfrak{q}_i\mathfrak{p}^n\Big)}{\varphi^2\Big(z+(\lambda_{i,j}^{(n)}-1)\Big|\mathfrak{q}_i\mathfrak{p}^n\Big)} \right)^{x_i}$$

where for fixed i and n the numbers

$$\lambda_{i,j}^{(n)}, \quad j = 1, ..., N(\mathfrak{q}_i) - 1$$

are a complete system of prime residue classen mod $\mathfrak{q}_i$ satisfying

$$\lambda_{i,j}^{(n)} \equiv 1 \bmod \mathfrak{p}^n.$$

Herewith we can prove the following two Theorems:

Theorem 3. *Let $\mathfrak{p}$ and $\mathfrak{q} = \mathfrak{q}_1 \cdot \dots \cdot \mathfrak{q}_s$ with $\mathfrak{q}_i$ as above. Then the functions $E_n^*(z)$ are in $\mathbb{C}_{\mathfrak{q}\mathfrak{p}^n}$ for $n \geq 0$ and satisfy the Normrelation*

$$E_n^*(z) = \mathbf{N}_{\mathbb{C}_{\mathfrak{q}\mathfrak{p}^{n+1}}/\mathbb{C}_{\mathfrak{q}\mathfrak{p}^n}}\left(E_{n+1}^*(z)\right) = \prod_{\xi \in \mathfrak{q}\mathfrak{p}^n \bmod \mathfrak{q}\mathfrak{p}^{n+1}} E_{n+1}^*(z+\xi).$$

Theorem 4. *Let $\mathfrak{p}$ and $\mathfrak{q} = \mathfrak{q}_1 \cdot \dots \cdot \mathfrak{q}_s$ with $\mathfrak{q}_i$ as above and let Φ denote the Euler function in K. Then*

(1) $E_n^*(1) \in K_{\mathfrak{p}^n}$ *for* $n \geq 0$,

(2) $E_n^*(1) \sim \mathfrak{p}^{\frac{w(\mathfrak{p}^n)}{[K_{\mathfrak{p}^n}:K_{(1)}]}}$ *for* $n \geq 1$,

(3) $E_n^*(1) = \mathbf{N}_{K_{\mathfrak{p}^{n+1}}/K_{\mathfrak{p}^n}}\left(E_{n+1}^*(1)\right)^{\frac{w(\mathfrak{p}^n)}{w(\mathfrak{p}^{n+1})}} = \prod_{\substack{\xi \in \mathfrak{q}\mathfrak{p}^n \\ \bmod \mathfrak{q}\mathfrak{p}^{n+1}}} E_{n+1}^*(1+\xi)$

for $n \geq 1$,

(4) $\mathbf{N}_{K_{\mathfrak{p}^n}}/K_{(1)}\left(E_n^*(1)\right) \sim \mathfrak{p}^{w(\mathfrak{p}^n)}$.

Remark: The constructions of the above theorems can clearly be generalized to any integral ideal $\mathfrak{a}$ prime to $\mathfrak{q}$ instead of $\mathfrak{p}^n$ with obvious norm relations for two ideals $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{a} \mid \mathfrak{b}$. Of course for a composite ideal $\mathfrak{a}$ the singular values will be units.

2. Proofs

Proposition. *Let* $\Gamma = [\omega, 1]$, $\hat{\Gamma} = [\frac{\omega}{n_1}, \frac{1}{n_2}]$ *be complex lattices,* $\Im(\omega) > 0$, $n_1, n_2 \in \mathbb{N}$. *We consider the following system of representatives for* $\hat{\Gamma}/\Gamma$*:*

$$\xi = \frac{x\omega}{n_1} + \frac{y}{n_2}, \quad x = 0, .., n_1 - 1, \; y = 0, ...n_2 - 1.$$

Expressing Δ *by the* η*-function,* $\Delta = (2\pi i)^{12}\eta^{24}$*, we define the 12-th roots of* $\Delta(\Gamma)$ *and* $\Delta(\hat{\Gamma})$ *by*

$$\sqrt[12]{\Delta(\Gamma)} := (2\pi i)\eta(\omega)^2, \qquad \sqrt[12]{\Delta(\hat{\Gamma})} := (2\pi i)\eta(\tfrac{n_1\omega}{n_2})^2 n_2$$

and we set

$$l_\Gamma(z, \xi) = 2\pi i(z_1\xi_2 - z_2\xi_1).$$

Then

$$\prod_{\xi} e^{-\frac{1}{2}l_\Gamma(z,\xi)}\varphi(z + \xi|\Gamma) = \zeta\varphi(z|\hat{\Gamma})$$

with

$$\zeta = -\zeta_4^{n_1n_2+n_1}\zeta_8^{(n_1-1)(n_2-1)}$$

($\zeta_n := e^{\frac{2\pi i}{n}}$*). Furthermore, dividing both sides of the product formula by* $\varphi(z|\Gamma)$*, the limit for* $z \to 0$ *yields*

$$\prod_{\xi \neq 0} \varphi(\xi|\Gamma) = \zeta \frac{\sqrt[12]{\Delta(\hat{\Gamma})}}{\sqrt[12]{\Delta(\Gamma)}}.$$

Proof. The assertion of the Proposition is obtained by multiplying the q-expansions of the functions involved. Using the notations

$$Q_w = e^{2\pi i w}, \; Q_w^{\frac{1}{2}} = e^{\pi i w}, \; q = Q_\omega, \; \hat{q} = Q_{\frac{n_2\omega}{n_1}}$$

the q-expansions of $\varphi(w|\Gamma)$ and $\varphi(z|\hat{\Gamma})$ are given by

$$\varphi(z + \xi|\Gamma) = Q_{z+\xi}^{\frac{1}{2}(z_1+\xi_1)}(Q_{z+\xi}^{\frac{1}{2}} - Q_{z+\xi}^{-\frac{1}{2}})q^{\frac{1}{12}} \prod_{n=1}^{\infty}(1 - q^n Q_{z+\xi})(1 - q^n Q_{z+\xi}^{-1}),$$

$$\varphi(z|\hat{\Gamma}) = Q_{n_2z}^{\frac{1}{2}n_1z_1}(Q_{n_2z}^{\frac{1}{2}} - Q_{n_2z}^{-\frac{1}{2}})\hat{q}^{\frac{1}{12}} \prod_{n=1}^{\infty}(1 - \hat{q}^n Q_{n_2z})(1 - \hat{q}^n Q_{n_2z}^{-1}).$$

So the product in the Proposition is of the form

$$\prod_{\xi} e^{-\frac{1}{2}l_\Gamma(z,\xi)}\varphi(z + \xi|\Gamma) = f_1f_2f_3$$

with

$$f_1 = e^{\frac{2\pi i}{2}(\sum_{x,y}(z+\xi)(z_1+\xi_1)-z_1\xi_2+z_2\xi_1)},$$

$$f_2 = \prod_{x,y} q^{\frac{1}{12}}(Q_{z+\xi}^{\frac{1}{2}} - Q_{z+\xi}^{-\frac{1}{2}}),$$

$$f_3 = \prod_{n=1}^{\infty}\prod_{x,y}(1-q^n Q_{z+\xi})(1-q^n Q_{z+\xi}^{-1}).$$

Using the formulas $\sum_{k=1}^{m-1} k = \frac{m(m-1)}{2}$ and $\sum_{k=1}^{m-1} k^2 = \frac{m(m-1)(2m-1)}{6}$ we then obtain

$$f_1 = \zeta_8^{(n_1-1)(n_2-1)} Q_{n_2 z}^{\frac{n_1 z_1}{2}} Q_{n_2 z}^{\frac{n_1-1}{2}} \hat{q}^{\frac{(n_1-1)(2n_1-1)}{12}}, \quad \hat{q} = q^{\frac{n_1}{n_2}}.$$

Further, using the identity $\prod_{y=0}^{n_2-1}(a - b\zeta_{n_2}^y) = a^{n_2} - b^{n_2}$ we can write f_2 in the form

$$f_2 = -\zeta_4^{n_1 n_2 + n_1} Q_{n_2 z}^{-\frac{n_1-1}{2}} \hat{q}^{n_1 n_2 - \frac{n_1(n_1-1)}{4}} \prod_{x=1}^{n_1-1}(1-\hat{q}^x Q_{n_2 z})$$

and in the same way

$$f_3 = \left(\prod_{k=n_1}^{\infty}(1-\hat{q}^k Q_{n_2 z})\right)\left(\prod_{k=1}^{\infty}(1-\hat{q}^k Q_{n_2 z}^{-1})\right).$$

Now, putting together the identities for f_1, f_2, f_3 we can easily derive our assertion. □

Proof of Theorem 1. First we observe that the assertion of the Proposition is also valid for arbitrary lattices $\Gamma \subset \hat{\Gamma}$, arbitrary systems $\{\xi\}$ of representatives and other normalization of the 12-th root of Δ, with possibly another constant ζ. This follows from the homogeneity and the transformation formula of the φ-function:

$$\varphi(\lambda z|\lambda\Gamma) = \varphi(z|\Gamma),$$

$$\varphi(z+\tau|\Gamma) = \psi(\tau)e^{\frac{1}{2}l_\Gamma(\tau,z)}\varphi(z|\Gamma) \quad \text{for } \tau \in \Gamma$$

with

$$\psi(\tau) = \begin{cases} 1, & \text{if} \quad \tau \in 2\Gamma, \\ -1, & \text{if} \quad \tau \in \Gamma \setminus 2\Gamma. \end{cases}$$

Considering the fact that $l_\Gamma(z,\xi)$ is linear in z we obtain from the generalized version of the Proposition just explained:

$$\prod_{\xi\in\mathfrak{qp}^n \bmod \mathfrak{qp}^{n+1}} E_{n+1}(z+\xi) = \frac{\varphi\left(z-\gamma_{n+1}|\,\mathfrak{qp}^n\right)\varphi\left(z+\gamma_{n+1}|\,\mathfrak{qp}^n\right)}{\varphi^2\left(z|\,\mathfrak{qp}^n\right)}.$$

Herein on the right γ_{n+1} can be replaced by γ_n using the transformation law of φ, because $\gamma_{n+1}\equiv\gamma_n \bmod \mathfrak{qp}^n$. This proves the formula of Theorem 1. □

Proof of Theorem 2. By reciprocity law of complex multiplication we know

$$\varphi(\delta|\mathfrak{qp}^n)\in K_{12N(\mathfrak{qp}^n)^2} \quad \text{for } \delta\in\mathfrak{O}_K$$

for every choice of basis in $\mathfrak{qp}^n$. Further, as can be found in [B-Sch], the action of a Frobenius automorphism $\sigma(\lambda)$ of $K_{12N(\mathfrak{qp}^n)^2}$ belonging to an integral principal ideal (λ) of $\mathfrak{O}_K$ prime to $12N(\mathfrak{qp}^n)$ is of the form

$$\varphi(\delta|\mathfrak{qp}^n)^{\sigma(\lambda)} = \epsilon\;\varphi(\delta\lambda|\mathfrak{qp}^n)$$

with a root of unity ϵ independent of δ. This implies

$$E_n(\delta)\in K_{12N(\mathfrak{qp}^n)^2}$$

and

$$E_n(\delta)^{\sigma(\lambda)} = \frac{\varphi\left(\delta\lambda-\gamma_n\lambda|\,\mathfrak{qp}^n\right)\varphi\left(\delta\lambda+\gamma_n\lambda|\,\mathfrak{qp}^n\right)}{\varphi^2\left(\delta\lambda|\,\mathfrak{qp}^n\right)} \quad \text{for } \delta\in\mathfrak{O}_K\setminus\{0\}$$

with λ having the above properties. For $\lambda=1+\tau,\ \tau\in\mathfrak{qp}^n$, the φ-values in the numerator on the right side can be simplified by the transformation law of φ:

$$\begin{aligned}\varphi\left(\delta\lambda\pm\gamma_n\lambda|\,\mathfrak{qp}^n\right) &= \varphi\left(\delta\lambda\pm\gamma_n\pm\gamma_n\tau|\,\mathfrak{qp}^n\right)\\ &= \psi(\tau\gamma_n)e^{\frac{1}{2}l(\delta\lambda\pm\gamma_n,\pm\gamma_n\tau)}\varphi\left(\delta\lambda\pm\gamma_n|\,\mathfrak{qp}^n\right)\end{aligned}$$

with $l=l_{\mathfrak{qp}^n}$. So

$$E_n(\delta)^{\sigma(\lambda)} = e^{l(\gamma_n,\gamma_n\tau)}E_n(\delta\lambda).$$

Herein, using the rule $l(a,bc)=l(a\overline{b},c)$,

$$l(\gamma_n,\gamma_n\tau) = l(\gamma_n\overline{\gamma}_n,\tau)\in 2\pi i\mathbb{Z}$$

because $\gamma_n\overline{\gamma}_n,\tau\in\mathfrak{qp}^n$, whence

$$E_n(\delta)^{\sigma(\lambda)} = E_n(\delta\lambda).$$

Now, considering the fact that E_n is elliptic with respect to $\mathfrak{qp}^n$, it follows

$$E_n(1)^{\sigma(\lambda)} = E_n(1) \quad \text{for } \lambda\equiv 1 \bmod \mathfrak{qp}^n$$

and we can conclude that $E_n(1)$ is in $K_{\mathfrak{qp}^n}$, because

$$\mathrm{Gal}(K_{12N(\mathfrak{qp}^n)^2}/K_{\mathfrak{qp}^n}) = \{\sigma(\lambda)\mid\lambda\equiv 1 \bmod \mathfrak{qp}^n \text{ and prime to } N(\mathfrak{qp})\}.$$

The third assertion of Theorem 2 is obtained similarly: We have

$$\mathrm{Gal}(K_{\mathfrak{q}\mathfrak{p}^{n+1}}/K_{\mathfrak{q}\mathfrak{p}^n}) = \{\sigma(1+\xi) \mid \xi \in \mathfrak{q}\mathfrak{p}^n \bmod \mathfrak{q}\mathfrak{p}^{n+1}\}$$

and

$$E_{n+1}(1)^{\sigma(1+\xi)}) = e^{l(\gamma_{n+1}\overline{\gamma_{n+1}},\xi)}E_{n+1}(1+\xi)$$

with $l = l_{\mathfrak{q}\mathfrak{p}^{n+1}}$, where of course $\sigma(1+\xi)$ denotes the Frobenius automorphism of $K_{\mathfrak{q}\mathfrak{p}^{n+1}}$ belonging to $(1+\xi)$. Again herein $l(\gamma_{n+1}\overline{\gamma_{n+1}},\xi)$ is in $2\pi i\mathbb{Z}$ because $\xi \in \mathfrak{q}\mathfrak{p}^n$ and because $\gamma_{n+1}\overline{\gamma_{n+1}}$ is even in $\mathfrak{q}\mathfrak{p}^{n+1}\overline{\mathfrak{p}}^{n+1}$, whence

$$E_{n+1}(1)^{\sigma(1+\xi)}) = E_{n+1}(1+\xi),$$

which proves the third assertion.

Finally, the second assertion of Theorem 2 follows from the factorisation of the singular φ-values [Sch1]:

$$\varphi(\delta|\mathfrak{q}\mathfrak{p}^n) \sim \begin{cases} 1, & \text{if} \quad o(\delta,\mathfrak{q}\mathfrak{p}^n) \text{ is composite,} \\ \mathfrak{p}^{\frac{1}{\Phi(\mathfrak{p}^r)}}, & \text{if} \quad o(\delta,\mathfrak{q}\mathfrak{p}^n) = \mathfrak{p}^r, \quad r \in \mathbb{N} \end{cases}$$

for every choice of basis in $\mathfrak{q}\mathfrak{p}^n$. Herein $\delta \in K \setminus \{0\}$ and $o(\delta,\mathfrak{q}\mathfrak{p}^n)$ denotes the denominator of the ideal $\frac{\delta}{\mathfrak{q}\mathfrak{p}^n}$. This factorisation implies that the first φ factor in the numerator of the definition of $E_n(1)$ has the factorisation $\mathfrak{p}^{\frac{1}{\Phi(\mathfrak{p}^n)}}$, whereas the other φ values are units. □

Proof of Theorem 3 and 4. The proof of Theorem 3 is completely analogous to the proof of Theorem 1. The first and second assertion of Theorem 4 have already been explained. The third assertion can easily be proved using the same arguments as in the proof of Theorem 2. □

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