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A class–field theoretical calculation

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Résumé. Dans cet article, nous donnons la caractérisation complète des sous-groupes de p–torsion de certains groupes de classes d'idèles associés à des corps de fonctions de charactéristique p. Nous utilisons ce résultat pour répondre à une question qui a surgi dans le contexte de l'approche employée par Tan [6] pour résoudre un important cas particulier d'une généralisation d'une conjecture de Gross [4] sur des valeurs spéciales des fonctions L.

Abstract. In this paper, we give the complete characterization of the p–torsion subgroups of certain idèle–class groups associated to characteristic p function fields. As an application, we answer a question which arose in the context of Tan's approach [6] to an important particular case of a generalization of a conjecture of Gross [4] on special values of L–functions.

1. Notation and motivation

Let p be a prime number. As usual, the term *characteristic p function field* (equivalently, characteristic p global field) refers to a finite extension of a field $\mathbb{F}_p(T)$, where $\mathbb{F}_p$ is the finite field of p elements and T is a variable. Let K/k be a finite abelian extension of characteristic p function fields, of Galois group $\Gamma := \mathrm{Gal}(K/k)$. Let S be a finite, nonempty set of primes in k, containing all the primes which ramify in K/k. We denote by S_K the set of primes in K dividing primes in S. Let $K_S^{\mathrm{ab},p}$ be the maximal pro–p abelian extension of K, unramified outside S_K. Since S_K is Γ–invariant, $K_S^{\mathrm{ab},p}/k$ is a Galois extension. Let $G := \mathrm{Gal}(K_S^{\mathrm{ab},p}/k)$, and $H := \mathrm{Gal}(K_S^{\mathrm{ab},p}/K)$. As usual, the group Γ acts by lift–and–conjugation on H. More precisely, $\gamma * h := \tilde{\gamma} h \tilde{\gamma}^{-1}$, for all $h \in H$ and $\gamma \in \Gamma$, where $\tilde{\gamma}$ denotes any lift of γ to G with respect to the usual epimorphism $G \longrightarrow \Gamma$. This way, since H is an abelian group, H is endowed with a natural $\mathbb{Z}[\Gamma]$–module structure.

In what follows, we denote by I_Γ the augmentation ideal in the group ring $\mathbb{Z}[\Gamma]$, i.e. the ideal of $\mathbb{Z}[\Gamma]$ generated by $\{\gamma - 1 \,|\, \gamma \in \Gamma\}$. Let $[G, G]$ denote

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the commutator subgroup of G, generated by the commutators $[x,y] = xyx^{-1}y^{-1}$ of all the elements $x, y \in G$. Since, by definition $(\gamma - 1) * h = \tilde{\gamma} h \tilde{\gamma}^{-1} h^{-1} = [\tilde{\gamma}, h]$, for all $\gamma \in \Gamma$ and $h \in H$, we have an inclusion

$$I_\Gamma \cdot H \subseteq [G,G]\,.$$

In [6], the following question arises in the context of Tan's approach to an important particular case (the so–called "p–primary part in characteristic p"–case) of a generalization of a conjecture of Gross [4].

Question. Under what conditions do we have an equality

$$[G,G] = I_\Gamma \cdot H\ ?$$

In §§2–3 below, we use class–field theory to show that the answer to the Question above depends on the p–torsion subgroup of a certain idèle–class group associated to K. In §3, we use class–field theory and Galois cohomology to calculate this p–torsion subgroup explicitly. Based on this calculation, in §4 we settle the Question stated above. In §5, we give a sufficient condition for the equality $[G,G] = I_\Gamma \cdot H$ to hold true, for abstract groups G, H, and Γ, not necessarily arising in the number–theoretical context described above.

2. Group theoretical considerations

Throughout this section, H, G, and Γ are arbitrary abstract groups, fitting into a short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1\,,$$

with H and Γ abelian. As above, Γ is viewed as acting on H via lift–and–conjugation, and this action endows H with a $\mathbb{Z}[\Gamma]$–module structure.

Proposition 2.1. *If $H/I_\Gamma \cdot H$ has no torsion, then one has an equality*

$$[G,G] = I_\Gamma \cdot H\,.$$

Proof. We have a short exact sequence of groups

$$1 \longrightarrow H/I_\Gamma \cdot H \longrightarrow G/I_\Gamma \cdot H \longrightarrow \Gamma \longrightarrow 1. \tag{2.1}$$

We will need the following lemma, which was suggested to the author by Tan [7].

Lemma 2.1. *Let $1 \longrightarrow \overline{H} \longrightarrow \overline{G} \xrightarrow{\pi} \overline{\Gamma} \longrightarrow 1$ be an exact sequence of groups, such that*

1. *$\overline{\Gamma}$ is a torsion abelian group.*
2. *$\overline{H}$ is a non–torsion abelian group.*
3. *$\overline{\Gamma}$ acts trivially on $\overline{H}$ (via the usual lift–and–conjugation action).*

Then, $\overline{G}$ is abelian.

Proof of Lemma 2.1. Hypothesis (3) above implies right away that $\overline{H} \subseteq \mathcal{Z}(\overline{G})$, where $\mathcal{Z}(\overline{G})$ denotes the center of $\overline{G}$. Let $a, b \in \overline{G}$. We will show that $[a, b] = 1$. Since $\overline{\Gamma}$ is abelian, $\pi([x, y]) = 1$ and therefore $[x, y] \in \overline{H}$, for all $x, y \in \overline{G}$. This shows that, if we denote by B the subgroup of G generated by b, then we have a function

$$f_a : B \longrightarrow \overline{H}\,,$$

defined by $f_a(x) = [a, x]$, for all $x \in B$. We claim that f_a is a group morphism. Indeed, if $x, y \in B$ we have the equalities

$$\begin{aligned} f_a(x) \cdot f_a(y) &= (axa^{-1}x^{-1})(aya^{-1})y^{-1} = (aya^{-1})(axa^{-1}x^{-1})y^{-1} = \\ &= ayxa^{-1}x^{-1}y^{-1} = f_a(yx) = f_a(xy)\,, \end{aligned}$$

which prove our claim. The second equality above is a consequence of $[a, x] \in \overline{H}$ and $\overline{H} \subseteq \mathcal{Z}(\overline{G})$. The last equality follows from the fact that B is abelian. Since $\overline{H} \subseteq \mathcal{Z}(\overline{G})$, we have $B \cap \overline{H} \subseteq \ker(f_a)$. Therefore, the image $\Im(f_a)$ of f_a is isomorphic to a quotient of $B/B \cap \overline{H}$. Since $B/B \cap \overline{H}$ is isomorphic to a subgroup of $\overline{\Gamma}$, hypothesis (2) shows that $\Im(f_a)$ is a torsion subgroup $\overline{H}$. Hypothesis (1) implies that $\Im(f_a)$ is a torsion subgroup of $\overline{H}$, while hypothesis (2) implies that $\Im(f_a)$ is trivial. Therefore $f_a(b) = [a, b] = 1$. This concludes the proof of Lemma 2.1. □

An alternative proof for Lemma 2.1. In what follows, we give a second, homological in nature and very enlightening proof for Lemma 2.1, suggested to us by the referee. Let $\overline{V} := \overline{H} \otimes_{\mathbb{Z}} \mathbb{Q}$. Hypothesis (1) implies that we have an exact sequence of $\mathbb{Z}[\overline{\Gamma}]$–modules

$$0 \longrightarrow \overline{H} \longrightarrow \overline{V} \longrightarrow \overline{V}/\overline{H} \longrightarrow 0\,,$$

with $\overline{\Gamma}$ acting trivially on each term (see hypothesis (3)). Hypotheses (3) and (1) imply that

$$\mathrm{H}^1(\overline{\Gamma}, \overline{V}) = \mathrm{Hom}(\overline{\Gamma}, \overline{V}) = 0\,, \qquad \mathrm{H}^1(\overline{\Gamma}, \overline{H}) = \mathrm{Hom}(\overline{\Gamma}, \overline{H}) = 0\,.$$

Consequently, the usual coboundary maps in the long–exact sequence of $\overline{\Gamma}$–cohomology groups associated to the last short exact sequence induce a group isomorphism

$$\mathrm{H}^1(\overline{\Gamma}, \overline{V}/\overline{H}) \xrightarrow{\sim} \mathrm{H}^2(\overline{\Gamma}, \overline{H})\,.$$

Let $\mathfrak{h} \in \mathrm{H}^1(\overline{\Gamma}, \overline{V}/\overline{H}) = \mathrm{Hom}(\overline{\Gamma}, \overline{V}/\overline{H})$ correspond to the extension class of $\overline{G}$ in $\mathrm{H}^2(\overline{\Gamma}, \overline{H})$ via this coboundary isomorphism. Then $\overline{G}$ is isomorphic to the pull–back of $\overline{V} \longrightarrow \overline{V}/\overline{H}$ along $\mathfrak{h} : \overline{\Gamma} \longrightarrow \overline{V}/\overline{H}$. Since $\overline{\Gamma}$ and $\overline{V}$ are abelian, so is $\overline{G}$. □

Now, we return to the proof of Proposition 2.1. The exact sequence (2.1) satisfies properties (1), (2), and (3) in Lemma 2.1, and therefore $G/I_\Gamma \cdot H$ is abelian. This shows that

$$[G,G] \subseteq I_\Gamma \cdot H\,.$$

As the reverse inclusion is obviously true, we obtain the desired equality

$$[G,G] = I_\Gamma \cdot H\,.$$

This concludes the proof of Proposition 2.1. □

If A is an (additive) abelian group, $\widehat{A}$ denotes the pro-p completion of A, namely

$$\widehat{A} := \varprojlim A/p^n A\,,$$

where $\varprojlim$ denotes the usual projective limit with respect to the canonical surjections $A/p^{n+1}A \longrightarrow A/p^n A$. Also, $A[p^\infty]$ will denote the subgroup of A consisting of all its p-power order elements. In what follows, we refer to $A[p^\infty]$ as the p–torsion subgroup of A. We have the following.

Lemma 2.2. *Let A be an abelian group. Then, the following hold true.*

(1) *The inclusion $A[p^\infty] \subseteq A$ induces a group isomorphism*

$$\widehat{A[p^\infty]}[p^\infty] \xrightarrow{\sim} \widehat{A}[p^\infty]\,.$$

(2) *If A has a trivial p-torsion subgroup, then $\widehat{A}$ has a trivial p-torsion subgroup.*

Proof. First, we prove (2). Let $\alpha = (\widehat{a}_n)_n \in \widehat{A}$, where $a_n \in A$ and $\widehat{a}_n$ is the class of a_n in $A/p^n A$. By definition, we have $\pi_n(\widehat{a}_n) = \widehat{a_{n-1}}$, where $\pi_n : A/p^n A \longrightarrow A/p^{n-1}A$ is the canonical projection, for all n. Assume that $p \cdot \alpha = 0$ in $\widehat{A}$. This implies that $pa_n \in p^n A$, for all $n \geq 1$. Since A has no p–torsion, this implies that $a_n \in p^{n-1}A$, for all $n \geq 1$, which shows that $\widehat{a_{n-1}} = \widehat{0}$ in $A/p^{n-1}A$, for all $n \geq 2$. This implies that $\alpha = 0$ in $\widehat{A}$.

Next, we prove (1). Let $B := A/A[p^\infty]$. We have an exact sequence of groups.

$$0 \longrightarrow A[p^\infty] \longrightarrow A \longrightarrow B \longrightarrow 0\,.$$

We have an obvious equality $p^n A \cap A[p^\infty] = p^n A[p^\infty]$, for all n. This implies that, for every n, we obtain an exact sequence of groups.

$$0 \longrightarrow A[p^\infty]/p^n A[p^\infty] \longrightarrow A/p^n A \longrightarrow B/p^n B \longrightarrow 0\,.$$

Since the canonical projections are surjective, the projective limit of these exact sequences with respect to the canonical projections leads to the exact sequence

$$0 \longrightarrow \widehat{A[p^\infty]} \longrightarrow \widehat{A} \longrightarrow \widehat{B} \longrightarrow 0\,.$$

By definition, B has a trivial p–torsion subgroup. Statement (2) in the Lemma implies that $\widehat{B}$ has a trivial p–torsion subgroup as well. Consequently, the last exact sequence leads to the desired isomorphism $\widehat{A[p^\infty]}[p^\infty] \xrightarrow{\sim} \widehat{A}[p^\infty]$. □

3. Class–field theoretical considerations

Now, we return to the notations and definitions in §1. Proposition 2.1 above shows that, since H is a pro–p group, in order to settle the Question stated above, it is important to characterize the p–torsion subgroup of $H/I_\Gamma \cdot H$. In particular, if one shows that $H/I_\Gamma \cdot H$ has no p–torsion then Proposition 2.1 above leads to the desired equality $[G,G] = I_\Gamma \cdot H$. In the current section, we use class–field theory to identify $H/I_\Gamma H$ with the pro–p completion of an idéle–class group of K and fully describe its torsion subgroup. As usual, let C_K denote the idèle-class group of K, i.e.

$$C_K := J_K/K^\times ,$$

where J_K is the group of idèles of K. For the definitions and properties of J_K and C_K, as well as class–field theory in idèlic and Galois–cohomological language, the reader is referred to the classical texts [1] and [3].

If restricted to the context of characteristic p global fields, global class–field theory (see [1], Chpt. VIII, §3, or [3], Chpt. VII, §5.5) shows that the global Artin map induces a topological group isomorphism between the profinite completion of C_K and the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian extension K^{ab} of K. For every prime w of K, U_w denotes the group of w–local units of the completion K_w of K with respect to w. Let $\prod U_w$ denote the (closed) subgroup of J_K, consisting of all those idèles with local component 1 at all $w \in S_K$ and local component belonging to U_w, for all $w \notin S_K$. Since $S \neq \emptyset$, we have $K^\times \cap \prod U_w = \{1\}$, and therefore we can view $\prod U_w$ as a subgroup of C_K, by identifying it with its image via the injective group morphism $\prod U_w \longrightarrow C_K$.

By global class–field theory and the definition of H, the global Artin map induces a topological group isomorphism of Γ–modules between the pro-p completion of $C_K/\prod U_w$ and H. Consequently, the global Artin map induces a topological group isomorphism between the pro-p completion of the quotient $C_K/(I_\Gamma \cdot C_K)\prod U_w$ and the group $H/I_\Gamma \cdot H$,

$$C_K/(I_\Gamma \widehat{\cdot C_K)} \prod U_w \xrightarrow{\sim} H/I_\Gamma \cdot H . \tag{3.1}$$

Our next goal is to prove the following theorem which gives a full description of the p–torsion of the group $C_K/(I_\Gamma \cdot C_K)\prod U_w$.

Theorem 3.1. *The p–torsion subgroup of $C_K/(I_\Gamma \cdot C_K)\cdot \prod U_w$ is isomorphic to $\wedge^2\Gamma^{(p)}$, where $\Gamma^{(p)}$ is the p-Sylow subgroup of Γ.*

Before proceeding to proving Theorem 3.1, we need to make several Galois–cohomological considerations. For a Γ–module A and an $i \in \mathbb{Z}$, we denote by $\widehat{\mathrm{H}}^i(\Gamma, A)$ the i–th Tate cohomology group of Γ with coefficients in A (see [3], Chpt. IV, §6 for the definitions). Also, $A[N_\Gamma]$ denotes the subgroup of A annihilated by the norm element $N_\Gamma \in \mathbb{Z}[\Gamma]$, where $N_\Gamma := \sum_{\gamma\in\Gamma}\gamma$. Obviously, $I_\Gamma \cdot A \subseteq A[N_\Gamma]$. By definition, $\widehat{H}^{-1}(\Gamma, A) = A[N_\Gamma]/I_\Gamma \cdot A$. Also, $\widehat{H}^0(\Gamma, A) = A^\Gamma/N_\Gamma \cdot A$, where A^Γ is the maximal subgroup of A fixed by Γ. We have an exact sequence of abelian groups.

(3.2)

$$1 \longrightarrow \frac{C_K[N_\Gamma]\cdot\prod U_w}{(I_\Gamma\cdot C_K)\cdot\prod U_w} \longrightarrow \frac{C_K}{(I_\Gamma\cdot C_K)\prod U_w} \longrightarrow \frac{C_K}{C_K[N_\Gamma]\cdot\prod U_w} \longrightarrow 1\,.$$

However, since K/k is unramified away from S, Shapiro's Lemma ([3], Prop. 2, p. 99) combined with the cohomological triviality of groups of units in unramified Galois extensions of local fields (see [3], Chpt. VII, §7) implies that

$$\widehat{\mathrm{H}}^i(\Gamma, \prod U_w) = \prod_{v\notin S}\widehat{\mathrm{H}}^i(\Gamma_v, U_{w(v)}) = 0\,, \quad \text{for all } i\in\mathbb{Z}\,, \tag{3.3}$$

where Γ_v denotes the decomposition group of v in K/k and $w(v)$ is a fixed prime in K dividing v, for all $v \notin S$. Equality (3.3) for $i = 1$ gives the following.

$$I_\Gamma\cdot\prod U_w = (\prod U_w)[N_\Gamma]\,.$$

This implies immediately that the left–most nonzero term of the exact sequence (3.2) is in fact isomorphic to $C_K[N_\Gamma]/I_\Gamma\cdot C_K = \widehat{H}^{-1}(\Gamma, C_K)$. Therefore, we obtain an exact sequence of groups.

(3.4)

$$1 \longrightarrow \widehat{\mathrm{H}}^{-1}(\Gamma, C_K) \longrightarrow C_K/(I_\Gamma\cdot C_K)\cdot\prod U_w \longrightarrow C_K/C_K[N_\Gamma]\cdot\prod U_w \longrightarrow 1\,.$$

Consequently, proving Theorem 3.1 amounts to studying the p–torsion subgroups of the two end–terms of exact sequence (3.4). This is accomplished by the next two lemmas.

Lemma 3.1. *The p-torsion subgroup of* $\widehat{\mathrm{H}}^{-1}(\Gamma, C_K)$ *is isomorphic to* $\wedge^2\Gamma^{(p)}$*.*

Proof. Global class-field theory (see [3], Chpt. VII, §11.3) gives an isomorphism

$$\widehat{\mathrm{H}}^{-1}(\Gamma, C_K) \xrightarrow{\sim} \widehat{\mathrm{H}}^{-3}(\Gamma, \mathbb{Z})\,.$$

On the other hand, one has an equality

$$\widehat{\mathrm{H}}^{-3}(\Gamma, \mathbb{Z}) = \mathrm{H}_2(\Gamma, \mathbb{Z})\,.$$

Theorem 6.4 (iii) in [2] gives an isomorphism

$$\mathrm{H}_2(\Gamma, \mathbb{Z}) \xrightarrow{\sim} \wedge^2\Gamma\,,$$

where the exterior product is taken over $\mathbb{Z}$. Taking p–torsion of both sides in the last equality concludes the proof of Lemma 3.1. □

In order to describe the p–torsion subgroup of the right–most nonzero term of exact sequence (3.4), we need the following result, proved by Kisilevsky in [5] and shown to imply the classical Leopoldt Conjecture for characteristic p function fields.

Theorem 3.2 (Kisilevsky)**.** *Let k be an arbitrary characteristic p function field. Let v a prime in k and k_v the completion of k with respect to v. If $x \in k$ is the p–power of an element in k_v, then x is the p–power of an element in k.*

Proof. See [5] □

Lemma 3.2. *The p–torsion subgroup of $C_K/C_K[N_\Gamma] \cdot \prod U_w$ is trivial.*

Proof. Let us assume that $C_K/C_K[N_\Gamma] \cdot \prod U_w$ has p-torsion. Let $j \in J_K$ such that its class $\widehat{j} \in C_K$ gives rise to an element of order p in the quotient $C_K/C_K[N_\Gamma] \cdot \prod U_w$. This means that there exist $\rho \in J_K$, $u \in \prod U_w$, $x \in K^\times$, and $y \in k^\times$, such that

(1) $j^p = \rho \cdot u \cdot x$ in J_K.
(2) $\mathrm{N}_\Gamma(\rho) = y$.

By taking norms in (1) above, we obtain

$$N_\Gamma(j)^p = N_\Gamma(u) \cdot y N_\Gamma(x)\,.$$

However, since $S \neq \emptyset$, this implies right away that $y \cdot N_\Gamma(x)$ is a p-power locally, at primes in S. Theorem 3.2 above implies that there exists $z \in k^\times$, such that $y \cdot N_\Gamma(x) = z^p$. This shows that $N_\Gamma(j)^p = N_\Gamma(u) \cdot z^p$, and therefore $N_\Gamma(u) = \theta^p$, for some $\theta \in \prod U_v$, where the product is taken over all primes v of k which are not in S, and U_v denotes the unit group of the completion k_v of k at v. However, equality (3.3) for $i = 0$ implies that $\widehat{H}^0(\Gamma, \prod U_w) = 0$ and therefore

$$\prod U_v = (\prod U_w)^\Gamma = N_\Gamma(\prod U_w)\,.$$

This shows that there exists $u' \in \prod U_w$ such that $\theta = N_\Gamma(u')$. This obviously implies that

$$N_\Gamma(j) = N_\Gamma(u') \cdot z\,, \text{ with } u' \in \prod U_w \text{ and } z \in k^\times\,.$$

The last equality shows that $\widehat{j/u'} \in C_K[N_\Gamma]$ and therefore $\widehat{j} \in C_K[N_\Gamma] \cdot \prod U_w$. This shows that the element $\widehat{j} \in C_K$ gives rise to the trivial class in the quotient $C_K/C_K[N_\Gamma] \cdot \prod U_w$. □

Proof of Theorem 3.1. This is a direct consequence of exact sequence (3.4), Lemma 3.1 and Lemma 3.2 above. □

The next corollary fully describes the torsion subgroup $T\,(H/I_\Gamma\cdot H)$ of $H/I_\Gamma\cdot H$.

Corollary 3.1. (1) *One has an isomorphism of groups*

$$\mathrm{T}\,(H/I_\Gamma\cdot H)\xrightarrow{\sim}\wedge^2\Gamma^{(p)}\,.$$

(2) *$H/I_\Gamma\cdot H$ has no torsion if and only if $\Gamma^{(p)}$ is cyclic.*

Proof. Let us first remark that since H is a pro–p group, its torsion subgroup and p–torsion subgroup are identical. With this in mind, statement (1) is a direct consequence of isomorphism (2), Theorem 3.1, Lemma 2.2 applied to the group $A:=C_K/(I_\Gamma\cdot C_K)\cdot\prod U_w$, and the finiteness of Γ. Statement (2) is a direct consequence of (1) □

4. The answer to the Question stated in §1

We work under the hypotheses and with the notations of §§1 and 3. The following theorem provides a full answer to the Question raised in §1.

Theorem 4.1. *The following statements are equivalent*

(1) $[G,G]=I_\Gamma\cdot H$.

(2) *The p–Sylow subgroup $\Gamma^{(p)}$ of Γ is cyclic.*

Proof. The implication (2) $\Longrightarrow$ (1) is a direct consequence of Corollary 3.1 (2) and Proposition 2.1. Now, let us assume that $[G,G]=I_\Gamma\cdot H$. As in §1, let $k_S^{\mathrm{ab},p}$ be the maximal pro–p abelian extension of k, unramified outside of S. Let L' be the maximal subfield of $K_S^{\mathrm{ab},p}$ fixed by $[G,G]$. Then, under the present hypothesis, we have

$$\mathrm{Gal}(L'/K)\xrightarrow{\sim}H/[G,G]=H/I_\Gamma\cdot H\,. \tag{4.1}$$

By definition, L' is the maximal subfield of $K_S^{\mathrm{ab},p}$ which is an abelian extension of k. Obviously, we have an inclusion $k_S^{\mathrm{ab},p}\subseteq L'$. Also, if we denote by K' the maximal subfield of K fixed by $\Gamma^{(p)}$, then K'/k and $k_S^{\mathrm{ab},p}/k$ are linearly disjoint extensions of k and their compositum $K'\cdot k_S^{\mathrm{ab},p}$ inside L equals L'. Consequently, we have a group isomorphism

$$\mathrm{Gal}(L'/k)\xrightarrow{\sim}\Gamma/\Gamma^{(p)}\times\mathrm{Gal}(k_S^{\mathrm{ab},p}/k)\,.$$

The isomorphism above combined with (4.1) and the fact that H is a pro–p group shows that $H/I_\Gamma\cdot H$ is isomorphic to a subgroup of $\mathrm{Gal}(k_S^{\mathrm{ab},p}/k)$ (via the usual map restricting automorphisms of $K_S^{\mathrm{ab},p}$ to automorphisms of $k_S^{\mathrm{ab},p}$). This shows that if we prove that $\mathrm{Gal}(k_S^{\mathrm{ab},p}/k)$ has a trivial torsion subgroup, then $H/I_\Gamma\cdot H$ has a trivial torsion subgroup which, via Corollary 3.1(2) implies that, indeed, $\Gamma^{(p)}$ is cyclic. However, the global Artin map for k induces an isomorphism

$$J_k/\widehat{k^\times \cdot \prod U_v} \xrightarrow{\sim} \mathrm{Gal}(k_S^{\mathrm{ab},p}/k)\,.$$

Therefore, Lemma 2.2 implies that it suffices to show that the group $J_k/k^\times \cdot \prod U_v$ has no p–torsion. This follows immediately by applying once again Kisilevsky's Theorem 3.2. Indeed, assume that $j \in J_k$ has the property that $j^p = x \cdot u$, where $x \in k^\times$ and $u \in \prod U_v$. Since S is non–empty, this implies that x is a p–power in $k_v^\times$, for all $v \in S$. Theorem 3.2 implies that $x = y^p$, for some $y \in k^\times$. This implies that $u = \theta^p$, with $\theta = j/x$, $\theta \in \prod U_v$. Consequently, $j = y \cdot \theta \in k^\times \cdot \prod U_v$, and therefore the class $\widehat{j}$ of j in the quotient $J_k/k^\times \cdot \prod U_v$ is trivial. This concludes the proof of the implication (1) $\Longrightarrow$ (2). □

5. More group theory (final thoughts)

We conclude with a short purely group–theoretical section providing a sufficient condition for the equality $[G,G] = I_\Gamma \cdot H$ to hold true, where G, H, and Γ are abstract groups.

Lemma 5.1. *Let us assume that we have an exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1\,,$$

with H and Γ abelian and H normal in G. Let us assume that π has a set–theoretic section $s : \Gamma \longrightarrow G$, such that $s(x) \cdot s(y) = s(y) \cdot s(x)$, for all x, y in Γ. Then, we have an equality $[G\,,\,G] = I_\Gamma \cdot H$.

Proof. It suffices to show that $[G,G] \subseteq I_\Gamma \cdot H$. Let α, β be two elements in G. Let $x, y \in \Gamma$, and $a, b \in H$, such that $\alpha = s(x)a$ and $\beta = s(y)b$. Since $s(y)^{-1}s(x)^{-1} = s(x)^{-1}s(y)^{-1}$, we have

$$\begin{aligned}[\alpha,\beta] =& s(x)as(y)ba^{-1}s(x)^{-1}b^{-1}s(y)^{-1} = \{s(x)as(x)^{-1}\}\cdot \\ &\cdot \{s(x)s(y)bs(y)^{-1}s(x)^{-1}\} \cdot \{s(y)s(x)a^{-1}s(x)^{-1}s(y)^{-1}\}\cdot \\ &\cdot \{s(y)b^{-1}s(y)^{-1}\}\,.\end{aligned}$$

Let us denote by m, n, p, and q respectively the elements appearing inside braces to the right of the second equality above. Since H is normal in G, we have $m, n, p, q \in H$. Since H is assumed to be abelian and $a, b \in H$, the equalities above give

$$\begin{aligned}[\alpha,\beta] &= \{ma^{-1}\} \cdot \{nb^{-1}\} \cdot \{pa\} \cdot \{qb\} = \\ &= [s(x),a] \cdot [s(x)s(y),b] \cdot [s(y)s(x),a^{-1}] \cdot [s(y),b^{-1}]\,.\end{aligned}$$

Let us now recall that s is a section of π. Therefore, there exists an element $\mu \in H$, such that $s(x)s(y) = s(xy)\mu$. Since H is abelian, this implies that

$[s(x)s(y), b] = [s(xy)\mu, b] = [s(xy), b]$ and $[s(y)s(x), a^{-1}] = [s(yx), a^{-1}]$. We obtain

$$\begin{aligned}[\alpha, \beta] &= [s(x), a] \cdot [s(xy), b] \cdot [s(yx), a^{-1}] \cdot [s(y), b] \\ &= \{(x-1) * a\} \cdot \{(xy-1) * b\} \cdot \{(yx-1) * a^{-1}\} \cdot \{(y-1) * b^{-1}\}.\end{aligned}$$

This shows that $[\alpha, \beta] \in I_\Gamma \cdot H$, which concludes the proof of Lemma 5.1. $\square$

Corollary 5.1. *Assume that we have an exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1,$$

with H and Γ abelian and H normal in G. Assume that either (1) or (2) below hold.

(1) Γ *is cyclic.*

(2) *The exact sequence above is split.*

Then, we have an equality $[G, G] = I_\Gamma \cdot H$.

Proof. It is very easy to check that if either one of the conditions above is satisfied, one can construct a set–theoretic section s for π, such that $s(x)s(y) = s(y)s(x)$, for all $x, y \in \Gamma$. (Under condition (2), one can actually find a group morphism section s. Such a section satisfies the commutativity property automatically, since Γ is assumed to be abelian.) The corollary is then a consequence of Lemma 5.1. $\square$

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