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Jet spaces as nonrigid Carnot groups

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Abstract. We define a product on the jet spaces $J^k(\mathbb{R}^m, \mathbb{R}^n)$ which makes them Carnot groups. The Carnot group contact structure coincides with the classical contact structure in the Lie-Bäcklund setting. Therefore, by prolongation, they are nonrigid Carnot groups, meaning that the space of contact maps is infinite dimensional. We also show that strata dimensions are not rigidity invariants. This is demonstrated by constructing two distinct Carnot groups with strata dimensions $(3, 2, 1)$ but with opposite rigidity.
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1. Introduction

A Carnot group G is a connected, simply connected, stratified nilpotent Lie group, equipped with a left-invariant sub-Riemannian metric, defined on the left-invariant sub-bundle of the tangent bundle corresponding to the first level of the stratification. The sub-bundle is called the horizontal bundle and the metric is called the Carnot–Carathéodory metric. Diffeomorphisms which preserve the horizontal bundle are called contact maps and G is said to be rigid when the space of contact maps is finite dimensional. Quasiconformal maps are defined with respect to the Carnot–Carathéodory metric and the definition implies they must also be contact maps in some weak sense. Carnot groups are naturally equipped with a family of dilations which, together with left translations, provide trivial examples of contact maps which in the rigid cases tend to be the only examples.

Rigidity arises from the fact that contact maps are P-differentiable, a concept due to Pansu [12]. The cases studied in the literature, e.g., [4], [12], [14], suggest that rigidity is common and according to [6], this is cause for concern. The euclidean spaces, the real and complex Heisenberg groups and the model filiform groups are the established examples of nonrigid groups, see [13], [7], [8], [16] and [17].

Recently, Tyson [18] asked the question: Are there Carnot groups of step 3 or higher which are nonrigid and support quasiconformal maps which are not conformal? The answer is yes, the simplest examples being the model filiform

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groups. In [4], it was observed that the four dimensional real model filiform group is not rigid and arises as the nilpotent part of the Iwasawa decomposition of $\mathrm{Sp}(2,\mathbb{R})$. The analogous part of $\mathrm{Sp}(2n,\mathbb{R})$, $n>1$, is rigid. In [20], all model filiform groups are shown to be nonrigid. The Heisenberg and model filiform groups are related by the fact that they are the generic jet spaces $J^1(\mathbb{R}^m,\mathbb{R}^n)$ and $J^k(\mathbb{R},\mathbb{R})$. This suggests that all generic jet spaces might be nonrigid Carnot groups. This is in fact the case and is the subject of this paper.

The generic jet spaces $J^k(\mathbb{R}^m,\mathbb{R}^n)$ are fundamental to the geometric study of partial differential equations and arise in the literature as examples of sub-Riemannian or Carnot–Carathéodory manifolds, of which in some sense Carnot groups are the ideal models. These manifolds are equipped with a distribution given by a frame of vector fields which generate the tangent space at each point by Lie brackets. Again, a transformation of the manifold is a contact transformation if it preserves the distribution and the question of rigidity applies. There is a classical rigidity theorem of Bäcklund which shows that jet spaces are nonrigid, however the contact condition is somewhat restrictive.

In this paper we construct an explicit multiplication on the generic jet spaces so that they become Carnot groups. The multiplication gives rise to a contact structure which coincides exactly with the jet space contact structure thus providing a large family of nonrigid Carnot groups supporting a nontrivial quasiconformal mapping theory. The difficulty in determining the multiplication arises from the complexity of the Baker–Campbell–Hausdorff formula.

2. Carnot Groups

A nilpotent Lie algebra $\mathfrak{g}$ is said to admit an n-step *stratification* if $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_n$, such that $\mathfrak{g}_{j+1}=[\mathfrak{g}_1,\mathfrak{g}_j]$, where $j=1,\ldots,n-1$, and $\mathfrak{g}_n$ is contained in the center $Z(\mathfrak{g})$. A *Carnot group* is a connected, simply connected nilpotent Lie group G, with stratified Lie algebra equipped with an inner product such that $\mathfrak{g}_i\perp\mathfrak{g}_j$ when $i\neq j$.

For simply connected nilpotent Lie groups, the exponential map $\exp:\mathfrak{g}\to G$ is a diffeomorphism which becomes an isomorphism $(\mathfrak{g},*)\to G$ when we define

$$X*Y=\exp^{-1}(\exp(X)\exp(Y)).$$

The Baker–Campbell–Hausdorff formula is the explicit expression

$$X*Y=\sum_{n>0}\frac{(-1)^{n+1}}{n}\sum_{\substack{0<p_i+q_i\\1\le i\le n}}C_{p,q}^{-1}(\mathrm{ad}X)^{p_1}(\mathrm{ad}Y)^{q_1}\ldots(\mathrm{ad}X)^{p_n}(\mathrm{ad}Y)^{q_n-1}Y$$

where $(\mathrm{ad}X)Y=[X,Y]$, $C_{p,q}=p_1!q_1!\ldots p_n!q_n!\sum_{i=1}^n p_i+q_i$ and the last term is $(\mathrm{ad}X)^{p_n-1}X$ when $q_n=0$. The expansion to order 4 takes the form

$$\begin{aligned}X*Y=X+Y+\frac{1}{2}[X,Y]&+\frac{1}{12}\left([X,[X,Y]]+[Y,[Y,X]]\right)\\&+\frac{1}{48}\left([Y,[X,[Y,X]]]+[X,[Y,[X,Y]]]\right)+\ldots.\end{aligned}$$

Choosing a basis for $\mathfrak{g}$ identifies $(\mathfrak{g},*)$ with $\mathbb{R}^{\dim\mathfrak{g}}$ and $X*Y$ becomes polynomial of degree at most $n-1$. A coordinate system of this type is said to be *normal of the first kind*.

In a similar fashion we obtain *normal coordinates of the second kind*. Given a basis $\{e_j\}_{j=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$, the map $\Phi : \mathfrak{g} \to G$ given by

$$X = \sum_j x_j e_j \xrightarrow{\Phi} \prod_j \exp(x_j e_j)$$

is a diffeomorphism [19, p. 86], which becomes an isomorphism $(\mathfrak{g}, \odot) \to G$ when we define

$$X \odot Y = \Phi^{-1}(\Phi(X)\Phi(Y)).$$

As before, $X \odot Y$ becomes polynomial of degree at most $n-1$.

Left translation, denoted $\tau_X Y$, is the analogue of translation in euclidean spaces. Specifically $\tau_X Y = X * Y$ in coordinates of the first kind and $\tau_X Y = X \odot Y$ in coordinates of the second kind. An important feature of Carnot groups is an analogue of dilation. For $t > 0$, the *dilation* $\delta_t : \mathfrak{g} \to \mathfrak{g}$ is given by $\delta_t(X) = \sum_{j=1}^n t^j X_j$ where $X = \sum_{j=1}^n X_j$ with $X_j \in \mathfrak{g}_j$, which defines dilation on G via the coordinate systems.

The sub-bundle of the tangent bundle given by left translation of $\mathfrak{g}_1$ is called the horizontal space or contact structure, and a contact transformation is a transformation which preserves the contact structure pointwise. Left translations and dilations are contact transformations.

3. Jet Spaces

3.1. Introduction.. In this section we establish the standard apparatus of jet spaces, see for example [3], [10], [11], [16] and [17].

A function $\mathsf{f} : \mathbb{R}^m \to \mathbb{R}$ has $d(m,k) = \binom{m+k-1}{k}$ distinct k-th order partial derivatives

$$\partial_I \mathsf{f}(x_0) = \frac{\partial^k \mathsf{f}}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}(x_0)$$

where the k-index, $I = (i_1, \dots, i_m)$ satisfies $|I| = i_1 + \dots + i_m = k$. We denote the set of k-indexes by $I(k)$ and let

$$\tilde{I}(k) = I(0) \cup \dots \cup I(k).$$

For $I \in \tilde{I}(k)$ and $t \in \mathbb{R}^m$ we define

$$I! = i_1! i_2! \dots i_m! \quad \text{and} \quad t^I = (t^1)^{i_1}(t^2)^{i_2} \dots (t^m)^{i_m},$$

moreover the k-th order Taylor polynomial of f at x_0 is given by

$$T_{x_0}^k(\mathsf{f})(t) = \sum_{I \in \tilde{I}(k)} \partial_I \mathsf{f}(x_0) \frac{(t-x_0)^I}{I!}.$$

If $D \subseteq \mathbb{R}^m$ is open and $p \in D$, then two functions $\mathsf{f}_1, \mathsf{f}_2 \in C^k(D, \mathbb{R})$ are defined to be equivalent at x_0, denoted $\mathsf{f}_1 \sim_{x_0} \mathsf{f}_2$, if and only if $T_{x_0}^k(\mathsf{f}_1) = T_{x_0}^k(\mathsf{f}_2)$. The k-jet space over D is given by

$$J^k(D, \mathbb{R}) = \cup_{x_0 \in D} C^k(D, \mathbb{R}) / \sim_{x_0} \tag{1}$$

where elements are denoted $j^k_{x_0}(\mathsf{f})$. It comes equipped with the following projections

$$x : J^k(D,\mathbb{R}) \to D \text{ and } \pi^k_j : J^k(D,\mathbb{R}) \to J^{k-j}(D,\mathbb{R}), \quad j = 1,\ldots,k,$$

where

$$x(j^k_{x_0}(\mathsf{f})) = x_0 \quad \text{and} \quad \pi^k_j(j^k_{x_0}(\mathsf{f})) = j^{k-j}_{x_0}(\mathsf{f}).$$

Global coordinates are given by $\psi^{(k)} = (x, u^{(k)})$ where

$$u_I(j^k_{x_0}(\mathsf{f})) = \partial_I \mathsf{f}(x_0), \quad I \in \tilde{I}(k),$$

and

$$u^{(k)} = \{u_I \mid I \in \tilde{I}(k)\}.$$

It follows that

$$J^k(D,\mathbb{R}) \equiv D \times \mathbb{R}^{d(m,0)} \times \mathbb{R}^{d(m,1)} \times \cdots \times \mathbb{R}^{d(m,k)}.$$

If $\mathsf{f} = (\mathsf{f}^1,\ldots,\mathsf{f}^n)$ is a map $\mathsf{f} : D \to \mathbb{R}^n$ then we apply the jet apparatus to the coordinate functions f^ℓ. Thus global coordinates are denoted by $\psi^{(k)} = (x, u^{(k)})$, where

$$x(j^k_{x_0}(\mathsf{f})) = x_0 \quad \text{and} \quad u^\ell_I(j^k_{x_0}(\mathsf{f})) = \partial_I \mathsf{f}^\ell(x), \quad I \in \tilde{I}(k), \quad \ell = 1,\ldots,n,$$

and

$$u^{(k)} = \{u^\ell_I \mid I \in \tilde{I}(k),\ \ell = 1,\ldots,n\}.$$

It follows that

$$J^k(D,\mathbb{R}^n) \equiv D \times \mathbb{R}^{nd(m,0)} \times \mathbb{R}^{nd(m,1)} \times \cdots \times \mathbb{R}^{nd(m,k)}.$$

When making comparisons between jet spaces of different orders, we add the superscript (t) to coordinate expressions on $J^t(\mathbb{R}^m,\mathbb{R}^n)$. In particular we replace x by $x^{(t)}$ and we use

$$u^{(t)} = \{u^{(t),\ell}_I \mid I \in \tilde{I}(t),\ \ell = 1,\ldots,n\}.$$

This notation expresses the compatibility of the coordinates with the projections π^t_s, that is:

$$x^{(t)} = x^{(t-s)} \circ \pi^t_s \quad \text{and} \quad u^{(t),\ell}_J = u^{(t-s),\ell}_J \circ \pi^t_s, \quad \text{when} \quad |J| \leq t-s. \tag{2}$$

We also use the notation

$$\bar{\pi}^t_s = \psi^{(t-s)} \circ \pi^t_s \circ (\psi^{(t)})^{-1}.$$

3.2. Contact structure.. The k-jet of a map $\mathsf{f} \in C^k(D,\mathbb{R}^n)$ is the section $x_0 \to j^k_{x_0}(\mathsf{f})$ of the bundle $x : J^k(D,\mathbb{R}^n) \to D$. A contact form θ on $J^k(D,\mathbb{R})$ is a one form satisfying $s^*\theta = 0$ for all k-jets s. By the chain rule, the contact forms are framed by the set

$$\Big\{\omega^\ell_I = du^\ell_I - \sum_{j=1}^m u^\ell_{I+e_j} dx^j \mid I \in \tilde{I}(k-1)\ ,\ \ell = 1,\ldots,n\Big\}, \tag{3}$$

and, see [5], a section s of $x : J^k(D, \mathbb{R}^n) \to D$ is a k-jet if and only if $s^*\omega_I^\ell = 0$ for all $I \in \tilde{I}(k-1)$ and $\ell = 1, \dots, n$.

The horizontal tangent bundle $\mathcal{H}^k$ is defined pointwise by

$$\mathcal{H}_p^k = \left\{ v \in T_p J^k(D, \mathbb{R}^n) \mid \omega_I^\ell(v) = 0, \quad I \in \tilde{I}(k-1), \quad \ell = 1, \dots, n \right\}.$$

In coordinates,

$$v = \sum_{j=1}^{m} dx^j(v) X_j^{(k)} + \sum_{\ell=1}^{n} \sum_{I \in I(k)} du_I^\ell(v) \frac{\partial}{\partial u_I^\ell},$$

where

$$X_j^{(k)} = \frac{\partial}{\partial x^j} + \sum_{\ell=1}^{n} \sum_{I \in \tilde{I}(k-1)} u_{I+e_j}^\ell \frac{\partial}{\partial u_I^\ell}, \quad j = 1, \dots, m,$$

and it follows that

$$\mathcal{H}^k = \operatorname{span}\left\{ X_j^{(k)} \mid j = 1, \dots, m \right\} \oplus \operatorname{span}\left\{ \frac{\partial}{\partial u_I^\ell} \mid I \in I(k)\ ,\ \ell = 1, \dots, n \right\}.$$

The nontrivial commutators are

$$\left[\frac{\partial}{\partial u_{I+e_j}^\ell}, X_j^{(k)} \right] = \frac{\partial}{\partial u_I^\ell}, \quad I \in \tilde{I}(k-1),\ \ell = 1, \dots, n.$$

If $L_0 = \mathcal{H}^k$ and

$$L_j = \operatorname{span}\{ \frac{\partial}{\partial u_I^\ell} \mid I \in I(k-j),\ \ell = 1, \dots, n \},$$

where $j \geq 1$, then $L_j = [L_0, L_{j-1}]$, where $j = 1, \dots, k$. It follows that

$$\mathfrak{X}^k = L_0 \oplus \cdots \oplus L_k$$

is a $(k+1)$-step stratified nilpotent Lie algebra of vector fields which span $TJ^k(D, \mathbb{R}^n)$ pointwise.

Corresponding to the abstract Lie algebra defined by $\mathfrak{X}^k$, there is a Carnot group $G^{(k)}(m,n)$, unique up to isomorphism, constructed via the Baker–Campbell–Hausdorff formula. As is shown later, in the case $D = \mathbb{R}^m$, we can explicitly determine a multiplication $\odot$ on $J^k(\mathbb{R}^m, \mathbb{R}^n)$ such that $(J^k(\mathbb{R}^m, \mathbb{R}^n), \odot)$ is a Carnot group isomorphic with $G^{(k)}(m,n)$ and the group induced contact structure agrees with the jet contact structure.

3.3. Contact Transformations.. A diffeomorphism f of some domain $D \subseteq J^k(\mathbb{R}^m, \mathbb{R}^n)$ is called a contact transformation if $f_* \mathcal{H}_p^k = \mathcal{H}_{f(p)}^k$. Equivalently, f is a contact transformation if it preserves contact forms, i.e., if θ is a contact form then $f^*\theta$ is a contact form.

Let $v \in \mathcal{H}_p^k$ and

$$\psi^{(k)} \circ f \circ (\psi^{(k)})^{-1}(x, u^{(k)}) = (\xi(x, u^{(k)}), \eta^{(k)}(x, u^{(k)})),$$

where

$$\eta^{(k)}(x, u^{(k)}) = \left\{ \eta^\ell_J(x, u^{(k)}) \mid J \in \tilde{I}(k),\ \ell = 1, \dots, n \right\}.$$

Then

$$dx^j(f_* v) = d\xi^j(\psi^{(k)}_* v) = \sum_i (X^{(k)}_i \xi^j) dx^i(v) + \sum_q \sum_{I \in I(k)} \frac{\partial \xi^j}{\partial u^q_I} du^q_I(v)$$

and

$$du^\ell_J(f_* v) = d\eta^\ell_J(\psi^{(k)}_* v) = \sum_i (X^{(k)}_i \eta^\ell_J) dx^i(v) + \sum_q \sum_{I \in I(k)} \frac{\partial \eta^\ell_J}{\partial u^q_I} du^q_I(v).$$

If $f_* v \in \mathcal{H}^k_{f(p)}$ then $du^\ell_J(f_* v) = \sum_j (u^\ell_{J+e_j} \circ f(p)) dx^j(f_* v)$, hence a contact diffeomorphism satisfies the contact conditions:

$$X^{(k)}_i \eta^\ell_J = \sum_j \eta^\ell_{J+e_j}(X^{(k)}_i \xi^j), \quad J \in \tilde{I}(k-1), \quad \ell = 1, \dots, n, \tag{4}$$

$$\frac{\partial \eta^\ell_J}{\partial u^q_I} = \sum_j \eta^\ell_{J+e_j} \frac{\partial \xi^j}{\partial u^q_I}, \quad J \in \tilde{I}(k-1), \quad I \in I(k), \quad \ell = 1, \dots, n. \tag{5}$$

In the case $n = 1$ we drop the superscript ℓ.

3.4. Prolongation. From a contact transformation f on $\Omega \subseteq J^k(\mathbb{R}^m, \mathbb{R}^n)$ we can construct a domain $\Omega_1 \subset J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$ and a map $\mathrm{pr}(f) : \Omega_1 \to \mathrm{pr}(f)(\Omega_1) \subseteq J^{k+1}(\mathbb{R}^m, \mathbb{R}^n)$, called the first prolongation of f, uniquely determined by the following conditions:

- $\mathrm{pr}(f)$ is a contact transformation (6)
- $\pi^{k+1}_1 \circ \mathrm{pr}(f) = f \circ \pi^{k+1}_1$. (7)

Let $\bar{\pi}^{k+1}_1 = \psi^{(k)} \circ \pi^{k+1}_1 \circ (\psi^{(k+1)})^{-1}$ and

$$\psi^{(k+1)} \circ \mathrm{pr}(f) \circ (\psi^{(k+1)})^{-1} = (\xi^{(k+1)}, \eta^{(k+1)}),$$

then (7) and the compatibility conditions (2) imply

$$\xi^{(k+1),j} = \xi^{(k),j} \circ \bar{\pi}^{k+1}_1, \tag{8}$$

and

$$\eta^{(k+1),\ell}_J = \eta^{(k),\ell}_J \circ \bar{\pi}^{k+1}_1 \tag{9}$$

when $|J| \le k$. When $|J| = k+1$, the definition of the coordinate functions $\eta^{(k+1),\ell}_J$ is given by the contact conditions

$$\omega^{(k+1),\ell}_I(\mathrm{pr}(f)_* X^{(k+1)}_i) = 0, \quad |I| = k, \quad \ell = 1, \dots, n, \quad i = 1, \dots, m. \tag{10}$$

In coordinates, these conditions give the matrix equation

$$\left[X^{(k+1)}_i (\eta^{(k),\ell}_I \circ \bar{\pi}^{k+1}_1) \right]_i = \left[X^{(k+1)}_i (\xi^{(k),j} \circ \bar{\pi}^{k+1}_1) \right]_{ij} \left[\eta^{(k+1),\ell}_{I+e_i} \right]_i$$

which serves to define the coordinate functions $\eta_J^{(k+1),\ell}$, where $|J| = k+1$, uniquely on $\Omega_1 = (\psi^{(k+1)})^{-1}(W)$ where

$$W = \left\{ (x^{(k+1)}, u^{(k+1)}) \in \psi^{(k+1)}\left((\pi_1^{k+1})^{-1}(\Omega)\right) \;\Big|\; \det\left[X_i^{(k+1)}(\xi^{(k),j} \circ \bar{\pi}_1^{k+1})\right]_{ij} \neq 0 \right\}.$$

It remains to be checked that $\mathrm{pr}(f)$ is a contact transformation. To this end, note that the compatibility conditions (2), imply that

$$dx^{(k+1),i} = (\pi_1^{k+1})^* dx^{(k),i} = dx^{(k),i} \circ (\pi_1^{k+1})_*, \tag{11}$$

and, when $|J| \le k$, that

$$du_J^{(k+1),\ell} = (\pi_1^{k+1})^* du_J^{(k),\ell} = du_J^{(k),\ell} \circ (\pi_1^{k+1})_*. \tag{12}$$

It follows that

$$\omega_J^{(k+1),\ell} = (\pi_1^{k+1})^* \omega_J^{(k),\ell} = \omega_J^{(k),\ell} \circ (\pi_1^{k+1})_* \tag{13}$$

when $|J| \le k-1$. From (11), (12) and (13) we have $(\pi_1^{k+1})_* : \mathcal{H}^{k+1} \to \mathcal{H}^k$. In particular

$$(\pi_1^{k+1})_* \frac{\partial}{\partial u_I^{(k+1),\ell}} = \begin{cases} \frac{\partial}{\partial u_I^{(k),\ell}} & |I| \le k \\ 0 & |I| = k+1 \end{cases} \tag{14}$$

and

$$(\pi_1^{k+1})_* X_j^{(k+1)} = X^{(k)} + \sum_\ell \sum_{|I|=k} du_I^{(k),\ell}\left((\pi_1^{k+1})_* X_j^{(k+1)}\right) \frac{\partial}{\partial u_I^{(k),\ell}}. \tag{15}$$

From (2) and (13), we have

$$\omega_J^{(k+1),\ell} \circ \mathrm{pr}(f)_* = \omega_J^{(k),\ell} \circ f_* \circ (\pi_1^{k+1})_* \tag{16}$$

when $|J| \le k-1$, hence (15) and (16), together with the fact that f is a contact transformation, imply

$$\omega_J^{(k+1),\ell}\left(\mathrm{pr}(f)_* X_j^{(k+1)}\right) = 0 \tag{17}$$

when $|J| \le k-1$. Furthermore, for $|I| = k+1$, (14) and (16), together with the fact that f is a contact transformation, show that

$$\omega_J^{(k+1),\ell}\left(\mathrm{pr}(f)_* \frac{\partial}{\partial u_I^{(k+1),\ell}}\right) = 0 \tag{18}$$

when $|J| \le k-1$.

For $|J| = k$, (11), (12) and (2) give

$$\omega_J^{(k+1),\ell} \circ \mathrm{pr}(f)_* = du_J^{(k),\ell} \circ f_* \circ (\pi_1^{k+1})_* - \sum_j u_{J+e_j}^{(k+1),\ell} \circ \mathrm{pr}(f)\; dx^{(k),j} \circ f_* \circ (\pi_1^{k+1})_*,$$

which, by (14), gives (18) when $|J| = k$ and $|I| = k+1$. It follows that $\mathrm{pr}(f)$ is a contact transformation. Iterating the procedure defines the higher order prolongation $\mathrm{pr}^{\ell}(f)$ with domain Ω_ℓ.

Prolongation gives rise to two particular types of contact transformations, known as point and Lie tangent transformations. A point transformation is a prolongation of a diffeomorphism of some $D \subseteq J^0(\mathbb{R}^m, \mathbb{R}^n) \equiv \mathbb{R}^m \times \mathbb{R}^n$, and a Lie tangent transformation is a prolongation of a contact transformation on some $D \subseteq J^1(\mathbb{R}^m, \mathbb{R}^n)$. It turns out that Lie tangent transformations can form a larger class than point transformations, but there are no other contact transformations beyond Lie tangent transformations, this fact is Bäcklund's theorem.

Theorem 3.1. (Bäcklund, [2]) *If $n > 1$, then every contact transformation on $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is the k-th order prolongation of a point transformation on $J^0(\mathbb{R}^m, \mathbb{R}^n)$. If $n = 1$, then every contact transformation on $J^k(\mathbb{R}^m, \mathbb{R})$ is the $(k-1)$-th order prolongation of a contact transformation on $J^1(\mathbb{R}^m, \mathbb{R})$.*

Bäcklund's proof is purely geometric, but other treatments, e.g., [1], tend to be at the infinitesimal level. Other useful references include [16] and [17]. Bäcklund's theorem can be derived directly from the Lie algebra $\mathfrak{X}^k$ using Cartan's formula or Cauchy characteristics, and is thus a consequence of the Carnot structure. This observation suggests a Bäcklund type theorem might be possible for Carnot groups generally.

4. Group Structure

4.1. Introduction. In what follows we obtain a multiplication, denoted $\odot$, for the jet spaces $J^k(\mathbb{R}^m, \mathbb{R}^n)$. The particular examples $J^1(\mathbb{R}^m, \mathbb{R}^n)$ and $J^k(\mathbb{R}, \mathbb{R})$ are simple enough that we can produce $\odot$ from second kind coordinates using the Baker-Campbell-Hausdorff formula. Owing to the complexity of the Baker–Campbell–Hausdorff formula, this approach is in general difficult. However, the left translation arising from $\odot$ must be a contact automorphism, and thus, also a point transformation. The examples suggest how to construct the coordinate maps ξ and η^ℓ, which through prolongation, define $\odot$.

4.2. Example: $J^1(\mathbb{R}^m, \mathbb{R}^n)$. In this case

$$\mathcal{H}^1 = \mathrm{span}\left\{ X_j^{(1)} \mid j = 1, \ldots, m \right\} \oplus \mathrm{span}\left\{ \frac{\partial}{\partial u_{e_j}^\ell} \mid \ell = 1, \ldots, n,\ j = 1, \ldots, m \right\}$$

where

$$X_j^{(1)} = \frac{\partial}{\partial x^j} + \sum_{\ell=1}^{n} u_{e_j}^\ell \frac{\partial}{\partial u_0^\ell}, \quad j = 1, \ldots, m$$

and the nontrivial commutators are

$$\left[\frac{\partial}{\partial u_{e_j}^\ell}, X_j^{(1)} \right] = \frac{\partial}{\partial u_0^\ell}.$$

If $L_0 = \mathcal{H}^1$ and $L_1 = \mathrm{span}\{\frac{\partial}{\partial u_0^\ell}\}$ then $L_1 = [L_0, L_0]$. It follows that

$$\mathfrak{X}^1 = L_0 \oplus L_1$$

is a 2-step stratified nilpotent Lie algebra of vector fields which span $TJ^1(\mathbb{R}^m, \mathbb{R}^n)$ pointwise.

Let $\mathfrak{g}$ denote the abstract Lie algebra over $\mathbb{R}$ isomorphic with $\mathfrak{X}^1$. Denote the basis by

$$\{e_1^{(1)}, \ldots, e_m^{(1)}, e_1^1, \ldots, e_m^1, \ldots, e_1^n, \ldots, e_m^n, e^1, \ldots, e^n\}$$

where the nontrivial commutator relations are

$$\left[e_j^\ell, e_j^{(1)}\right] = e^\ell$$

and the isomorphism is given by $X_j^{(1)} \leftrightarrow e_j^{(1)}$, $\frac{\partial}{\partial u_{e_j}^\ell} \leftrightarrow e_j^\ell$ and $\frac{\partial}{\partial u^\ell} \leftrightarrow e^\ell$. The map

$$\sum x_j e_j^{(1)} + \sum u_j^\ell e_j^\ell + \sum u^\ell e^\ell \to m(x, u^{(1)})$$

where

$$m(x, u^{(1)}) = \begin{pmatrix} 0 & \ldots & 0 & u_1^1 & \ldots & u_m^1 & u^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & u_1^n & \ldots & u_m^n & u^n \\ 0 & \ldots & 0 & 0 & \ldots & 0 & x^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & x^m \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

is a Lie algebra isomorphism giving a matrix model of $\mathfrak{g}$. In coordinates of the second kind we have

$$\Phi(m(x, u^{(1)})) = \begin{pmatrix} 1 & \ldots & 0 & u_1^1 & \ldots & u_m^1 & u^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 1 & u_1^n & \ldots & u_m^n & u^n \\ 0 & \ldots & 0 & 1 & \ldots & 0 & x^1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 & x^m \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

and it follows that the second kind coordinate multiplication

$$(x, u^{(1)}) \odot (y, v^{(1)}) = (z, w^{(1)})$$

is defined by $z = x + y$, $w_{e_j}^\ell = v_{e_j}^\ell + u_{e_j}^\ell$ and

$$w^\ell = u^\ell + v^\ell + \sum_{j=1}^m u_{e_j}^\ell y_j. \tag{19}$$

4.3. Example: $J^k(\mathbb{R}, \mathbb{R})$**.** In this case

$$\mathcal{H}^k = \operatorname{span}\{X^{(k)}, \frac{\partial}{\partial u_k}\}$$

where

$$X^{(k)} = \frac{\partial}{\partial x} + \sum_{j=0}^{k-1} u_{j+1}\frac{\partial}{\partial u_j}$$

and the commutator relations are

$$\left[\frac{\partial}{\partial u_j}, X^{(k)}\right] = \frac{\partial}{\partial u_{j-1}}, \quad j = 1,\dots,k.$$

If $L_0 = \mathcal{H}^k$ and $L_j = \operatorname{span}\{\frac{\partial}{\partial u_{k-j}}\}$, $j \geq 1$, then $L_j = [L_0, L_{j-1}]$, $j = 1,\dots,k$ and it follows that

$$\mathfrak{X}^k = L_0 \oplus \cdots \oplus L_k$$

is a $(k+1)$-step stratified nilpotent Lie algebra of vector fields which span $T_pJ^k(\mathbb{R},\mathbb{R})$ for every point p.

Let $\mathfrak{g}^{(k)}$ denote the abstract Lie algebra over $\mathbb{R}$ isomorphic with $\mathfrak{X}^k$. Denote the basis by $\{e^{(k)}, e_k, \dots, e_0\}$ where the nontrivial commutators are $[e_j, e^{(k)}] = e_{j-1}$, when $j = 1,\dots,k$ and the isomorphism is given by the correspondence $X^{(k)} \leftrightarrow e^{(k)}$ and $e_j \leftrightarrow \frac{\partial}{\partial u_j}$. Note that $\mathfrak{g}^{(1)}$ is the Heisenberg algebra, $\mathfrak{g}^{(2)}$ is the Engel algebra, and in general $\mathfrak{g}^{(k)}$ goes by the names model filiform algebra and Goursat algebra.

The map

$$xe^{(k)} + \sum u_j e_j \to \begin{pmatrix} 0 & -x & 0 & \cdots & 0 & u_0 \\ 0 & 0 & -x & \cdots & 0 & u_1 \\ 0 & 0 & 0 & \cdots & 0 & u_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -x & u_{k-1} \\ 0 & 0 & 0 & \cdots & 0 & u_k \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a Lie algebra isomorphism giving a matrix model of $\mathfrak{g}^{(k)}$. In coordinates of the second kind, the elements of the corresponding connected, simply connected Lie group $G^{(k)}$ take the form

$$\exp(xe^{(k)})\exp(u_k e_k + \cdots + u_0 e_0).$$

Multiplication in second kind coordinates, denoted

$$(x, u_k, \dots, u_0) \odot (y, v_k, \dots, v_0) = (z, w_k, \dots, w_0),$$

can be found by solving

$$\exp(ze^{(k)})\exp(\sum_j w_j e_j) = \exp(xe^{(k)})\exp(\sum_j u_j e_j)\exp(ye^{(k)})\exp(\sum_j v_j e_j) \quad (20)$$

for $(z, w_k, \dots, w_0)$. Using the matrix model we have

$$\exp(xe^{(k)}) \sim \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$\begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-x) & (-x)^2/2! & (-x)^3/3! & \cdots & (-x)^k\ /(k-0)! & 0 \\ 0 & 1 & (-x) & (-x)^2/2! & \cdots & (-x)^{k-1}/(k-1)! & 0 \\ 0 & 0 & 1 & (-x) & \cdots & (-x)^{k-2}/(k-2)! & 0 \\ 0 & 0 & 0 & 1 & \cdots & (-x)^{k-3}/(k-3)! & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-x) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$\exp(\sum_j u_j e_j) \sim \begin{pmatrix} \mathrm{Id} & V(u) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & u_0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & u_1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & u_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & u_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & u_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Substituting these expressions into (20) gives

$$\begin{pmatrix} A(z) & A(z)V(w) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A(x)A(y) & A(x)A(y)V(v) + A(x)V(u) \\ 0 & 1 \end{pmatrix}. \tag{21}$$

From (21), we have

$$A(z) = A(x)A(y) \quad \text{and} \quad V(w) = V(v) + A(y)^{-1}V(u).$$

It follows that

$$z = x + y, \quad w_s = v_s + u_s + \sum_{j=s+1}^{k} u_j \frac{y^{j-s}}{(j-s)!}, \quad s = 0, \ldots, k \tag{22}$$

For each $(x, u^{(k)})$ the previous formula defines a contact transformation in the variable $(y, v^{(k)})$ and is thus the prolongation of the point transformation

$$(y, v_0) \to \Big(x + y, v_0 + \sum_{j=0}^{k} u_j \frac{y^j}{j!}\Big). \tag{23}$$

4.4. Multiplication. Guided by (23), we first construct multiplication on the jet spaces $J^k(\mathbb{R}^m, \mathbb{R})$ and then follow (19) to extend it to $J^k(\mathbb{R}^m, \mathbb{R}^n)$. To this end we establish some notation. We write

$$(x, u^{(k)}) \odot (y, v^{(k)}) = (x + y, uv^{(k)})$$

where

$$(y, v^{(k)}) = j_y^k(\mathsf{f}), \quad \mathsf{f}(t) = \sum_{I \in \tilde{I}(k)} v_I \frac{(t-y)^I}{I!},$$
$$(x, u^{(k)}) = j_x^k(\mathsf{g}), \quad \mathsf{g}(t) = \sum_{I \in \tilde{I}(k)} u_I \frac{(t-x)^I}{I!},$$
$$(x+y, uv^{(k)}) = j_{x+y}^k(\mathsf{h}), \quad \mathsf{h}(t) = \sum_{I \in \tilde{I}(k)} uv_I \frac{(t-y-x)^I}{I!}.$$

We write $J \leq I$ if $j_\ell \leq i_\ell$ for all ℓ then

$$\frac{\partial t^I}{\partial t^J} = \left(\frac{\partial}{\partial t^1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial t^m}\right)^{j_m} (t^1)^{i_1}(t^2)^{i_2} \dots (t^m)^{i_m} = \begin{cases} \frac{I!}{(I-J)!} t^{I-J} & \text{if } J \leq I \\ 0 & \text{otherwise.} \end{cases}$$

Guided by (19), (22) and (23), we define

$$uv_I = v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} = \frac{\partial}{\partial t^I}\mathsf{f}(t)\Big|_{t=y} + \frac{\partial}{\partial t^I}\mathsf{g}(t)\Big|_{t=y+x}. \tag{24}$$

In particular, vu_I is the I-th coordinate function η_I, of the prolonged point transformation

$$(y, v_0) \to (x+y, v_0 + \sum_{0 \leq J} u_J \frac{y^J}{J!}).$$

To prove associativity, we use the notation

$$\Big((z, w^{(k)}) \odot (x, u^{(k)})\Big) \odot (y, v^{(k)}) = (z+x+y, (wu)v^{(k)})$$

and

$$(z, w^{(k)}) \odot \Big((x, u^{(k)}) \odot (y, v^{(k)})\Big) = (z+x+y, w(uv)^{(k)}).$$

By definition,

$$\begin{aligned}(wu)v_I &= v_I + \sum_{I \leq J} wu_J \frac{y^{J-I}}{(J-I)!} \\ &= v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} + \sum_{I \leq J} \sum_{J \leq K} w_K \frac{x^{K-J}}{(K-J)!} \frac{y^{J-I}}{(J-I)!}\end{aligned}$$

and

$$\begin{aligned}w(uv)_I &= uv_I + \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!} \\ &= v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!} + \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!}.\end{aligned}$$

Hence associativity will follow if $(wu)v_I - w(uv)_I = 0$, where

$$(wu)v_I - w(uv)_I = \sum_{I \leq J} \sum_{J \leq K} w_K \frac{x^{K-J}}{(K-J)!} \frac{y^{J-I}}{(J-I)!} - \sum_{I \leq J} w_J \frac{(x+y)^{J-I}}{(J-I)!}. \tag{25}$$

Using the multi-index binomial formula

$$(x+y)^{J-I} = \sum_{0\le K\le J-I} \frac{(J-I)!}{(J-I-K)!K!} x^{J-I-K} y^K,$$

the sum in (25) becomes

$$\sum_{I\le J}\sum_{J\le K} w_K \frac{x^{K-J}}{(K-J)!}\frac{y^{J-I}}{(J-I)!} - \sum_{I\le J}\sum_{K\le J-I} w_J \frac{x^{J-I-K}y^K}{(J-I-K)!K!}. \tag{26}$$

Exchanging J and K in the first sum of (26) and changing K to $K-I$ in the second sum of (26), we obtain

$$(wu)v_I - w(uv)_I = \sum_{I\le K}\sum_{K\le J} w_J \frac{x^{J-K}}{(J-K)!}\frac{y^{K-I}}{(K-I)!} - \sum_{I\le J}\sum_{I\le K\le J} w_J \frac{x^{J-K}y^{K-I}}{(J-K)!(K-I)!}.$$

If $S_1(I) = \{(J,K) \mid I \le K \text{ and } K \le J\}$ and $S_2(I) = \{(J,K) \mid I \le J \text{ and } I \le K \le J\}$, then $S_1(I) \subset S_2(I)$ and $S_2(I) \subset S_1(I)$, hence the right hand side of the previous expression is zero.

From (24), the point $(y, v^{(k)})$, where

$$y = -x \quad \text{and} \quad v_I = -\sum_{I\le J} (-1)^{|J-I|} u_J \frac{x^{J-I}}{(J-I)!},$$

defines a right inverse of $(x, u^{(k)})$. Since the multiplication is associative, the right inverse is also a left inverse.

The distribution induced by the left translation under $\odot$ is exactly $\mathcal{H}^k$. Indeed it follows that

$$\frac{\partial}{\partial y^j} uv_J\Big|_{(0,0)} = \begin{cases} u_{J+e_j} & \text{if} \quad |J| = 0,\dots,k-1 \\ 0 & \text{if} \quad |J| = k \end{cases}$$

and

$$\frac{\partial}{\partial v_I} uv_J\Big|_{(0,0)} = \begin{cases} 1 & \text{if} \quad I = J \\ 0 & \text{otherwise,} \end{cases}$$

implying that

$$L_{(x,u^{(k)})*}\Big(\frac{\partial}{\partial y^j}\Big|_{(0,0)}\Big) = X_j^{(k)}\Big|_{(x,u^{(k)})} \quad \text{and} \quad L_{(x,u^{(k)})*}\Big(\frac{\partial}{\partial u_I}\Big|_{(0,0)}\Big) = \frac{\partial}{\partial u_I}\Big|_{(x,u^{(k)})}.$$

Multiplication on $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is obtained by applying the multiplication on $J^k(\mathbb{R}^m, \mathbb{R})$ to the coordinate functions, i.e., define

$$uv_I^\ell = v_I^\ell + \sum_{I\le J} u_J^\ell \frac{y^{J-I}}{(J-I)!} = \frac{\partial}{\partial t^I}\mathsf{f}^\ell(t)\Big|_{t=y} + \frac{\partial}{\partial t^I}\mathsf{g}^\ell(t)\Big|_{t=y+x}.$$

5. Rigidity and Strata Dimension

The folk law rule of thumb is that noncommutativity should reflect rigidity in the sense that a high degree of noncommutativity should imply more rigidity. The problem here is that we don't know what the measure of noncommutativity should be. An obvious consideration is that a measure of noncommutativity or rigidity should depend on the step and the dimensions of the strata. However such data tells us almost nothing, as we can construct distinct Carnot groups with strata dimensions $(3, 2, 1)$ but with opposite rigidity.

For example, using the vector field method, as in [7] and [20], it easy to check that the Carnot group corresponding to the Lie algebra $\mathfrak{n}(4, \mathbb{R})$, the strictly upper triangular 4×4 real matrices, is rigid with strata dimensions $(3, 2, 1)$.

For the nonrigid example we use Grassmanian prolongation (see [9]): Let $\Sigma(k, M)$ be a distribution of k dimensional subspaces of an n dimensional manifold M, i.e, if $p \in M$ then $\Sigma(k, M)_p$ is a k dimensional subspaces of T_pM and for some neighborhood U of p there exist smooth vector fields $X_1, \ldots, X_k$ such that

$$\Sigma(k, M)_q = \mathrm{span}\{X_1(q), \ldots, X_k(q)\}, \quad q \in U.$$

The study of ℓ dimensional submanifolds of M which are tangent to $\Sigma(k, M)$ gives rise to the bundle $\mathrm{Gr}(\ell, \Sigma(k, M)) \to M$ where each fibre $\mathrm{Gr}(\ell, \Sigma(k, M))_p$ is the Grassmannian of ℓ dimensional subspaces $\Lambda_p \subset \Sigma(k, M)_p$. The elements of $\mathrm{Gr}(\ell, \Sigma(k, M))_p$ represent the possible tangent spaces of the submanifolds.

A curve through $(p, \Lambda_p) \in \mathrm{Gr}(\ell, \Sigma(k, M))$ has the form $(\gamma(t), \Lambda_{\gamma(t)})$, where $\gamma(0) = p$, and is defined to be horizontal at (p, Λ_p) if $\dot\gamma(0) \in \Lambda_p$. These curves define a subspace of $T_{(p,\Lambda_p)}\mathrm{Gr}(\ell, \Sigma(k, M))$ and the collection of all these subspaces defines a distribution $\Sigma(\mathrm{Gr}(\ell, \Sigma(k, M)))$ on $\mathrm{Gr}(\ell, \Sigma(k, M))$. The Grassman bundle $\mathrm{Gr}(\ell, \Sigma(k, M))$, together with the distribution $\Sigma(\mathrm{Gr}(\ell, \Sigma(k, M)))$, is called the Grassman prolongation of $\Sigma(k, M)$. A contact map of M lifts to a contact map of $\mathrm{Gr}(\ell, \Sigma(k, M))$ via $f(p, \Lambda_p) = (f(p), f_*\Lambda_p)$ so that M and $\mathrm{Gr}(\ell, \Sigma(k, M))$ share the same rigidity.

Consider the Carnot group G with Lie algebra given by

$$\mathrm{span}\{X_1, X_2, X_3, X_4\}$$

and nontrivial brackets $[X_1, X_2] = [X_1, X_3] = X_4$. The horizontal space is

$$\mathcal{H} = \mathrm{span}\{X_1, X_2, X_3\}$$

and the strata dimensions are $(3, 1)$. In second kind coordinates, we have

$$X_1 = \frac{\partial}{\partial x_1} - (x_2 + x_3)\frac{\partial}{\partial x_4} \qquad X_2 = \frac{\partial}{\partial x_2} \qquad X_3 = \frac{\partial}{\partial x_3} \qquad X_4 = \frac{\partial}{\partial x_4}$$

with corresponding dual forms dx_1, dx_2, dx_3, and $dx_4 + (x_2 + x_3)dx_1$, and $\mathcal{H} = \Sigma(3, G)$. A vector field $V = \sum v_i X_i$ induces a contact flow if $[\mathcal{H}, V] = 0 \mod \mathcal{H}$ which implies that

$$X_1 v_4 + v_2 + v_3 = 0, \quad X_2 v_4 - v_1 = 0 \quad \text{and} \quad X_3 v_4 - v_1 = 0.$$

It follows that

$$V = (X_2 v_4)X_1 + v_2 X_2 - (X_1 v_4 + v_2)X_3 + v_4 X_4$$

with v_2 arbitrary and $v_4 = P(x_1, x_2 + x_3, x_4)$ for any suitably smooth P. We conclude that G is nonrigid.

We Grassman prolong G by 1-dimensional subspaces of the form

$$\operatorname{span}\{X_1 + tX_2 + sX_3\} \subset \mathcal{H}.$$

Thus we define $\gamma = (x_1, x_2, x_3, x_4, t, s)$ to be horizontal if

$$\dot{x}_1 \neq 0, \quad \dot{x}_4 = -(x_2 + x_3)\dot{x}_1 \quad \text{and} \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \lambda(1, t, s),$$

equivalently if $\dot{\gamma} = \dot{x}_1(X_1 + tX_2 + sX_3) + \dot{t}\frac{\partial}{\partial t} + \dot{s}\frac{\partial}{\partial s}$. It follows that

$$\widetilde{\mathcal{H}} = \Sigma(\mathrm{Gr}(1, \Sigma(3, G))) = \operatorname{span}\left\{\tilde{X}_1,\ T,\ S\right\}$$

where $\tilde{X}_1 = X_1 + tX_2 + sX_3$, $T = \frac{\partial}{\partial t}$ and $S = \frac{\partial}{\partial s}$, moreover the nontrivial brackets are

$$[T, \tilde{X}_1] = X_2,\ [S, \tilde{X}_1] = X_3,\ [\tilde{X}_1, X_2] = [X_1, X_2] = X_4,\ [\tilde{X}_1, X_3] = [X_1, X_3] = X_4.$$

By construction, the corresponding Carnot group is nonrigid with strata dimensions $(3, 2, 1)$.

6. Further Consequences

By Bäcklund's theorem, the quasiconformal automorphism groups of the jet spaces must consist of point transformations and be polynomial in all but the base variables. Except for the complications that might arise from the analytic definition of quasiconformality, it is feasible that we can calculate these quasiconformal automorphisms. In work in preparation we investigate the quasiconformal mappings of $J^k(\mathbb{R}, \mathbb{R})$ obtaining explicitly the quasiconformal automorphisms as well as Liouville's theorem.

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