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Nilpotent Lie algebras of Maximal Rank and of Kac-Moody Type $D_4^{(3)}$

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Abstract. Let $\mathfrak{g}$ be the Kac-Moody algebra associated with the twisted affine Cartan matrix $D_4^{(3)}$. Each nilpotent Lie algebra of maximal rank and of type $D_4^{(3)}$ is isomorphic to a quotient of the positive part of $\mathfrak{g}$. We determine the isomorphism classes of nilpotent Lie algebras of maximal rank and of type $D_4^{(3)}$.
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1. Introduction

In [12] Santharoubane associated canonically a Kac-Moody algebra $\mathfrak{g}(A)$ with each nilpotent Lie algebra $\mathfrak{L}$ of maximal rank, where A is a generalized Cartan matrix.

There are 3 families of Kac-Moody algebras: finite, affine and indefinite, and the second family is divided in two subfamilies: non-twisted affine and twisted affine.

The study of nilpotent Lie algebras of maximal rank associated with the finite Kac-Moody algebras (i.e. the finite-dimensional simple Lie algebras) was already done (see [3], [5] and [4]).

At present, several authors are studying the nilpotent Lie algebras of maximal rank associated with non-twisted affine Kac-Moody algebras (see [1, 2], [6, 7], [10] and [11]).

In this paper we study the nilpotent Lie algebras associated with the twisted affine Kac-Moody algebra $\mathfrak{g}(D_4^{(3)})$. The main result we get is the following: there are exactly 88 infinite series (up to isomorphism) with discrete parameters and 1 infinite series with continuous parameter of nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_4^{(3)}$.

I thank the referee for pointing out an error which caused me to miss in the original version of this paper the 5 infinite series in the last 2 lines of Theorem 4.1a below. This same mistake appears also in [6], [7], [1], [2] and [10]. The corrections to [6] and [7] will be given elsewhere.

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2. The Classification Method for nilpotent Lie algebras of maximal rank

Let $\mathfrak{L}$ be a finite-dimensional nilpotent Lie algebra, $Der\mathfrak{L}$ its derivation algebra, $Aut\mathfrak{L}$ its automorphism group. A *torus* on $\mathfrak{L}$ is a commutative subalgebra of $Der\mathfrak{L}$ whose elements are semi-simple. All maximal (for the inclusion) tori on $\mathfrak{L}$ have the same dimension called the *rank* of $\mathfrak{L}$. The rank r of $\mathfrak{L}$ is less than the dimension ℓ of $\mathfrak{L}/[\mathfrak{L},\mathfrak{L}]$; one says that $\mathfrak{L}$ is of *maximal rank* if $r=\ell$.

A matrix $A=(a_{ij})_{i,j=1}^{\ell}$ with entries in $\mathbb{Z}$ is called a *generalized Cartan matrix* if it satisfies the following conditions:

1. $a_{ii}=2$ for $i=1,\ldots,\ell$;
2. $a_{ij}\leq 0$ for $i\neq j$;
3. $a_{ij}=0 \Longleftrightarrow a_{ji}=0$.

If $\mathfrak{L}$ is of maximal rank ℓ, then we can associate a generalized Cartan matrix $A=(a_{ij})_{1\leq i,j\leq \ell}$ with $\mathfrak{L}$ (see 3.2. of [11]), we say that $\mathfrak{L}$ is of *Kac-Moody type* A.

Let $A=(a_{ij})_{1\leq i,j\leq\ell}$ be a generalized Cartan matrix, $\mathfrak{g}(A)$ be the Kac-Moody algebra associated with A, $\mathfrak{n}_+$ be the positive part of $\mathfrak{g}(A)$, Δ_+ the positive root system, $\mathfrak{g}_\alpha$ be the root subspace associated with $\alpha\in\Delta_+$, $G=\mathfrak{S}_\ell(A)$ be the automorphism group of the Dynkin diagram of A and $\alpha_1,\ldots,\alpha_\ell$ the simple roots. Let $\mathfrak{n}_{++}$ be the ideal of $\mathfrak{n}_+$ defined by

$$\mathfrak{n}_+ = \Big(\bigoplus_{\substack{1\leq i\neq j\leq \ell\\ 0\leq k\leq -a_{ji}}} \mathfrak{g}_{\alpha_i+k\alpha_j}\Big)\oplus\mathfrak{n}_{++}.$$

Let $\mathcal{I}(\mathfrak{n}_{++})$ be the set of ideals of $\mathfrak{n}_+$ included in $\mathfrak{n}_{++}$ and stable under the action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$. The group G acts on $\mathfrak{n}_+$ as an automorphism group by $\sigma e_i = e_{\sigma i}$ $(i=0,\ldots,\ell)$ where $e_0,\cdots,e_\ell$ are the Chevalley generators of $\mathfrak{n}_+$. The group G acts on $\mathcal{I}(\mathfrak{n}_{++})$.

According to previous definitions and 6.3 of [11], the mapping

$$G.\mathfrak{a}\mapsto \mathfrak{n}_+/\mathfrak{a}$$

is a bijection from the set of G-orbits of $\mathcal{I}(\mathfrak{n}_{++})$ onto a set of representatives of the isomorphism classes of nilpotent Lie algebras of maximal rank and of Kac-Moody type A.

By the above result, our main problem of finding all nilpotent Lie algebras of maximal rank and of Kac-Moody type A is equivalent to the concrete problem of finding some ideals of the positive part of the Kac-Moody algebra $\mathfrak{g}(A)$, up to the action of the automorphism group G.

3. The Kac-Moody algebra associated with $D_4^{(3)}$

We consider the generalized Cartan matrix $D_4^{(3)}$ (see table Aff3, p.55 of [9]). Let $\mathfrak{g}'=\mathfrak{g}(D_4)$, where D_4 is a finite Cartan matrix and its Dynkin diagram is:

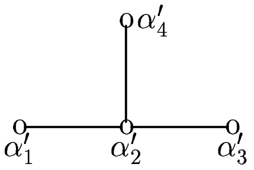
$S(D_4) \equiv$ 

Figure 1: Dynkin diagram of D_4

Let $\overline{\mu}$ be an automorphism of the Dynkin diagram of D_4 of order 3. Since there are two such automorphism which are equivalent, we choose one of them:

$$\overline{\mu}(\alpha'_1) = \alpha'_3, \quad \overline{\mu}(\alpha'_2) = \alpha'_2, \quad \overline{\mu}(\alpha'_3) = \alpha'_4, \quad \overline{\mu}(\alpha'_4) = \alpha'_1.$$

Let μ be the corresponding automorphism of $\mathfrak{g}'$. Set $\epsilon = exp\frac{2\pi i}{3}$. Then each eigenvalue of μ has the form ϵ^j, $j \in \mathbb{Z}/3\mathbb{Z}$, and since μ is diagonalizable, we have the decomposition

$$\mathfrak{g}' = \bigoplus_{j\in\mathbb{Z}/3\mathbb{Z}} \mathfrak{g}'_j, \tag{1}$$

where $\mathfrak{g}'_j$ is the eigenspace of μ for the eigenvalue ϵ^j.

Fix a non-degenerate invariant symmetric bilinear $\mathbb{C}$-valued form $(\cdot|\cdot)$ on $\mathfrak{g}'$. Let $L = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t. We consider the following Lie algebra:

$$L(\mathfrak{g}', \mu) = \bigoplus_{j\in\mathbb{Z}} L(\mathfrak{g}', \mu)_j,$$

where $L(\mathfrak{g}', \mu)_j = t^j \otimes \mathfrak{g}'_{j_{\mathrm{mod}\ 3}}$.

The Kac-Moody algebra associated with the affine matrix $D_4^{(3)}$ is a *twisted affine algebra* and is defined by (see Chap. 8 of [9]):

$$\widehat{L}(\mathfrak{g}', \mu) = L(\mathfrak{g}', \mu) \oplus \mathbb{C}c' \oplus \mathbb{C}d'$$

with the bracket defined as follows:

$$[t^k \otimes x \oplus \lambda c' \oplus \mu d', t^{k_1} \otimes y \oplus \lambda_1 c' \oplus \mu_1 d'] =$$
$$(t^{k+k_1} \otimes [x, y] + \mu k_1 t^{k_1} \otimes y - \mu_1 k t^k \otimes x) \oplus k\delta_{k,-k_1}(x|y)c'$$

where $x, y \in \mathfrak{g}'; \lambda, \mu, \lambda_1, \mu_1 \in \mathbb{C}$.

Let $\mathfrak{h}'$ be the Cartan subalgebra of $\mathfrak{g}'$, Δ' the root system, $\{\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4\}$ the root basis, $\{\alpha'^{\vee}_1, \alpha'^{\vee}_2, \alpha'^{\vee}_3, \alpha'^{\vee}_4\}$ the coroot basis, E'_1, E'_2, E'_3, E'_4, F'_1, F'_2, F'_3, F'_4 the Chevalley generators. Let $\mathfrak{g}' = \oplus_{\alpha'\in\Delta'}\mathfrak{g}'_{\alpha'}$ be the root subspace decomposition of $\mathfrak{g}'$. We have $dim\mathfrak{h}' = 4$, the coroot basis is a basis of $\mathfrak{h}'$, and the root subspaces are unidimensional, $\mathfrak{g}'_{\alpha'} = \mathbb{C}E'_{\alpha'}$. The positive root system is:

$$\begin{aligned}\Delta'_+ \ = \{ \ & \alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_1 + \alpha'_2, \alpha'_2 + \alpha'_3, \alpha'_2 + \alpha'_4, \alpha'_1 + \alpha'_2 + \alpha'_3, \alpha'_1 + \alpha'_2 + \alpha'_4, \\ & \alpha'_2 + \alpha'_3 + \alpha'_4, \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4, \alpha'_1 + 2\alpha'_2 + \alpha'_3 + \alpha'_4\}.\end{aligned}$$

We introduce the following elements of $\mathfrak{g}'$:

$$\begin{aligned}
&\theta_0 = \alpha'_1 + \alpha'_2 + \alpha'_3, \\
&H_0 = -2(\alpha'^{\vee}_1 + \alpha'^{\vee}_3 + \alpha'^{\vee}_4) - 3\alpha'^{\vee}_2, \quad H_1 = \alpha'^{\vee}_2, \quad H_2 = \alpha'^{\vee}_1 + \alpha'^{\vee}_3 + \alpha'^{\vee}_4, \\
&E_0 = E'_{-\theta_0} + \epsilon^2 E'_{-\overline{\mu}(\theta_0)} + \epsilon E'_{-\overline{\mu}^2(\theta_0)}, \quad E_1 = E'_2, \quad E_2 = E'_1 + E'_3 + E'_4, \\
&F_0 = -E'_{\theta_0} - \epsilon E'_{\overline{\mu}(\theta_0)} - \epsilon^2 E'_{\overline{\mu}^2(\theta_0)}, \quad F_1 = F'_2, \quad F_2 = F'_1 + F'_3 + F'_4.
\end{aligned}$$

The $\mathbb{Z}/3\mathbb{Z}$-gradation of $\mathfrak{g}'$ described in (1) is $\mathfrak{g}' = \mathfrak{g}'_{\overline{0}} \oplus \mathfrak{g}'_{\overline{1}} \oplus \mathfrak{g}'_{\overline{2}}$, $\mathfrak{h}'_{\overline{s}} = \mathfrak{h}' \cap \mathfrak{g}_{\overline{s}}$ and $\mathfrak{g}'_{\overline{s}} = \mathfrak{h}'_{\overline{s}} \oplus \left(\bigoplus_{\alpha\in\Delta_{\overline{s}}} \mathfrak{g}'_{\overline{s},\alpha}\right)$ for $s = 0,1,2$, where $\Delta_{\overline{s}} = \Delta_{\overline{s},+} \cup \Delta_{\overline{s},-}$, $\Delta_{\overline{s},-} = \{-\alpha \,/\, \alpha \in \Delta_{\overline{s},+}\}$ and

$$\begin{array}{rcl}\Delta_{\overline{0},+} &=& \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, 2\alpha_1+3\alpha_2\}\\ \Delta_{\overline{1},+} &=& \{\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\} = \Delta_{\overline{2},+}\end{array}$$

with $\alpha_1 = \alpha'_2$ and $\alpha_2 = \frac{1}{3}(\alpha'_1 + \alpha'_3 + \alpha'_4)$.

Set $\mathfrak{h} = \mathfrak{h}'_{\overline{0}} \oplus \mathbb{C}c' \oplus \mathbb{C}d'$ and define $\delta \in \mathfrak{h}^*$ by $\delta|_{\mathfrak{h}'_{\overline{0}}\oplus\mathbb{C}c'} = 0$, $\delta(d') = 1$. Set $e_0 = t\otimes E_0$, $f_0 = t^{-1}\otimes F_0$, $e_i = 1\otimes E_i$, $f_i = 1\otimes F_i$, $(i=1,2)$. Then we have:

$$[e_i, f_i] = 1\otimes H_i\,(i=1,2);\quad [e_0,f_0] = 3c' + 1\otimes H_0.$$

We describe the root system and the root space decomposition of $\widehat{L}(\mathfrak{g}',\mu)$ with respect to $\mathfrak{h}$:

$$\Delta = \{j\delta+\gamma\,;\, j\in\mathbb{Z},\, \gamma\in\Delta_{\overline{s}},\, j\equiv s \bmod 3,\, s=0,1,2\} \cup \{j\delta\,;\, j\in\mathbb{Z}-\{0\}\};$$

$$\widehat{L}(\mathfrak{g}',\mu) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} L(\mathfrak{g}',\mu)_\alpha\right),$$

where

$$L(\mathfrak{g}',\mu)_{j\delta+\gamma} = t^j \otimes \mathfrak{g}'_{\overline{s},\gamma},\;\; L(\mathfrak{g}',\mu)_{j\delta} = t^j\otimes\mathfrak{h}'_{\overline{s}}.$$

We set

$$\Pi = \{\alpha_0 = \delta-\theta_0, \alpha_1, \alpha_2\}$$

and

$$\Pi^\vee = \{\alpha_0^\vee = 3c' + 1\otimes H_0, \alpha_i^\vee = 1\otimes H_i\,(i=1,2)\}.$$

$\mathfrak{h}$ is the Cartan subalgebra, e_0, e_1, e_2, f_0, f_1, f_2 are the Chevalley generators and Π and $\Pi^\vee$ are, respectively, the root basis and the coroot basis of $\widehat{L}(\mathfrak{g}',\mu)$, which we denote by $\mathfrak{g}$ from now on.

The positive part $\mathfrak{n}_+$ of $\mathfrak{g}$ is

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \widehat{L}(\mathfrak{g}',\mu)_\alpha = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$$

and the positive root system of $\mathfrak{g}$ is

$$\Delta_+ = \Delta_{\overline{0},+} \cup \{j\delta+\gamma\,;\, j\geq 1,\, \gamma\in\Delta_{\overline{s}}\cup\{0\},\, j\equiv s \bmod 3\}.$$

If we set

$$\Delta^0 = \Delta_{\overline{0},+} \cup \{\delta+\gamma\,;\,\gamma\in\Delta_{\overline{1}}\} \cup \{2\delta+\gamma\,;\,\gamma\in\Delta_{\overline{2}}\} \cup \{3\delta+\gamma\,;\,\gamma\in\Delta_{\overline{0},+}\} \cup \{\delta, 2\delta, 3\delta\}$$

we have $\Delta_+ = \bigsqcup_{j\geq 0} \Delta^j$, where $\Delta^j = \{3j\delta + \gamma;\, \gamma \in \Delta^0\}$ if $j \geq 1$.

Since $\delta = \alpha_0 + \alpha_1 + 2\alpha_2$ we have $\Delta^0 = \{\alpha_0, \alpha_1, \ldots, \alpha_{26}\}$, with
$\alpha_3 = \alpha_0 + \alpha_2$, $\alpha_4 = \alpha_1 + \alpha_2$,
$\alpha_5 = \alpha_0 + \alpha_1 + \alpha_2$, $\alpha_6 = \alpha_1 + 2\alpha_2$,
$\alpha_7 = \delta = \alpha_0 + \alpha_1 + 2\alpha_2$, $\alpha_8 = \alpha_1 + 3\alpha_2$,
$\alpha_9 = 2\alpha_0 + \alpha_1 + 2\alpha_2$, $\alpha_{10} = \alpha_0 + \alpha_1 + 3\alpha_2$, $\alpha_{11} = 2\alpha_1 + 3\alpha_2$,
$\alpha_{12} = 2\alpha_0 + \alpha_1 + 3\alpha_2$, $\alpha_{13} = \alpha_0 + 2\alpha_1 + 3\alpha_2$,
$\alpha_{14} = 3\alpha_0 + \alpha_1 + 3\alpha_2$, $\alpha_{15} = 2\alpha_0 + 2\alpha_1 + 3\alpha_2$, $\alpha_{16} = \alpha_0 + 2\alpha_1 + 4\alpha_2$,
$\alpha_{17} = 3\alpha_0 + 2\alpha_1 + 3\alpha_2$, $\alpha_{18} = 2\delta = 2\alpha_0 + 2\alpha_1 + 4\alpha_2$,
$\alpha_{19} = 3\alpha_0 + 2\alpha_1 + 4\alpha_2$, $\alpha_{20} = 2\alpha_0 + 2\alpha_1 + 5\alpha_2$,
$\alpha_{21} = 3\alpha_0 + 2\alpha_1 + 5\alpha_2$, $\alpha_{22} = 2\alpha_0 + 3\alpha_1 + 5\alpha_2$,
$\alpha_{23} = 3\alpha_0 + 3\alpha_1 + 5\alpha_2$, $\alpha_{24} = 3\alpha_0 + 2\alpha_1 + 6\alpha_2$, $\alpha_{25} = 2\alpha_0 + 3\alpha_1 + 6\alpha_2$,
$\alpha_{26} = 3\delta = 3\alpha_0 + 3\alpha_1 + 6\alpha_2$.

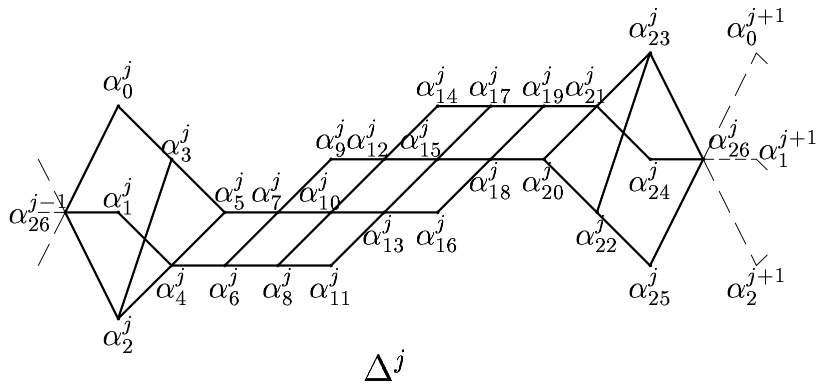


Figure 2: Positive roots of $D_4^{(3)}$

where $\alpha_i^j = 3j\delta + \alpha_i$, with $j \geq 1$ and $\alpha_i \in \Delta^0$.

We order these roots by $\alpha_i^j < \alpha_k^l$ iff $j < l$ or $j = l$ and $i < k$.

4. The nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_4^{(3)}$

We have to obtain all G-orbits of ideals of the positive part of the Kac-Moody algebra $\mathfrak{g}(D_4^{(3)})$ included in $\mathfrak{n}_{++}$.

The Dynkin diagram of $D_4^{(3)}$ is:

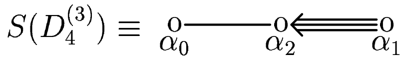


Figure 3: Dynkin diagram of $D_4^{(3)}$

Then the automorphism group G of the Dynkin diagram is $\{id\}$ and each G-orbit has an unique ideal. Therefore we can identify the sets $\mathcal{I}(\mathfrak{n}_{++})$ and $\mathcal{I}(\mathfrak{n}_{++})/G$.

Let Δ_{++} be defined by :

$$\Delta_+ = \{\alpha_i + k\alpha_j\,;\, 0 \le i \ne j \le \ell, 0 \le k \le -a_{ji}\} \cup \Delta_{++}$$

Since

$$D_4^{(3)} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$$

we have:

$$\Delta_{++} = \widetilde{\Delta}^0 \sqcup \Delta^1 \sqcup \Delta^2 \sqcup \cdots$$

with $\widetilde{\Delta}^0 = \{\alpha_5, \alpha_7, \alpha_9, \alpha_{10}, \ldots, \alpha_{26}\}$.

If $I \subseteq \Delta_{++}$ then one can write:

$$I = \bigsqcup_{j\in\mathbb{N}} I^j$$

with $I^0 = I \cap \widetilde{\Delta}^0$ and $I^j = I \cap \Delta^j$ if $j \ge 1$. We say that I^j is an ideal of Δ^j iff

$$(\alpha \in I^j, \alpha + \alpha_i \in \Delta^j) \Longrightarrow \alpha + \alpha_i \in I^j \quad \forall i = 0, 1, 2$$

We set $\mathcal{I}(\Delta_{++}) = \{I \subseteq \Delta_{++}\,;\, I^j \text{ is an ideal of } \Delta^j, \forall j \in \mathbb{N}\}$.

If $\mathfrak{a} \in \mathcal{I}(\mathfrak{n}_{++})$ then one can write :

$$\mathfrak{a} = \bigoplus_{\alpha\in\Delta_\mathfrak{a}} \mathfrak{a} \cap \mathfrak{g}_\alpha$$

where $\Delta_\mathfrak{a} = \{\alpha \in \Delta_{++}; \mathfrak{a} \cap \mathfrak{g}_\alpha \ne (0)\}$. Then, as any root in $\Delta^j \setminus \{\alpha_{26}^j\}$ has multiplicity one, $\Delta_\mathfrak{a} \in \mathcal{I}(\Delta_{++})$.

We can define the map $\mathcal{I}(\mathfrak{n}_{++}) \xrightarrow{\varphi} \mathcal{I}(\Delta_{++})$ by setting $\varphi(\mathfrak{a}) = \Delta_\mathfrak{a}$. But this is not onto. There exists $j_\mathfrak{a} \in \mathbb{N}$ such that $3j_\mathfrak{a}\delta \notin \varphi(\mathfrak{a})$ and $3(j_\mathfrak{a}+1)\delta \in \varphi(\mathfrak{a})$. Then

$$\varphi(\mathfrak{a}) = \bigsqcup_{j\in\mathbb{N}} \varphi(\mathfrak{a})^j$$

with $\varphi(\mathfrak{a})^{j_\mathfrak{a}}$ any ideal of $\Delta^{j_\mathfrak{a}}$, $\varphi(\mathfrak{a})^{j_\mathfrak{a}+1}$ an ideal of $\Delta^{j_\mathfrak{a}+1}$ that depends on $\varphi(\mathfrak{a})^{j_\mathfrak{a}}$ and $\varphi(\mathfrak{a})^j = \Delta^j$ if $j > j_\mathfrak{a}+1$ (this results from the calculation in 4.2 of the different possibilities for $\varphi(\mathfrak{a})^{j_\mathfrak{a}+1}$ associated to a $\varphi(\mathfrak{a})^{j_\mathfrak{a}}$).

The map φ is not injective either. Since $\dim \mathfrak{g}_\alpha = 1$ if $\alpha \ne 3j\delta$, we have $\mathfrak{a}_\alpha = \mathfrak{g}_\alpha$ if $\alpha \in \varphi(\mathfrak{a})$ and $\alpha \ne 3j\delta$. But if $\alpha \in \varphi(\mathfrak{a})$ and $\alpha = 3j\delta$, it may be $\mathfrak{a}_\alpha \ne \mathfrak{g}_\alpha$ since $\dim \mathfrak{g}_{3j\delta} = 2$.

The first step for obtaining the nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_4^{(3)}$ is to determine $\operatorname{Im}\varphi$. The next step consists of determining $\varphi^{-1}(I)$ for each $I \in \operatorname{Im}\varphi \subset \mathcal{I}(\Delta_{++})$.

4.1. The image of φ.

If $I \in \operatorname{Im}\varphi$ there exists $j_I \in \mathbb{N}$ such that $3j_I\delta \notin I$ and $3(j_I+1)\delta \in I$. We have

$$I = I^{j_I} \sqcup I^{j_I+1} \sqcup \Delta^{j_I+2} \sqcup \Delta^{j_I+3} \sqcup \Delta^{j_I+4} \sqcup \cdots$$

where I^{j_I} and I^{j_I+1} are ideals of Δ^{j_I} and Δ^{j_I+1}, respectively. Then we have to determine all ideals of Δ^j, $\forall j \in \mathbb{N}$.

The map $J \mapsto J+3(j-1)\delta$, $j \geq 1$, is a bijection between the sets of ideals of Δ^1 and Δ^j. Therefore it is sufficient to determine all ideals of $\widetilde{\Delta}^0$ and all ideals of Δ^j for $j \neq 0$.

There are 80 series of ideals, we have 56 ideals of $\widetilde{\Delta}^0$ and 80 ideals of Δ^j, $\forall j \geq 1$. For $j \geq 1$ the ideals are: (we denote by $\langle \gamma_1, \gamma_2, \ldots \rangle$ the ideal generated by $\gamma_1 + 3j\delta, \gamma_2 + 3j\delta, \ldots$)

$$\begin{array}{lllll}
I_1^j = \langle \alpha_0 \rangle, & I_2^j = \langle \alpha_1 \rangle, & I_3^j = \langle \alpha_2 \rangle, & I_4^j = \langle \alpha_3 \rangle, & I_5^j = \langle \alpha_4 \rangle, \\
I_6^j = \langle \alpha_5 \rangle, & I_7^j = \langle \alpha_6 \rangle, & I_8^j = \langle \alpha_7 \rangle, & I_9^j = \langle \alpha_8 \rangle, & I_{10}^j = \langle \alpha_9 \rangle, \\
I_{11}^j = \langle \alpha_{10} \rangle, & I_{12}^j = \langle \alpha_{11} \rangle, & I_{13}^j = \langle \alpha_{12} \rangle, & I_{14}^j = \langle \alpha_{13} \rangle, & I_{15}^j = \langle \alpha_{14} \rangle, \\
I_{16}^j = \langle \alpha_{15} \rangle, & I_{17}^j = \langle \alpha_{16} \rangle, & I_{18}^j = \langle \alpha_{17} \rangle, & I_{19}^j = \langle \alpha_{18} \rangle, & I_{20}^j = \langle \alpha_{19} \rangle, \\
I_{21}^j = \langle \alpha_{20} \rangle, & I_{22}^j = \langle \alpha_{21} \rangle, & I_{23}^j = \langle \alpha_{22} \rangle, & I_{24}^j = \langle \alpha_{23} \rangle, & I_{25}^j = \langle \alpha_{24} \rangle, \\
I_{26}^j = \langle \alpha_{25} \rangle, & I_{27}^j = \langle \alpha_{26} \rangle, & & &
\end{array}$$

$$\begin{array}{llll}
I_{28}^j = \langle \alpha_0, \alpha_1 \rangle, & I_{29}^j = \langle \alpha_0, \alpha_2 \rangle, & I_{30}^j = \langle \alpha_0, \alpha_4 \rangle, & I_{31}^j = \langle \alpha_0, \alpha_6 \rangle, \\
I_{32}^j = \langle \alpha_0, \alpha_8 \rangle, & I_{33}^j = \langle \alpha_0, \alpha_{11} \rangle, & I_{34}^j = \langle \alpha_1, \alpha_2 \rangle, & I_{35}^j = \langle \alpha_1, \alpha_3 \rangle, \\
I_{36}^j = \langle \alpha_3, \alpha_4 \rangle, & I_{37}^j = \langle \alpha_3, \alpha_6 \rangle, & I_{38}^j = \langle \alpha_3, \alpha_8 \rangle, & I_{39}^j = \langle \alpha_3, \alpha_{11} \rangle, \\
I_{40}^j = \langle \alpha_5, \alpha_6 \rangle, & I_{41}^j = \langle \alpha_5, \alpha_8 \rangle, & I_{42}^j = \langle \alpha_5, \alpha_{11} \rangle, & I_{43}^j = \langle \alpha_7, \alpha_8 \rangle, \\
I_{44}^j = \langle \alpha_7, \alpha_{11} \rangle, & I_{45}^j = \langle \alpha_8, \alpha_9 \rangle, & I_{46}^j = \langle \alpha_9, \alpha_{10} \rangle, & I_{47}^j = \langle \alpha_9, \alpha_{11} \rangle, \\
I_{48}^j = \langle \alpha_9, \alpha_{13} \rangle, & I_{49}^j = \langle \alpha_9, \alpha_{16} \rangle, & I_{50}^j = \langle \alpha_{10}, \alpha_{11} \rangle, & I_{51}^j = \langle \alpha_{11}, \alpha_{12} \rangle,
\end{array}$$

$$\begin{array}{llll}
I_{52}^j = \langle \alpha_{11}, \alpha_{14} \rangle, & I_{53}^j = \langle \alpha_{12}, \alpha_{13} \rangle, & I_{54}^j = \langle \alpha_{12}, \alpha_{16} \rangle, & I_{55}^j = \langle \alpha_{13}, \alpha_{14} \rangle, \\
I_{56}^j = \langle \alpha_{14}, \alpha_{15} \rangle, & I_{57}^j = \langle \alpha_{14}, \alpha_{16} \rangle, & I_{58}^j = \langle \alpha_{14}, \alpha_{18} \rangle, & I_{59}^j = \langle \alpha_{14}, \alpha_{20} \rangle, \\
I_{60}^j = \langle \alpha_{14}, \alpha_{22} \rangle, & I_{61}^j = \langle \alpha_{14}, \alpha_{25} \rangle, & I_{62}^j = \langle \alpha_{15}, \alpha_{16} \rangle, & I_{63}^j = \langle \alpha_{16}, \alpha_{17} \rangle, \\
I_{64}^j = \langle \alpha_{17}, \alpha_{18} \rangle, & I_{65}^j = \langle \alpha_{17}, \alpha_{20} \rangle, & I_{66}^j = \langle \alpha_{17}, \alpha_{22} \rangle, & I_{67}^j = \langle \alpha_{17}, \alpha_{25} \rangle, \\
I_{68}^j = \langle \alpha_{19}, \alpha_{20} \rangle, & I_{69}^j = \langle \alpha_{19}, \alpha_{22} \rangle, & I_{70}^j = \langle \alpha_{19}, \alpha_{25} \rangle, & I_{71}^j = \langle \alpha_{21}, \alpha_{22} \rangle, \\
I_{72}^j = \langle \alpha_{21}, \alpha_{25} \rangle, & I_{73}^j = \langle \alpha_{22}, \alpha_{24} \rangle, & I_{74}^j = \langle \alpha_{23}, \alpha_{24} \rangle, & I_{75}^j = \langle \alpha_{23}, \alpha_{25} \rangle, \\
I_{76}^j = \langle \alpha_{24}, \alpha_{25} \rangle, & & &
\end{array}$$

$$I_{77}^j = \langle \alpha_0, \alpha_1, \alpha_2 \rangle, \quad I_{78}^j = \langle \alpha_9, \alpha_{10}, \alpha_{11} \rangle, \quad I_{79}^j = \langle \alpha_{14}, \alpha_{15}, \alpha_{16} \rangle,$$
$$I_{80}^j = \langle \alpha_{23}, \alpha_{24}, \alpha_{25} \rangle.$$

Moreover, for $j = 0$ we have the ideals I_i^0 with:
$i = 6, 8, 10, \ldots, 27, 42, 44, 46, \ldots, 76, 78, 79, 80$.

We haven't determined the image of φ yet. If $I \in \operatorname{Im} \varphi$, although I^{j_I} is any ideal of Δ^{j_I}, I^{j_I+1} is an ideal of Δ^{j_I+1} that depends on I^{j_I} and $\mathfrak{a}_{3(j_I+1)\delta}$. So, in order to obtain the ideals of $\mathfrak{n}_+$ included in $\mathfrak{n}_{++}$, we will realize a case-by-case study.

4.2. The ideals in $\mathfrak{n}_+$.

Let $I \in \operatorname{Im} \varphi$. Then there exists i such that $I^j = I_i^j$ and $3j\delta \notin I$, but $3(j+1)\delta \in I$. We define:

$$\begin{array}{rcl}
I'_p & = & \{\alpha \in I^{j+p} ; \mid \alpha \mid \leq \mid 3(j+p+1)\delta \mid -2\} \\
I''_p & = & \{\alpha \in I^{j+p} ; \mid \alpha \mid = \mid 3(j+p+1)\delta \mid -1\}
\end{array}$$

with $p = 0, 1$. Then we have the partition:

$$I = \underbrace{I'_0 \sqcup I''_0 \sqcup \{3(j+1)\delta\}}_{I^j} \sqcup \underbrace{I'_1 \sqcup I''_1 \sqcup \{3(j+2)\delta\}}_{I^{j+1}} \sqcup \left(\bigsqcup_{k>j+1} \Delta^k \right)$$

If $\mathfrak{a} \in \varphi^{-1}(I)$, then we have $\mathfrak{a} = \oplus_{\alpha\in I}\mathfrak{a}_\alpha$ with $\mathfrak{a}_\alpha = \mathfrak{a}\cap\mathfrak{g}_\alpha \neq (0)$.

The partition of I gives us a direct sum decomposition for $\mathfrak{a}$:

$$\mathfrak{a} = \left(\bigoplus_{\alpha\in I_0'}\mathfrak{a}_\alpha\right) \oplus \left(\bigoplus_{\alpha\in I_0''}\mathfrak{a}_\alpha\right) \oplus \mathfrak{a}_{3(j+1)\delta} \oplus \left(\bigoplus_{\alpha\in I_1'}\mathfrak{a}_\alpha\right) \oplus \left(\bigoplus_{\alpha\in I_1''}\mathfrak{a}_\alpha\right) \oplus \mathfrak{a}_{3(j+2)\delta}\oplus$$

$$\oplus \left(\bigoplus_{\alpha>3(j+2)\delta}\mathfrak{a}_\alpha\right)$$

Since $\dim\mathfrak{g}_\alpha = 1$ if $\alpha \neq 3k\delta$, we have $\mathfrak{a}_\alpha = \mathfrak{g}_\alpha$ if $\alpha \in I$ and $\alpha \neq 3k\delta$; as we shall see below, I_1'' has cardinal 3, hence $\mathfrak{a}_{3k\delta-\alpha_m} = \mathfrak{g}_{3k\delta-\alpha_m}$ for $m = 0, 1, 2$ and $k \geq j+2$, we have $\mathfrak{a}_{3k\delta} = \mathfrak{g}_{3k\delta}$ for $k \geq j+2$; it follows that:

$$\mathfrak{a} = \left(\bigoplus_{\alpha\in I_0'}\mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha\in I_0''}\mathfrak{g}_\alpha\right) \oplus \mathfrak{a}_{3(j+1)\delta} \oplus \left(\bigoplus_{\alpha\in I^{j+1}}\mathfrak{g}_\alpha\right) \oplus \mathfrak{b}$$

where

$$\mathfrak{b} = \bigoplus_{\alpha>3(j+2)\delta}\mathfrak{g}_\alpha.$$

Therefore we have to determine $\mathfrak{a}_{3(j+1)\delta} \subseteq \mathfrak{g}_{3(j+1)\delta}$, that depends on I_0'', and I^{j+1}, that depends on $\mathfrak{a}_{3(j+1)\delta}$. We have

$$\mathfrak{g}_{3(j+1)\delta} = t^{3(j+1)} \otimes \mathfrak{h}_{\overline{0}}' = \mathbb{C}t^{3(j+1)} \otimes H_1 \oplus \mathbb{C}t^{3(j+1)} \otimes H_2.$$

We denote $[\lambda_1, \lambda_2] = \mathbb{C}\left(\lambda_1 t^{3(j+1)} \otimes H_1 + \lambda_2 t^{3(j+1)} \otimes H_2\right)$ for $(\lambda_1, \lambda_2) \in \mathbb{C}^2$.

Let $n = \#I_0''$. Since

$$I_0'' \subseteq \{3(j+1)\delta - \alpha_m\,;\ m = 0, 1, 2\} = \{3j\delta + \alpha_m\,;\ m = 23, 24, 25\},$$

we have $0 \leq n \leq 3$. Then we have to consider 3 cases:

Case 1: $n = 0$. Then $I_0'' = \emptyset$ and $\mathfrak{a} = \mathfrak{a}_{3(j+1)\delta} \oplus (\oplus_{\alpha\in I^{j+1}}\mathfrak{g}_\alpha) \oplus \mathfrak{b}$. There is only one ideal of Δ^j in this case: $I^j = I^j_{27} = \langle 3(j+1)\delta\rangle = \{3(j+1)\delta\}$ with $j \geq 0$. There are 2 possibilities:

(1.a) $\dim \mathfrak{a}_{3(j+1)\delta} = 1$. Then $\mathfrak{a}_{3(j+1)\delta} = [\lambda_1, \lambda_2]$ for $(\lambda_1, \lambda_2) \in \mathbb{P}^1$. In order to obtain I^{j+1} we will considerate some subcases:

(1.a.1) If $(\lambda_1, \lambda_2) = (1, 0)$, i.e. $\mathfrak{a}_{3(j+1)\delta} = \mathbb{C}t^{3(j+1)} \otimes H_1$, then $I^{j+1}_{34} = \langle\alpha_1^{j+1}; \alpha_2^{j+1}\rangle \subseteq I^{j+1}$ since $[\mathfrak{g}_{\alpha_0}, \mathfrak{a}_{3(j+1)\delta}] = 0$, $[\mathfrak{g}_{\alpha_1}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$ and $[\mathfrak{g}_{\alpha_2}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$. Therefore $I^{j+1} = I^{j+1}_{34}$ or $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}^{j,(1,0)}_{27,1} = [1, 0] \oplus \left(\oplus_{\alpha>\alpha_0^{j+1}}\mathfrak{g}_\alpha\right)$$

or

$$\mathfrak{a} = \mathfrak{a}^{j,(1,0)}_{27} = [1, 0] \oplus \left(\oplus_{\alpha>3(j+1)\delta}\mathfrak{g}_\alpha\right),$$

respectively.

(1.a.2) If $(\lambda_1, \lambda_2) = (3,2)$, i.e. $\mathfrak{a}_{3(j+1)\delta} = \mathbb{C}\left(3t^{3(j+1)} \otimes H_1 + 2t^{3(j+1)} \otimes H_2\right)$, then $I_{29}^{j+1} = \langle \alpha_0^{j+1}; \alpha_2^{j+1} \rangle \subseteq I^{j+1}$ since $[\mathfrak{g}_{\alpha_0}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$, $[\mathfrak{g}_{\alpha_1}, \mathfrak{a}_{3(j+1)\delta}] = 0$ and $[\mathfrak{g}_{\alpha_2}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$. Therefore $I^{j+1} = I_{29}^{j+1}$ or $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}_{27,1}^{j,(3,2)} = [3,2] \oplus \mathfrak{g}_{\alpha_0^{j+1}} \oplus \left(\oplus_{\alpha > \alpha_1^{j+1}} \mathfrak{g}_\alpha\right)$$

or

$$\mathfrak{a} = \mathfrak{a}_{27}^{j,(3,2)} = [3,2] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right),$$

respectively.

(1.a.3) If $(\lambda_1, \lambda_2) = (2,1)$, i.e. $\mathfrak{a}_{3(j+1)\delta} = \mathbb{C}\left(2t^{3(j+1)} \otimes H_1 + t^{3(j+1)} \otimes H_2\right)$, then $I_{28}^{j+1} = \langle \alpha_0^{j+1}; \alpha_1^{j+1} \rangle \subseteq I^{j+1}$ since $[\mathfrak{g}_{\alpha_0}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$, $[\mathfrak{g}_{\alpha_1}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$ and $[\mathfrak{g}_{\alpha_2}, \mathfrak{a}_{3(j+1)\delta}] = 0$. Therefore $I^{j+1} = I_{28}^{j+1}$ or $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}_{27,1}^{j,(2,1)} = [2,1] \oplus \mathfrak{g}_{\alpha_0^{j+1}} \oplus \mathfrak{g}_{\alpha_1^{j+1}} \oplus \left(\oplus_{\alpha > \alpha_2^{j+1}} \mathfrak{g}_\alpha\right)$$

or

$$\mathfrak{a} = \mathfrak{a}_{27}^{j,(2,1)} = [2,1] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right),$$

respectively.

(1.a.4) For the remaining values of $(\lambda_1, \lambda_2) \in \mathbb{P}^1$, we have $I^{j+1} = \Delta^{j+1}$ since $[\mathfrak{g}_{\alpha_i}, \mathfrak{a}_{3(j+1)\delta}] \neq 0$ for $i = 0, 1, 2$. Then

$$\mathfrak{a} = \mathfrak{a}_{27}^{j,(\lambda_1,\lambda_2)} = [\lambda_1, \lambda_2] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right)$$

(1.b) $\dim \mathfrak{a}_{3(j+1)\delta} = 2$. Then $\mathfrak{a}_{3(j+1)\delta} = \mathfrak{g}_{3(j+1)\delta}$, $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}_{27}^{j} = \mathfrak{g}_{3(j+1)\delta} \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right) = \oplus_{\alpha \geq 3(j+1)\delta} \mathfrak{g}_\alpha.$$

So we have obtained the following set of ideals in $\mathfrak{n}_+$ for $n = 0$:

$$\{\mathfrak{a}_{27}^{j,(\lambda_1,\lambda_2)};\ (\lambda_1, \lambda_2) \in \mathbb{P}^1\} \cup \{\mathfrak{a}_{27,1}^{j,(1,0)}, \mathfrak{a}_{27,1}^{j,(3,2)}, \mathfrak{a}_{27,1}^{j,(2,1)}\} \cup \{\mathfrak{a}_{27}^{j}\}.$$

Case 2: $n = 1$. There are 3 ideals in this case: I_{24}^j, I_{25}^j, I_{26}^j with $j \geq 0$.

If we call γ the unique element of I_0'', then

$$\mathfrak{a} = \mathfrak{g}_\gamma \oplus \mathfrak{a}_{3(j+1)\delta} \oplus (\oplus_{\alpha \in I^{j+1}} \mathfrak{g}_\alpha) \oplus \mathfrak{b}.$$

$\mathfrak{a}$ contains the ideal generated by $\mathfrak{g}_\gamma$ and $\langle \mathfrak{g}_\gamma \rangle = \mathfrak{g}_\gamma \oplus \mathfrak{a}_\gamma \oplus \left(\oplus_{\alpha \in I_\gamma^{j+1}} \mathfrak{g}_\alpha\right) \oplus \mathfrak{b}$ with $\mathfrak{a}_\gamma$ a subspace of dimension 1 in $\mathfrak{g}_{3(j+1)\delta}$ and I_γ^{j+1} an ideal in Δ^{j+1} contained in I^{j+1}.

There are three ideals in this case and some possibilities for each one:

(2.a) dim $\mathfrak{a}_{3(j+1)\delta} = 1$. We have

$$\mathfrak{a}_\gamma = \begin{cases} [0,1] & \text{for } i = 24 \\ [1,0] & \text{for } i = 25 \\ [3,2] & \text{for } i = 26 \end{cases}$$

since

$$\gamma = \begin{cases} \alpha^j_{23} = 3j\delta + \alpha_{23} = 3(j+1)\delta - \alpha_2 & \text{for } I = I^j_{24} \\ \alpha^j_{24} = 3j\delta + \alpha_{24} = 3(j+1)\delta - \alpha_1 & \text{for } I = I^j_{25} \\ \alpha^j_{25} = 3j\delta + \alpha_{25} = 3(j+1)\delta - \alpha_0 & \text{for } I = I^j_{26} \end{cases}$$

Now we will determine I^{j+1}_γ:

$\boxed{\gamma = \alpha^j_{23}}$ Since $[\mathfrak{g}_{\alpha_i}, \mathfrak{a}_{\alpha^j_{23}}] \neq 0$ for $i = 0,1,2$, we have $I^{j+1}_{\alpha^j_{23}} = \Delta^{j+1}$. Therefore $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}^j_{24,1} = \mathfrak{g}_{\alpha^j_{23}} \oplus [0,1] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right).$$

$\boxed{\gamma = \alpha^j_{24}}$ Since $[\mathfrak{g}_{\alpha_0}, \mathfrak{a}_{\alpha^j_{24}}] = 0$, $[\mathfrak{g}_{\alpha_1}, \mathfrak{a}_{\alpha^j_{24}}] \neq 0$ and $[\mathfrak{g}_{\alpha_2}, \mathfrak{a}_{\alpha^j_{24}}] \neq 0$, we have $I^{j+1}_{\alpha^j_{24}} = I^{j+1}_{34} = \langle \alpha^{j+1}_1, \alpha^{j+1}_2 \rangle$. Therefore $I^{j+1} = \Delta^{j+1}$ or $I^{j+1} = I^{j+1}_{34}$ and

$$\mathfrak{a} = \mathfrak{a}^j_{25,1} = \mathfrak{g}_{\alpha^j_{24}} \oplus [1,0] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right)$$

or

$$\mathfrak{a} = \mathfrak{a}^j_{25,2} = \mathfrak{g}_{\alpha^j_{24}} \oplus [1,0] \oplus \left(\oplus_{\alpha > \alpha^{j+1}_0} \mathfrak{g}_\alpha\right),$$

respectively.

$\boxed{\gamma = \alpha^j_{25}}$ Since $[\mathfrak{g}_{\alpha_0}, \mathfrak{a}_{\alpha^j_{25}}] \neq 0$, $[\mathfrak{g}_{\alpha_1}, \mathfrak{a}_{\alpha^j_{25}}] = 0$ and $[\mathfrak{g}_{\alpha_2}, \mathfrak{a}_{\alpha^j_{25}}] \neq 0$, we have $I^{j+1}_{\alpha^j_{25}} = I^{j+1}_{29} = \langle \alpha^{j+1}_0, \alpha^{j+1}_2 \rangle$. Therefore $I^{j+1} = \Delta^{j+1}$ or $I^{j+1} = I^{j+1}_{29}$ and

$$\mathfrak{a} = \mathfrak{a}^j_{26,1} = \mathfrak{g}_{\alpha^j_{25}} \oplus [3,2] \oplus \left(\oplus_{\alpha > 3(j+1)\delta} \mathfrak{g}_\alpha\right)$$

or

$$\mathfrak{a} = \mathfrak{a}^j_{26,2} = \mathfrak{g}_{\alpha^j_{25}} \oplus [3,2] \oplus \mathfrak{g}_{\alpha^{j+1}_0} \oplus \left(\oplus_{\alpha > \alpha^{j+1}_1} \mathfrak{g}_\alpha\right),$$

respectively.

(2.b) dim $\mathfrak{a}_{3(j+1)\delta} = 2$. Then $\mathfrak{a}_{3(j+1)\delta} = \mathfrak{g}_{3(j+1)\delta}$, $I^{j+1} = \Delta^{j+1}$ and

$$\mathfrak{a} = \mathfrak{a}^j_i = \mathfrak{g}_{\alpha^{j+1}_{i-1}} \oplus \left(\oplus_{\alpha \geq 3(j+1)\delta} \mathfrak{g}_\alpha\right) \text{ for } i = 24, 25, 26.$$

So we have obtained the following set of ideals in $\mathfrak{n}_+$ for $n = 1$:

$$\{\mathfrak{a}^j_{i,1}\,;\, i = 24, 25, 26\} \cup \{\mathfrak{a}^j_{i,2}\,;\, i = 25, 26\} \cup \{\mathfrak{a}^j_i\,;\, i = 24, 25, 26\}.$$

Case 3: $n = 2, 3$. The other ideals are in this case.

$n = 2, 3$ implies $\mathfrak{a}_{3(j+1)\delta} = \mathfrak{g}_{3(j+1)\delta}$ and $I^{j+1} = \Delta^{j+1}$. Therefore

$$\mathfrak{a} = \mathfrak{a}_i^j = \oplus_{\alpha \in I} \mathfrak{g}_\alpha.$$

So we have obtained an unique ideal in $\mathfrak{n}_+$ for $n = 2, 3$ and each possible I_i^j: $\mathfrak{a}_i^j$.

We have obtained the ideals of $\mathfrak{n}_+$ included in $\mathfrak{n}_{++}$. Since $G = \{id\}$ we can identify the sets $\mathcal{I}(\mathfrak{n}_{++})$ and $\mathcal{I}(\mathfrak{n}_{++})/G$. Now we obtain a representative of each isomorphism class of nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_4^{(3)}$, building the quotient $\mathfrak{n}_+/\mathfrak{a}$ for each above obtained ideal $\mathfrak{a}$.

As a consequence of this study we have the following result:

Theorem 4.1. *Up to isomorphism there are exactly:*

(a) 88 *infinite series with discrete parameters:*

$$D_{4,i}^{(3),j} \begin{cases} \text{for } i = 6, 8, 10, \dots, 27, 42, 44, 46, \dots, 76, 78, 79, 80;\ j \geq 0 \\ \text{for the remaining values of } i;\ j \geq 1 \end{cases}$$

$$D_{4,i,1}^{(3),j} \ \{\ i = 24, 25, 26; j \geq 0$$

$$D_{4,i,2}^{(3),j} \ \{\ i = 25, 26; j \geq 0$$

$$D_{4,27,1}^{(3),j,(\lambda_1,\lambda_2)} \ \{\ (\lambda_1, \lambda_2) = (1, 0), (3, 2), (2, 1); j \geq 0$$

(b) 1 *infinite series with continuous parameter:*

$$D_{4,27}^{(3),j,(\lambda_1,\lambda_2)} \ \text{for } (\lambda_1, \lambda_2) \in \mathbb{P}^1; j \geq 0$$

of nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_4^{(3)}$.

5. An example

In this section we give explicitly the Lie algebra $D_{4,42}^{(3),0}$, which is the nilpotent Lie algebra of maximal rank and of Kac-Moody type $D_4^{(3)}$ associated with the ideal $\mathfrak{a}_{42}^0$ of $\mathfrak{n}_+$. Then we have:

$$D_{4,42}^{(3),0} = \mathfrak{n}_+/\mathfrak{a}_{42}^0$$

Since $\mathfrak{a}_{42}^0 = \oplus_{\alpha \in \langle \alpha_5, \alpha_{11} \rangle} \mathfrak{g}_\alpha$ we can identify $D_{4,42}^{(3),0}$ with the following Lie algebra:

$$\bigoplus_{\alpha \in \Delta_+ \setminus \langle \alpha_5, \alpha_{11} \rangle} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Delta_+ \setminus \Delta_{++}} \mathfrak{g}_\alpha = \left(\oplus_{i=0}^4 \mathfrak{g}_\alpha \right) \oplus \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_8}$$

where the root subspaces are:

$$\begin{array}{llll} \mathfrak{g}_{\alpha_0} = t \otimes \mathfrak{g}'_{\bar{1}, -\alpha_1 - 2\alpha_2} = \mathbb{C}e_0, & \mathfrak{g}_{\alpha_1} = 1 \otimes \mathfrak{g}'_{\bar{0}, \alpha_1} = \mathbb{C}e_1, \\ \mathfrak{g}_{\alpha_2} = 1 \otimes \mathfrak{g}'_{\bar{0}, \alpha_2} = \mathbb{C}e_2, & \mathfrak{g}_{\alpha_3} = t \otimes \mathfrak{g}'_{\bar{1}, -\alpha_1 - \alpha_2} = \mathbb{C}e_3, \\ \mathfrak{g}_{\alpha_4} = 1 \otimes \mathfrak{g}'_{\bar{0}, \alpha_1 + \alpha_2} = \mathbb{C}e_4, & \mathfrak{g}_{\alpha_6} = 1 \otimes \mathfrak{g}'_{\bar{0}, \alpha_1 + 2\alpha_2} = \mathbb{C}e_6, \\ \mathfrak{g}_{\alpha_8} = 1 \otimes \mathfrak{g}'_{\bar{0}, \alpha_1 + 3\alpha_2} = \mathbb{C}e_8. & \end{array}$$

We have

$$D_{4,42}^{(3),0} = \left(\oplus_{i=0}^{4}\mathbb{C}e_i\right) \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_8$$

with brackets

$$[e_0, e_2] = (1+\epsilon^2)e_3 = -\epsilon e_3, \quad [e_1, e_2] = -e_4, \quad [e_2, e_4] = 2e_6, \quad [e_2, e_6] = 3e_8.$$

This Lie algebra is the nilpotent Lie algebra of maximal rank and of Kac-Moody type $D_4^{(3)}$ of least dimension (up to isomorphism).

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