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Lengths of Involutions in Coxeter Groups

Sarah B. Perkins and Peter J. Rowley

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Abstract. Let t be an involution in a Coxeter group W. We determine the minimal and maximal (in the case of finite W) length of an involution in the conjugacy class of t.
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Let W be a finitely generated Coxeter group whose distinguished set – the set of fundamental reflections – is R. The length $l(w)$ of a non-trivial element w in W is defined to be

$$l(w) = \min\{l \in \mathbb{N} : w = r_1 r_2 \cdots r_l \text{ some } r_i \in R\}$$

and $l(1) = 0$. Suppose t is an involution in W, and let $C = t^W$ be the conjugacy class of t in W. The aim of this short paper is to determine the minimal and maximal (in which case W is assumed finite) length of an involution in C.

Associated to any Coxeter group W is the root system Φ, which is the disjoint union of its positive and negative roots (denoted Φ^+ and Φ^- respectively). The fundamental reflections $r \in R$ are in one-to-one correspondence with the fundamental roots $\alpha_r, r \in R$ and W acts faithfully on Φ (see [1]). For $w \in W$, define $N(w) := \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\}$, $I(w) := \{\alpha \in \Phi^+ : w \cdot \alpha = -\alpha\}$ and $\mathrm{Fix}(w) := \{\alpha \in \Phi^+ : w \cdot \alpha = \alpha\}$. It is well known that for each $w \in W$, $l(w) = |N(w)|$. For $J \subseteq R$, write W_J for the (Coxeter) group generated by J, Φ_J for its root system and, when it is finite, w_J for the unique longest element of W_J. Our main result is given in

Theorem 1.1. *Suppose t is an involution in W, and put $C = t^W$. We have*

(i) $\min_{s \in C}\{l(s)\} = |I(t)|$ *and if x is of minimal length in C, then $x = w_J$ for some $J \subseteq R$.*

(ii) *If W is finite, then* $\max_{s \in C}\{l(s)\} = |\Phi^+| - |\mathrm{Fix}(t)|$ *and for y of maximal length in C, $y = w_K w_R$ for some $K \subseteq R$.*

Put another way, Theorem 1.1 is saying that the maximum and minimum length in a conjugacy class of involutions may be obtained by examining the action on Φ

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of any one involution in that class. We remark that part (i) appears as Theorem A (a) in [3]. We include a (shorter, and different) proof here to emphasise the similarity between parts (i) and (ii).

Proof. Let t be an involution and $C = t^W$. Note that for any $t' \in C$, $|I(t')| = |I(t)|$ and $|\mathrm{Fix}(t')| = |\mathrm{Fix}(t)|$, because $t \cdot \alpha = \pm\alpha$ if and only if $t^g \cdot (g \cdot \alpha) = \pm(g \cdot \alpha)$, for each $g \in W$. It is clear from this that the length of any involution in C is at least $|I(t)|$ and at most $|\Phi^+| - |\mathrm{Fix}(t)|$. Let $r \in R$ with $\alpha_r \notin N(t)$, and suppose $\alpha_r \notin \mathrm{Fix}(t)$. It is well known that for any $w \in W$, $r \in R$, $l(wr) > l(w)$ if and only if $w \cdot \alpha_r \in \Phi^+$. We have $t \cdot \alpha_r \in \Phi^+ \setminus \{\alpha_r\}$, so $rt \cdot \alpha_r \in \Phi^+$. Therefore $l(rtr) > l(rt)$. Now $rt = (tr)^{-1}$, hence $l(rt) = l(tr) > l(t)$, since $\alpha_r \notin N(t)$. Thus $l(rtr) > l(t)$. Suppose now that $\alpha_r \in N(t)$ with $\alpha_r \notin I(t)$. We have $l(rtr) < l(rt)$ because $rt \cdot \alpha_r \in \Phi^-$, and $l(rt) = l(tr) < l(r)$ because $\alpha_r \in N(t)$. Thus $l(rtr) < l(t)$.

We have shown that if $\alpha_r \notin N(t)$, then either $l(rtr) > l(t)$ or $\alpha_r \in \mathrm{Fix}(t)$, and that if $\alpha_r \in N(t)$, then either $l(rtr) < l(t)$ or $\alpha_r \in I(t)$. Thus for each x of minimal length in C, there exists $J \subseteq R$ with $\alpha_r \in I(x)$ for each $r \in J$ and $\alpha_r \notin N(x)$ when $r \notin J$. Let $r \in J$. Then $w_J x \cdot \alpha_r = -w_J \cdot \alpha_r \in \Phi^+$. If $r \notin J$ then $w_J x \cdot \alpha_r \in \Phi^+$ unless $x \cdot \alpha_r \in \Phi_J^+$. But this would imply that $x^2 \cdot \alpha_r = -x \cdot \alpha_r \neq \alpha_r$, which is impossible. Thus $N(w_J x) = \emptyset$ and hence $x = w_J$. Now $N(x) = \Phi_J^+ = I(x)$ and so x has length $|I(t)|$ in C, which is minimal.

Similarly, when W is finite, for y of maximal length in C there exists $K \subseteq R$ with $\alpha_r \in \mathrm{Fix}(y)$ whenever $r \in K$, and $\alpha_r \in N(y)$ for $r \notin K$. We claim that $\mathrm{Fix}(y) = \Phi_K^+$. Certainly $\Phi_K^+ \subseteq \mathrm{Fix}(y)$. For the reverse inclusion, let $\alpha = \sum_{r\in R} \lambda_r \alpha_r \in \mathrm{Fix}(y)$ (where each $\lambda_r \geq 0$). Now $y \cdot \alpha_r \in \Phi^-$ for all $r \in R \setminus K$, so $\sum_{r \in R\setminus K} \lambda_r y \cdot \alpha_r$ is a negative linear combination of roots, say $-\sum_{r\in R} \mu_r \alpha_r$ for some $\mu_r \geq 0$. We have $\sum_{r\in R} \lambda_r \alpha_r = \alpha = y \cdot \alpha = \sum_{r\in K}(\lambda_r - \mu_r)\alpha_r - \sum_{r\in R\setminus K} \mu_r \alpha_r$. For $r \in R\setminus K$ then, we see that $\lambda_r = -\mu_r$. Hence $\lambda_r = \mu_r = 0$. Therefore $\alpha \in \Phi_K^+$ and so $\mathrm{Fix}(y) \subseteq \Phi_K^+$.

Now for $r \in K$, $w_K y \cdot \alpha_r = w_K \cdot \alpha_r \in \Phi^-$. If $r \notin K$, $w_K y \cdot \alpha_r \in \Phi^+$ only when $y \cdot \alpha_r \in \Phi_K^-$, which is impossible. Consequently $N(w_K y) = \Phi^+$, that is $y = w_K w_R$ and $l(y) = |N(y)| = |\Phi^+| - |\Phi_K^+| = |\Phi^+| - |\mathrm{Fix}(y)|$ and this is the maximum possible length of an involution in C. ∎

We remark that it is necessary, as Proposition 1.3 shows, to assume, when W is irreducible, that W is finite in order for $\max_{s\in C}\{l(s)\}$ to be defined. We require the following lemma, which follows from the fact that the geometric representation of W is irreducible and faithful (see [1]).

Lemma 1.2. *([2], Lemma 2.3) Let W be an irreducible Coxeter group and let $\alpha \in \Phi$. Then W acts faithfully on the orbit $W \cdot \alpha$.*

Proposition 1.3. *Suppose W is an infinite irreducible Coxeter group. Then each conjugacy class of involutions in W contains elements of arbitrarily large length.*

Proof. Let t be an involution of W. Then, by Theorem 1.1, $I(t)$ is non-empty, so there exists $\alpha \in \Phi^+$ with $t \cdot \alpha = -\alpha$. Let $\beta \in W \cdot \alpha$. Then $\beta = w \cdot \alpha$ for some $w \in W$. Now $t^w \cdot \beta = wtw^{-1} \cdot (w \cdot \alpha) = -\beta$, whence $\beta \in N(t^w)$. Thus $W \cdot \alpha \subseteq \cup_{w \in W} N(t^w)$. Each element t^w has finite length, but $W \cdot \alpha$ is infinite, by Lemma 1.2, hence the conjugacy class of t must be infinite. Consequently, since there can only be finitely many elements of a given length in W, the conjugacy class of t must contain elements of arbitrarily large length. ■

References

[1] J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge Studies in Advanced Mathematics **29**, 1990.

[2] Perkins, S. B., and P. J. Rowley, *Bad Upward Elements in Infinite Coxeter Groups*, Manchester Centre for Pure Mathematics Preprint, 2000/6.

[3] Richardson, R. W., *Conjugacy classes of involutions in Coxeter groups*, Bull. Austral. Math. Soc. **30** (1982), 1–15.

S. B. Perkins and P. J. Rowley
Department of Mathematics,
UMIST, PO Box 88,
Manchester M60 1QD
United Kingdom
Sarah.Perkins@umist.ac.uk

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