Journal of Lie Theory
Volume **13** (2003) 291–306

The Exponential Map and Differential Equations on Real Lie Groups

Martin Moskowitz*and Richard Sacksteder

Communicated by K. H. Hofmann

Abstract. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, $\exp_{\mathrm{G}} : \mathfrak{g} \longrightarrow G$ the exponential map and $E(G)$ its range. $E^n(G)$ will denote the set of all n-fold products of elements of $E(G)$. G is called *exponential* if $E(G) = E^1(G) = G$. Since most real (or complex) connected Lie groups are not exponential, it is of interest to know that the weaker conclusion $E^2(G) = G$ is always true (Theorem 5.6). This result will be applied to prove Theorem 6.4, a generalized version of Floquet-Lyapunov theory for Lie groups. It will then be seen the property that a Lie group is exponential is equivalent to the existence of a special form of Floquet-Lyapunov theory for it (Corollary 6.3). Theorem 2.8, generalizes the well-known fact that connected nilpotent Lie groups are exponential. Our methods also provide alternative proofs of some known results by arguments which seem simpler and more natural than the usual ones. Among these is part of the classical Dixmier-Saito result, Theorem 5.8.

The method employed here stems from the earliest techniques of Lie theory. It exploits connections between the exponential map and differential equations on G, starting from the observation that the one parameter subgroup $g(t) = \exp_{\mathrm{G}}(t\gamma)$ corresponding to an element $\gamma \in \mathfrak{g}$ satisfies the differential equations

$$g'(t) = \mathrm{dL}_{g(t)}\gamma \quad \text{and} \quad g'(t) = \mathrm{dR}_{g(t)}\gamma$$

on G with the initial condition $g(0) = \mathrm{e_G}$. More generally here it will be necessary to consider differential equations corresponding to certain time dependent vector fields on G, or equivalently, certain time dependent cross-sections of the tangent bundle of G.

1. Introduction

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\exp_{\mathrm{G}} : \mathfrak{g} \longrightarrow G$ the exponential map. The groups discussed here will usually be connected and the underlying field is to be taken as real except in a few cases where it is explicitly stated to be complex. We denote by $\mathrm{e_G}$ the identity element and by $E(G)$ the range of the exponential map. (The subscripts on $\exp_{\mathrm{G}}$ and $\mathrm{e_G}$ will sometimes be omitted

*The research for this article was done while the first author was a recipient of a Collaborative Grant from the Research Foundation of CUNY. He would like to take this opportunity to thank the RF for its generous support.

ISSN 0949–5932 / \$2.50

when no confusion is possible.) $E(G)$ is a canonically defined subset of G which is invariant under inversion and all automorphisms of G. G is called *exponential* if $\exp_G$ is surjective, that is, if every element of G lies on a 1-parameter subgroup. Also, $E^n(G)$ will denote the set of all n-fold products of elements of G lying on one parameter subgroups. Thus G is exponential means that $E^1(G) = E(G) = G$. Other notation that will be used is: L_g and R_g stand for the left and right translations by the element g in G and the derivatives of these maps will be written as dL_g and dR_g.

Compact connected Lie groups are always exponential, since exp commutes with conjugation and any compact connected Lie group is the union of the conjugates of a maximal torus which is exponential. In the late fifties Dixmier [2] and Saito [19] independently proved that for simply connected solvable groups the exponential map is surjective (iff it is injective iff it is bijective iff it is a global diffeomorphism) and all of these are equivalent to the condition that the adjoint representation of the Lie algebra have no non-trivial purely imaginary roots. The question of which connected, noncompact real Lie groups, particularly semisimple and mixed groups, are exponential has been studied rather intensively in recent years. For example, the main result in [14], together with [15], shows that any noncompact, centerless, real rank 1, simple group, other than the exceptional one, is exponential. (D. Djokovic and Nguyêñ Thăńg in [3] show that the exceptional group fails to be exponential). In semisimple groups of higher rank and in mixed groups the situation is less favorable. A thorough summary of the status of these questions, as of 1997, in given in [4]. More recent results showing the complexity of the situation can be found in [17], [18], and [24].

Since most real (or complex) connected Lie groups are not exponential, it is of interest to know that the weaker conclusion $E^2(G) = G$ is always true, as will be shown by one of our main results, Theorem 5.6. This will then be applied to prove Theorem 6.4, a generalized version of Floquet-Lyapunov theory for Lie groups. It will be seen the property that a Lie group is exponential is equivalent to the existence of a special form of Floquet-Lyapunov theory for it (Corollary 6.3). Another result, Theorem 2.8, generalizes the well-known fact that connected nilpotent Lie groups are exponential. Our methods also provide alternative proofs of some known results by arguments which seem simpler and more natural than the usual ones. Among these is part of the Dixmier-Saito result, Therorem 5.8. The authors would like to take the opportunity here to thank the referees for pointing out a gap in an earlier version of the paper.

The method employed here stems from the earliest techniques of Lie theory. It exploits connections between the exponential map and differential equations on G, starting from the observation that the one parameter subgroup $g(t) = \exp_G(t\gamma)$ corresponding to an element $\gamma \in \mathfrak{g}$ satisfies the differential equations

$$g'(t) = \mathrm{dL}_{g(t)}\gamma \quad \text{and} \quad g'(t) = \mathrm{dR}_{g(t)}\gamma$$

on G with the initial condition $g(0) = \mathrm{e}_G$. More generally here it will be necessary to consider differential equations corresponding to time dependent vector fields on G, or equivalently, time dependent cross-sections of the tangent bundle of G. The cross-sections needed here are of a special form that can be represented as follows: Let $\gamma(t)$ be a C^1 function of the real parameter t with values in the Lie algebra

$\mathfrak{g}$. Then a cross-section is defined by $\mathrm{dL}_g\gamma(t)$ or $\mathrm{dR}_g\gamma(t)$ and the corresponding differential equations are

$$g'(t) = \mathrm{dL}_{g(t)}\gamma(t) \tag{1}$$

and

$$g'(t) = \mathrm{dR}_{g(t)}\gamma(t). \tag{2}$$

The existence, uniqueness, and smoothness of solutions of such equations present no difficulties, but usually it is not possible to give explicit formulas for them. In the next section we discuss the choices of $\gamma(t)$ that are of interest here and the properties of the corresponding solutions.

2. Properties of Solutions

In what follows: G will denote a connected Lie group, W a closed connected, normal subgroup, $H = G/W$ the quotient group, and $\pi : G \longrightarrow H = G/W$ the projection. The corresponding map of Lie algebras is denoted by $\pi' : \mathfrak{g} \longrightarrow \mathfrak{h} = \mathfrak{g}/\mathfrak{w}$. Then corresponding, for example, to equation (1) there is the differential equation

$$h'(t) = \mathrm{dL}_{h(t)}\pi'\gamma(t) \tag{3}$$

on H.

Proposition 2.1. *If $g(t)$ is the solution of equation* (1) *with $g(0) = \mathrm{e_G}$, then $h(t) = \pi g(t)$ is the solution to equation* (3) *with $h(0) = \mathrm{e_H}$.*

Proof. Since $\pi(gg') = \pi(g)\pi(g')$, holding g fixed and differentiating π at $g' = \mathrm{e_G}$ with respect to γ gives $\mathrm{d}\pi_g\mathrm{dL}_g\gamma = \mathrm{dL}_{\pi g}\mathrm{d}\pi_{\mathrm{e_G}}\gamma$. Noting that $\mathrm{d}\pi_{\mathrm{e_G}}\gamma = \pi'\gamma$ and setting $g = g(t)$ gives

$$\mathrm{d}\pi_{g(t)}\mathrm{dL}_{g(t)}\gamma = \mathrm{dL}_{\pi g(t)}\pi'\gamma = \mathrm{dL}_{h(t)}\pi'\gamma.$$

This shows that the image of the tangent to the solution to (1) is the tangent to the solution to (3), which implies the conclusion of the proposition. ■

Other standing notation will be: D will denote a derivation of $\mathfrak{g}$. D is then a linear map of $\mathfrak{g}$ into itself as is $\mathrm{Exp}(tD)$ for any real t. Then we shall be concerned with differential equations like (1) where $\gamma(t) = \mathrm{Exp}(tD)\gamma$ for γ a fixed element of $\mathfrak{g}$. The next proposition is well-known and is the source of the interest in derivations here.

Proposition 2.2. *The automorphism group of $\mathfrak{g}$, $Aut(\mathfrak{g})$, is a linear Lie group with Lie algebra $Der(\mathfrak{g})$, the set of derivations of $\mathfrak{g}$. A derivation D in $Der(\mathfrak{g})$ corresponds to the one-parameter subgroup $M(t) = \exp_{Aut(\mathfrak{g})}(tD) = \mathrm{Exp}(tD)$ of $Aut(\mathfrak{g})$.*

The following proposition is proved by an obvious formal argument, which we omit.

Proposition 2.3. *If $D(\mathfrak{w}) \subseteq \mathfrak{w}$, a derivation D_π of $\mathfrak{h}$ is defined unambiguously by $D_\pi \pi' \gamma = \pi' D\gamma$ for $\gamma \in \mathfrak{g}$. Moreover, if $\gamma(t) = \mathrm{Exp}(tD)\gamma$ then $\pi'(\gamma(t)) = \mathrm{Exp}(tD_\pi)\pi'\gamma$.*

Suppose that $g(t)$ is the solution to (1) with $g(0) = \mathrm{e_G}$ and $\gamma(t) = \mathrm{Exp}(tD)\gamma$. A map $\mathcal{E}_G^D : \mathfrak{g} \longrightarrow G$ is defined by setting $\mathcal{E}_G^D(\gamma) = g(1)$. When $D = 0$, this is just the exponential map. If $D(\mathfrak{w}) \subseteq \mathfrak{w}$, there are maps $\mathcal{E}_H^{D_\pi}$ and $\mathcal{E}_W^D$ defined similarly, where it is understood that the derivation for W is the restriction of D and for H is D_π as in Proposition 2.3. The next proposition also needs no proof.

Proposition 2.4. *Suppose that $D(\mathfrak{w}) \subseteq \mathfrak{w}$ and $\mathcal{E}_W^D$ and $\mathcal{E}_H^{D_\pi}$ are onto. Then $G = \mathcal{E}_G^D(\mathfrak{g})\mathcal{E}_W^D(\mathfrak{w})$.*

Theorem 2.5. *Assume the hypotheses of* Proposition 2.4 *and that, in addition, W is a subgroup of the center of G. Then $G = \mathcal{E}_G^D(\mathfrak{g})$.*

Remarks. Note that if W is the connected component of the identity in the center of G, the hypothesis that $D(\mathfrak{w}) \subseteq \mathfrak{w}$ is automatically satisfied. For conditions to imply that $\mathcal{E}_W^D(\mathfrak{w}) = W$ when W is a vector group, see Lemma 2.7 below.

Proof. By Proposition 2.4 any any element of G is a product $g = \mathcal{E}_G^D(\phi)\mathcal{E}_W^D(\omega)$ for $\phi \in \mathfrak{g}$ and $\omega \in \mathfrak{w}$. Set $\gamma = \phi + \omega$, and note that $\mathrm{Exp}(tD)\gamma = \mathrm{Exp}(tD)\phi + \mathrm{Exp}(tD)\omega$. Let $w(t)$ and $h(t)$ denote respectively the solutions to

$$w'(t) = \mathrm{dL}_{w(t)}\mathrm{Exp}(tD)\omega, \text{ and } h'(t) = \mathrm{dL}_{h(t)}\mathrm{Exp}(tD)\phi$$

that pass through $\mathrm{e_G}$ when $t = 0$. Since D maps $\mathfrak{w}$ into itself, $w(t)$ is in W, hence $w(s)h(t) = h(t)w(s)$. This equality implies that $w'(t)h(t) = h(t)w'(t)$, so if $g(t) = w(t)h(t)$,

$$g'(t) = h(t)w'(t) + w(t)h'(t) = \mathrm{dL}_{h(t)}\mathrm{dL}_{w(t)}\mathrm{Exp}(tD)\omega +$$

$$\mathrm{dL}_{w(t)}\mathrm{dL}_{h(t)}\mathrm{Exp}(tD)\phi = \mathrm{dL}_{g(t)}\mathrm{Exp}(tD)\gamma.$$

Thus, $g(t)$ satisfies (1) with $\gamma(t) = \mathrm{Exp}(tD)\gamma$. Setting $t = 1$, it follows that $\mathcal{E}_G^D(\gamma) = g(1) = w(1)h(1) = \mathcal{E}_W^D(\omega)\mathcal{E}_G^D(\phi) = g$. That is g is in $\mathcal{E}_G^D(\mathfrak{g})$. ∎

The following is an analogue of a notion by the same name given in [16] (p. 325): An endomorphism of a real or complex vector space is said to be of *type E* if it has no eigenvalues equal to $2\pi i n$, where $n \neq 0$ is an integer. A real Lie algebra $\mathfrak{g}$ is said to be of *type E* if the adjoint map ad_γ corresponding to each of its elements γ is of *type E* as an endomorphism of $\mathfrak{g}$.

The easy proof of the next proposition is omitted.

Proposition 2.6. *Assume the conditions of* Proposition 2.3. *Suppose that D has eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ (repeated according to multiplicity) and that $\{\lambda_1, \ldots, \lambda_m\}$ are those that correspond to the ideal $\mathfrak{w}$. Then the eigenvalues of D_π are $\{\lambda_{m+1}, \ldots, \lambda_n\}$*

Lemma 2.7. *Suppose that W is a real simply connected Abelian Lie group and D is a derivation of $\mathfrak{w}$. Then $\mathcal{E}_W^D$ is one-to-one and onto if D is of type E. If D is not of type E, $\mathcal{E}_W^D$ is of codimension at least two in W.*

Proof. One can verify by direct computation that for $\omega \in \mathfrak{w}$, $\mathcal{E}_W^D(\omega) = \int_0^1 \mathrm{Exp}(tD)\mathrm{d}t\omega$. This equality can be written in terms of functional calculus as $\mathcal{E}_W^D(\omega) = F(D)\gamma$, where $F(z)$ is the holomorphic function $\int_0^1 \mathrm{e}^{tz}\mathrm{d}t$. The spectral mapping theorem shows that $F(D)$ has a zero eigenvalue if and only if D has an eigenvalue λ such that $F(\lambda) = 0$. But $F(0) = 1$ and $F(z) = (\mathrm{e}^z - 1)z^{-1}$ otherwise, so $F(D)$ has a 0 eigenvalue if and only if D is of type E. Since W is real and the non-real eigenvalues occur in conjugate pairs. Thus any zero eigenvalue of $F(D)$ has multiplicity at least two and $\mathcal{E}_W^D$ is of codimension at least two. ■

The next theorem is a generalization of the well-known result that a connected nilpotent Lie group is exponential, which is the case $D = 0$.

Theorem 2.8. *Suppose that G is a connected nilpotent group and D is a derivation of its Lie algebra $\mathfrak{g}$ which is of type E. Then $\mathcal{E}_G^D$ is onto. Conversely, if D is not of type E, $\mathcal{E}_G^D$ is the image of a subspace of $\mathfrak{g}$ of codimension at least two.*

Proof. Suppose that D is of type E, let W be the identity component of the center of G and $H = G/W$. Obviously, $D(\mathfrak{w}) \subseteq \mathfrak{w}$ so that Propositions 2.1 and 2.3 apply. The proof goes by induction on the dimension of G. Proposition 2.6 shows that the hypotheses of the theorem apply to the nilpotent Lie group H. $\mathcal{E}_W^D$ is onto by Lemma 2.7 and $\mathcal{E}_H^D$ is onto by the induction hypothesis. Theorem 2.5 shows that $\mathcal{E}_G^D$ is onto.

If D is not of type E, Proposition 2.6, shows that either $D|\mathfrak{w}$ or D_π is not of type E. In the first case, Lemma 2.7 applies to the Abelian Lie group W and there is a subspace of codimension two of $\mathfrak{w} \subseteq \mathfrak{g}$ that $\mathcal{E}_G^D$ maps to e_G, hence the conclusion follows. If D_π is not type E, the argument can be applied to the nilpotent group H and the identity component of its center. Continuing in this way, evenually a center will not be of type E, or the quotient will be Abelian. In either case the result follows. ■

The following lemma will be needed later.

Lemma 2.9. *Suppose that G is a simply connected solvable Lie group and that D is a derivation of $\mathfrak{g}$ of type E. If W is the commutator subgroup $[G, G]$, $G = \mathcal{E}_G^D(\mathfrak{g})\mathcal{E}_W^D(\mathfrak{w})$. Taking $D = 0$ gives $E^2(G) = G$.*

Proof. Now W and $H = G/W$ are simply connected (cf. [10] pp. 135-136). Clearly, $\mathfrak{w}$ is D-stable. Theorem 2.8 shows that $\mathcal{E}_W^D$ is onto, since W is nilpotent. Since H is abelian Proposition 2.6 and Lemma 2.7 show that $\mathcal{E}_H^D$ is onto. Proposition 2.4 yields the conclusion. ■

3. The exponential map on the semidirect product of Lie groups

The interest here in time-dependent vector fields defined, for example, by $\gamma(t) = \mathrm{Exp}(tD)\gamma$ stems from the exponential map on semi-direct products of Lie groups. Suppose W and H are connected Lie groups and $A : W \times H \to W$ defines an analytic homomorphism from H to $Aut(W)$, the group of automorphisms of W. Then A satisfies the functional relations:

$$A(ww', h) = A(w, h)A(w', h); \qquad A(\mathrm{e_W}, h) = \mathrm{e_W}. \tag{4}$$

$$A(w, hh') = A(A(w, h'), h); \qquad A(w, \mathrm{e_H}) = w. \tag{5}$$

The semidirect product Lie group structure is defined on the product manifold $W \times H = G$ by the multiplication law

$$(w, h)(w', h') = (wA(w', h), hh'). \tag{6}$$

Of course, the map A is the restriction to W of an inner automorphism of G. Lie algebras can be viewed as the tangent spaces at the identities of the corresponding groups. Thus, $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{h}$ as a vector space, but the bracket depends on the map A. Suppose that (ω, η) is in $\mathfrak{w} \oplus \mathfrak{h}$ and $\omega^* = (\omega, 0)$, $\eta^* = (0, \eta)$. To view $(\omega, \eta) = \omega^* + \eta^*$ as a left invariant vector field on G one first observes that by (6) $\mathrm{dL}_{(w,h)}(\omega, \eta) = (\mathrm{dL}_w(\partial_W A(e_{\mathrm{W}}, h))(\omega), \mathrm{dL}_h(\eta))$. Here ∂_W denotes partial differentiation with respect to the W-variables. Therefore by the remarks on one parameter subgroups in Section 1, if $\exp_{\mathrm{G}}(t(\omega, \eta)) = (w(t), h(t))$, then $h(t) = \exp_{\mathrm{H}}(t\eta)$ and $w(t)$ is the solution to the following initial value problem on W.

$$w'(t) = \mathrm{dL}_{w(t)}(\partial_W A(e_{\mathrm{W}}, h(t))(\omega); \qquad w(0) = \mathrm{e_W}. \tag{7}$$

To obtain alternate expressions for $w(t) = w(t, \eta, \omega)$ note that (5) implies that since $h(t)h(u) = h(t + u)$,

$$\partial_W A(e_{\mathrm{W}}, h(t + u)) = \partial_W A(e_{\mathrm{W}}, h(u)) \circ \partial_W A(e_{\mathrm{W}}, h(t)). \tag{8}$$

Setting $M(t, \eta) = \partial_W A(e_{\mathrm{W}}, h(t))$ (a map of $\mathfrak{w}$ to itself), we see by (5) that $M(0, \eta) = I$, the identity map. With this notation (8) becomes $M(t + u, \eta) = M(t, \eta)M(u, \eta)$. This means that $M(t, \eta)$ defines a one-parameter group of automorphisms of $\mathfrak{w}$ and therefore by Proposition 2.2, $M(t, \eta) = \mathrm{Exp}(tD)$, where $D = d/dtM(t, \eta)$ at $t = 0$ is a derivation of $\mathfrak{w}$. Noting that $M(t, \eta) = Ad_{h(t)}$ leads to $D = ad_{\eta*}$. (Here it is to be understood that $Ad_{h(t)}$ and $ad_{\eta*}$ are restricted to $\mathfrak{w}$.) Therefore (7) can be rewritten as

$$w'(t, \eta, \omega) = \mathrm{dL}_{w(t,\eta,\omega)}\mathrm{Exp}(tD)\omega; \qquad w(0, \eta, \omega) = \mathrm{e_W}. \tag{9}$$

The results of this section can be summarized as follows:

Theorem 3.1. *Let $G = W \times H$ be the semi-direct product of connected Lie groups W and H. Then any element η of $\mathfrak{h}$ determines a derivation $D = ad_\eta$ of $\mathfrak{g}$ hence also of $\mathfrak{w}$. If (ω, η) is an element of $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{h}$, the corresponding one-parameter subgroup of G is $g(t) = (w(t), h(t))$, where $h(t) = \exp_H(t\eta)$ and $w(t)$ is the solution to the initial value problem (9). In particular, $exp_G(\omega, \eta) = (\mathcal{E}_W^D(\omega), exp_H(\eta))$.*

4. Semisimple groups

A connected Lie group G is said to be of *compact type* if its Lie algebra, $\mathfrak{g}$ has a positive definite invariant symmetric bilinear form, or equivalently if G is locally isomorphic to some compact Lie group. In this case one also says $\mathfrak{g}$ is of compact type.

Lemma 4.1. *A connected Lie group of compact type is exponential.*

Proof. If the connected Lie group G is of compact type, then (see [6]) G is the direct product of a vector group V and a compact group C. Since V is central in G and its Lie algebra $\mathfrak{v}$ is central in $\mathfrak{g}$, it follows that $\exp_G(v+c) = \exp_G(v)\exp_G(c)$, where v and c are, respectively, in the Lie algebras of C and V. Since these commute, Lemma 4.1 follows from the fact that both compact and vector groups are exponential. ∎

Now let S be a connected semisimple Lie group and $\mathfrak{s}$ be its Lie algebra. The Iwasawa decomposition (see Wallach [23]) tells us $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and as a manifold S is a direct product of K, A and N, where K, A and N are the subgroups of S corresponding to $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Since N is normalized by A, AN is a simply connected solvable group all of whose roots are real. Therefore AN is of type E, that is, AN is exponential and $\exp_{AN}$ is a diffeomorphism by the Dixmier-Saito Theorem. The Lie algebra of K is of compact type. (We remark that if S were linear, or more generally had finite center, then K would actually be compact). In any case by Lemma 4.1 K is exponential. Hence, $E^2(S) = S$.

Definition: $E^+(G)$ will denote the subset of $E(G)$ consisting of those points $g = \exp_G(\gamma)$ such that for some choice of γ the map from γ to G is non-singular, that is, has non-vanishing Jacobian.

We now expand the remarks just above by showing that for a semisimple group S one has $S = E(S)E^+(S) = E^+(S)E(S)$. To do so we use a result of Harish-Chandra. It is Lemma 1 of [7] (see also Wallach [22] p. 53). Since this fact will play an important role in our argument for completeness we give a sketch of the proof.

Lemma 4.2. *Let $\mathfrak{h}$ be a semisimple Lie algebra of non-compact type with Iwasawa decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If $x = a + n$, where $a \in \mathfrak{a}$ and $n \in \mathfrak{n}$, then the characteristic polynomials of* ad_x *and* ad_a*, acting on $\mathfrak{h}$ are the same.*

Proof. First we make the following observation: Suppose T and U are block triangular operators on the same vector space V with identical block sizes and with T_i and U_i, $i = 1, \dots r$ the respective diagonal blocks. If, for each i, T_i and U_i are equal (or even conjugate), then T and U have the same characteristic polynomial.

Now let $\lambda_1 < \dots \lambda_r$ be an ordering of the roots and $\mathfrak{h}_{\lambda_i}$ be the corresponding root spaces. Put $\mathfrak{h}_i = \Sigma_{\lambda > \lambda_i} \mathfrak{h}_\lambda$, where $i = 1, \dots r$. Then

$$\mathfrak{h}_0 = \mathfrak{h} > \mathfrak{h}_1 > \dots \mathfrak{h}_r = (0),$$

these all being $\mathrm{ad}_{\mathfrak{a}+\mathfrak{n}}$ invariant. Since $[\mathfrak{n}, \mathfrak{h}_i] \subseteq \mathfrak{h}_{i+1}$, both $\mathrm{ad}_{\mathfrak{r}}$ and $\mathrm{ad}_{\mathfrak{a}}$ give the same endomorphism on each $\mathfrak{h}_i/\mathfrak{h}_{i+1}$. ∎

Lemma 4.3. *Let S be any real semisimple Lie group. Then $E(S)E^+(S) = E^+(S)E(S) = S$.*

Proof. Now the Lie algebra $\mathfrak{s}$ is a direct sum of a real compact semisimple Lie algebra $\mathfrak{k}_o$ with one of non-compact type, $\mathfrak{h}$. If K_0 is the subgroup corresponding to $\mathfrak{k}_0$ and KAN is the Iwasawa decomposition of H, the subgroup corresponding to $\mathfrak{h}$, then, as we saw, K_0K is exponential. As is well known ad_a acting on $\mathfrak{h}$ has only real eigenvalues. It follows from Lemma 4.2 that the same is true of ad_x for any $x \in \mathfrak{a} \oplus \mathfrak{n}$. Since $\mathfrak{k}_o$ and $\mathfrak{h}$ commute ad_x acting on $\mathfrak{s}$ also has only real eigenvalues. Because ad_x acting on $\mathfrak{s}$ has no purely imaginary eigenvalues, it follows from Corollary 5.2 below that $d(\exp_S)x$ is invertible for each $x \in \mathfrak{a} \oplus \mathfrak{n}$. Thus not only is everything in AN on a one parameter subgroup of AN and therefore of S, but actually $AN \subseteq E^+(S)$. Hence $E(S)E^+(S) = S$. Taking inverses proves the other equality in the statement of the lemma. ∎

5. The exponential map in general

The following classical theorem will be needed later. A proof using a connection can be found in Helgason [9]; another using a power series expansion is in Varadarajian [21] or Wallach [23]. There is another proof, known in the folklore of the subject, which is similar to the proof given by H. Hopf [11] for the discrete analogue. We give this proof below because it is more consistent with the methods used elsewhere in this paper.

Theorem 5.1. *Let G be a connected real Lie group and γ and $\zeta \in \mathfrak{g}$. The derivative of $\exp_{\mathrm{G}}(\gamma)$ evaluated at ζ is $\mathrm{dL}_{\exp(\gamma)}(\int_0^1 \mathrm{Exp}(-t\,\mathrm{ad}_\gamma)\mathrm{d}t(\zeta))$.*

Proof. Let $g(t) = \exp_{\mathrm{G}}(t\gamma)$, $k(s,t) = \exp_{\mathrm{G}}(t(\gamma + s\zeta))$, and $F(s,t) = g(-t)k(s,t)$. The partial derivative of F with respect to t is

$$\partial F/\partial t(s,t) = \mathrm{dL}_{g(-t)}\mathrm{dR}_{k(s,t)}(-\gamma) + \mathrm{dL}_{g(-t)}\mathrm{dR}_{k(s,t)}(\gamma + s\zeta) = \mathrm{dL}_{g(-t)}\mathrm{dR}_{k(s,t)}(s\zeta).$$

Here we have used (1) for $g(-t)$ and (2) for $k(s,t)$. The mixed partial is then $\partial^2 F/\partial t \partial s(0,t) = \mathrm{Ad}_{g(-t)}(\zeta)$, since $k(0,t) = g(t)$. The map $(t, \zeta) \mapsto \mathrm{Ad}_{g(-t)}(\zeta)$ defines a one-parameter group of automorphisms of $\mathfrak{g}$. Proposition 2.2, then gives $\mathrm{Ad}_{g(-t)}(\zeta) = \mathrm{Exp}(-t\,\mathrm{ad}_\gamma)(\zeta)$. Thus it has been shown that $\mathrm{Exp}(-t\,\mathrm{ad}_\gamma)(\zeta)$ is the derivative with respect to t of the composition of $\mathrm{dL}_{g(-t)}$ with the derivative of $\exp_{\mathrm{G}}(t\gamma)$ evaluated at ζ. Noting that the derivative at $t = 0$ is 0 determines the antiderivative and integrating from 0 to 1 gives the formula in the statement of the theorem. ∎

Corollary 5.2. *The map* $\exp_G$ *is non-singular at* γ *if and only if* ad_γ *is of type E.*

Lemma 5.3. *For any connected Lie group* G, $E^+(G)$ *is an open subset of* G *and its intersection with any one-parameter subgroup is open and dense.*

Proof. The openness is clear and the density is a consequence of Corollary 5.2, since any ray through the origin of $\mathfrak{g}$ contains a dense set of points γ such that ad_γ is of type E. ■

Lemma 5.4. *Let* H *be a connected subgroup of a Lie group* G. *Then*
$$E^+(H)E(H) = E(H)E^+(H) = H \cap (E^+(G)E^+(G)).$$

Proof. $H \cap (E^+(G)E^+(G)) \subseteq E^+(H)E(H) = E(H)E^+(H)$ is clear. Conversely, suppose that an element $h \in H$ is a product $h = h_1h_2$ of elements of $E(H)$ where one factor (e.g. h_1) is in $E^+(H)$. By Lemma 5.3 there is $h_2{}' \in H \cap E^+(G) \subseteq E^+(H)$ arbitrarily near h_2 so that $h_1{}' = h(h_2{}')^{-1}$ is arbitrarily near h_1, in particular, in $E^+(H)$. Now the argument just given can essentially be repeated (first picking $h_1{}''$ near $h_1{}'$) to give $h = h_1{}''h_2{}''$, with both factors in $H \cap E^+(G)$. ■

Lemmas 4.3 and 5.4 (taking $S = H = G$) give the following:

Corollary 5.5. *For any real semisimple Lie group* S, $E^+(S)E^+(S) = S$.

The next theorem is one of our main results.

Theorem 5.6. *For any real connected Lie group* G, $E^2(G) = G$.

Proof. No generality is lost by assuming that G is simply connected. Let R denote the radical of G. Then the Levi decomposition $G = R \times H$ is actually a semidirect product. The Levi factor H is semisimple, so it follows from Corollary 5.5 and Lemma 5.4 that every element h of H is a product h_1h_2 of two elements of $H \cap E^+(G)$. Now let $g = (r, h)$ be an arbitrary element of G with $h = h_1h_2$ as above and $h_i = \exp_H(\eta_i)$. By Lemma 2.9, (identifying G and W of the lemma respectively with R and $[R, R]$ here) it can be arranged that $r = r_1r_2$, where $r_1 = \mathcal{E}_R^{D_1}(\rho_1)$ and $r_2 = \mathcal{E}_{[R,R]}^{D_1}(\rho_2) \in [R, R]$, where $\rho \in [\mathfrak{r}, \mathfrak{r}]$. This means that $(r_1, h_1) = \exp_G(\rho_1, \eta_1) \in E(G)$ by Theorem 3.1. Also, since $[R, R]$ is nilpotent, Theorem 2.8 shows that for any r' in $[R, R]$ there is a ρ' in $[\mathfrak{r}, \mathfrak{r}]$ such that $r' = \mathcal{E}_{[R,R]}^{D_2}(\rho')$. Again by Theorem 3.1 $(r', h_2) = \exp_G(\rho', \eta_2) \in E(G)$. But r' can be chosen so that $A(r', h_1) = r_2$, because for fixed h_1, $A(\cdot, h_1)$ is an automorphism of R. Therefore, $(r, h) = (r_1r_2, h_1h_2) = (r_1A(r', h_1), h_1h_2) = (r_1, h_1)(r', h_2)$. ■

The next Lemma is needed for the proof of Theorem 5.8. For the proof of both these results it is assumed that the Lie algebra has a norm, but no special properties of the norm will be needed.

Lemma 5.7. *Suppose that* $\mathfrak{g}$ *is a solvable real Lie algebra of type E. Then there exists an* $a > 0$ *such that for every* $\gamma \in \mathfrak{g}$*, all of the eigenvalues of* ad_γ *are contained in the set* $S_a = \{z = x + iy \in \mathbb{C} : |x| \geq a|y|\}$*. Thus there is a constant* C *such that if* λ *is an eigenvalue of the linear map* ad*, the norms of the linear maps* $\mathrm{ad}_\gamma - \lambda\mathrm{I}$ *satisfy* $\|(\mathrm{ad}_\gamma - \lambda\mathrm{I})^k\| \leq C(\|\gamma\|^k)$ *for any* γ *and* $0 \leq k \leq \dim \mathfrak{g}$*.*

Proof. By Lie's theorem there is a basis of the complexification of $\mathfrak{g}$ such that the operators ad_γ are simultaneously in upper triangular form. So the diagonal elements, $\lambda_1, \ldots, \lambda_q$, are complex-valued linear functionals on $\mathfrak{g}$. Now $\lambda_j(\gamma) = R_j(\gamma) + iI_j(\gamma)$, where R_j and I_j are real-valued functionals. The hypothesis that $\mathfrak{g}$ is of type E implies that $I_j(\gamma) = 0$ whenever $R_j(\gamma) = 0$, or if N_j and M_j are the nullspaces of R_j and I_j respectively, $N_j \subseteq M_j$. This means that if $\hat{R}_j$ and $\hat{I}_j$ are the linear functionals induced on the quotient space $\mathfrak{g}/N_j$, then $\hat{R}_j(\gamma) = 0$ only if $\gamma = 0$. For $\|\gamma\| = 1$ the ratio $|\hat{I}_j(\gamma)/\hat{R}_j(\gamma)| \leq (a_j)^{-1} > 0$ by compactness, and by the homogeneity of the ratio the inequality holds for all $\gamma \neq 0$. If $a = \min\{a_1, \ldots, a_q\} > 0$, then all of the eigenvalues are contained in $S_a = \{z = x + iy \in \mathbb{C} : |x| \geq a|y|\}$. The linear maps $(\mathrm{ad}_\gamma - \lambda(\gamma)\mathrm{I})$ are uniformly bounded for $\|\gamma\| = 1$ by compactness. Since they are linear as functions of γ, they are bounded by a constant times $\|\gamma\|$ and the required estimates can be made. ∎

Theorem 5.8. (Dixmier-Saito [2], [19]) *If* G *is a real simply connected solvable group whose Lie algebra is of type E, then* G *is exponential.*

Proof. Suppose that $\mathfrak{g}$ is of type E. Let g be an arbitrary element of G. We need to show that $g = \exp_{\mathrm{G}}(\gamma)$ for some $\gamma \in \mathfrak{g}$. By Theorem 5.6, $g = \exp_{\mathrm{G}}(\gamma_1)\exp_{\mathrm{G}}(\eta)$ for some γ_1 and η in $\mathfrak{g}$. The idea of the proof is to use the implicit function theorem to determine $\gamma(t)$ such that

$$g \equiv \exp_{\mathrm{G}}(\gamma(t))\exp_{\mathrm{G}}(t\eta), \tag{10}$$

where $\gamma(1) = \gamma_1$. If it could be shown that $\gamma(t)$ can be continued until $t = 0$, one would get $g = \exp_{\mathrm{G}}(\gamma)$ for $\gamma = \gamma(0)$, proving that G is exponential.

Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. The $\gamma(t)$ constructed in the course of the argument below might *a priori* wander outside of $\mathfrak{g}$. At the end of the proof we shall observe that, in fact, $\gamma(t)$ must stay within $\mathfrak{g}$ for all t.

By Corollary 5.2 and the type E assumption, $\exp_{\mathrm{G}}(\gamma)$ is non-singular at every γ, so $\gamma(t)$ can be determined for t near 1. Differentiating (10), using Theorem 5.1, and (2) one gets

$$\mathrm{dL}_{\gamma(t))}\mathrm{dR}_{h(t)}[\int_0^1 \mathrm{Exp}(-u\,\mathrm{ad}_{\gamma(t)})\mathrm{d}u(\gamma'(t)) + \eta] = 0,$$

where $h(t) = \exp_{\mathrm{G}}(t\eta)$. Defining $F(z) = (1 - \mathrm{e}^{-z})z^{-1} = \int_0^1 \mathrm{e}^{-uz}\mathrm{d}u$, this equation is equivalent to

$$F(\mathrm{ad}_{\gamma(t)})(\gamma'(t)) = -\eta,$$

where $F(\mathrm{ad}_{\gamma(t)})$ is to be understood in the sense of functional calculus on finite dimensional spaces as is described in [5]. Alternatively, if $f(z) = (F(z))^{-1} = z(1 - \mathrm{e}^{-z})^{-1}$, then

$$\gamma'(t) = -f(\mathrm{ad}_{\gamma(t)})(\eta).$$

We will prove that the solution to (5.) can be continued to any t, in particular $t = 0$. According to a Theorem of Wintner (see [8] Theorem 5.1) it is sufficient to show that

$$\|f(\mathrm{ad}_\gamma)\| = O(\|\gamma\|). \tag{11}$$

We now establish the estimate (11).

Lemma 5.9. *Let $f(z) = z(1 - \mathrm{e}^{-z})^{-1}$ and $S_a = \{z = x + iy \in \mathbb{C} : |x| \geq a|y|\}$ for some $a > 0$. Then f and all of its derivatives, $f^{(k)}$, are bounded on compact subsets of S_a and as $|z| \longrightarrow \infty$ within S_a,*

$$f(z) = O(|z|), \quad f^{(1)}(z) = O(1), \quad \textit{and} \quad f^{(k)}(z) = O(|z|^{-n})$$

for $k > 1$ and any $n > 0$.

Proof. The boundedness statement is obvious because f has no singularities in S_a apart from a removable one at the origin. Also note that for $z = x + iy$ in S_a, $|x| \leq |z| \leq (1 + a)|x|$, so the asymptotic statements can be proved with $|z|$ replaced by $|x|$.

If $g(z) \equiv (1 - \mathrm{e}^{-z})^{-1}$ and $z = x + iy \in S_a$

$$|g(z)|^{-2} = (1 - \mathrm{e}^{-x})^2 + 4\sin^2(\frac{y}{2})\mathrm{e}^{-x} \geq (1 - \mathrm{e}^{-x})^2 \text{ and } |g(-z)|^{-2} \geq (\mathrm{e}^x - 1)^2,$$

$$\text{so } g'(z) \equiv g(-z)g(z) = O(\mathrm{e}^{-|z|}) \text{ as } |z| \longrightarrow \infty \text{ in } S_a.$$

Since $g(z)$ and $g(-z)$ are uniformly bounded in the complement of any neighborhood of the origin in S_a, the estimate for $f(z) = zg(z)$ in the statement of the lemma is clear. $f^{(1)}(z) = g(z) + zg'(z) = O(1)$, and this is the assertion for $f^{(1)}$. An induction argument shows that the higher derivatives are linear combinations of $g(z)^p g(-z)^q = g'(z)O(1)$ for p and $q \geq 1$ and such terms multiplied by z. But then the estimate above for $g'(z)$ gives the desired estimates for the higher derivatives. ■

To prove (11), let $\gamma \in \mathfrak{g}$ be given. By Lemma 5.7, the eigenvalues of ad_γ lie in S_a for some $a > 0$ independent of γ. Then we use Theorem VII.1.8 of [5], which shows that $f(\mathrm{ad}_\gamma)$ can be represented as follows: For every eigenvalue $\lambda_j = \lambda_j(\gamma)$, let E_j be the projection onto the corresponding invariant subspace. Then $f(\mathrm{ad}_\gamma)$ is a linear combination of terms $f(\lambda_j)E_j$ and $(\mathrm{ad}_\gamma - \lambda_j \mathrm{I})^k f^{(k)}(\lambda_j)E_j$ for k no bigger than the dimension of G and $\lambda_j = \lambda_j(\gamma)$ in the spectrum of ad_γ. Note that $\|(\mathrm{ad}_\gamma - \lambda_j I)^k E_j\| = O(\|\gamma\|^k)$ for all $k \geq 0$. But then by Lemma 5.9 and Lemma 5.7, $\|(\mathrm{ad}_\gamma - \lambda_j I)^k f^{(k)}(\lambda_j)E_j\| = O(\|\gamma\|)$, so (11) holds and continuation until $t = 0$ is possible.

To see that $\gamma(t)$ remains within $\mathfrak{g}$, note that $-f(\mathrm{ad}_\gamma)(\eta)$ can be regarded by translation as an element of the tangent space to $\mathfrak{g}_{\mathbb{C}}$ at γ. Then if γ and η are in $\mathfrak{g}$, $-f(\mathrm{ad}_\gamma)(\eta)$ is tangent to $\mathfrak{g}$. This is the case in view of the representation of $f(\mathrm{ad}_\gamma)$ discussed in the preceeding paragraph because the non-real eigenvalues of ad_γ occur in conjugate complex pairs, the corresponding invariant subspaces are complex conjugates of each other, and $f(\bar{z}) = \bar{f}(z)$. But a solution to a differential equation defined by a C^1 vector field that is tangent to a submanifold—in this case $\mathfrak{g}$—cannot leave the submanifold. Therefore, since $\gamma_1 \in \mathfrak{g}$, all $\gamma(t)$ must lie in $\mathfrak{g}$. This completes the proof that G is exponential. ■

6. Floquet-Lyapunov theory in Lie groups

Suppose that G is a real subgroup of $\mathrm{GL}(n, \mathbb{C})$, which here will be viewed as a real Lie group of dimension $2n^2$. Then the differential equations (1) and (2) considered in the introduction can be written

$$g'(t) = g(t)\gamma(t) \tag{12}$$

and

$$g'(t) = \gamma(t)g(t), \tag{13}$$

where $g(t)$ and $\gamma(t)$ are $n \times n$ matrices, with $g(t)$ nonsingular and the products on the right sides of the equations are given by ordinary matrix multiplication. For our purposes it suffices to consider the case where $\gamma(t)$ is C^1. Then standard theorems assure the existence of solutions satisfying $g(0) = g_0$ for any g_0 in G and all t.

The classical Floquet-Lyapunov theory shows that if $\gamma(t)$ is periodic $\gamma(t) \equiv \gamma(t + 2a)$ and $G = \mathrm{GL}(n, \mathbb{C})$ then the solutions satisfying $g(0) = \mathrm{e_G}$ can be written as $g(t) = \exp_{\mathrm{G}}(t\eta)p(t)$ for (12) or $g(t) = q(t)\exp_{\mathrm{G}}(t\zeta)$ for (13), where $p(t) \equiv p(t + 2a)$, $q(t) \equiv q(t + 2a)$ and η and ζ are constant matrices. However, many connected Lie groups have no faithful finite dimensional representation. Even if such a representation did exist the η or ζ obtained by the Floquet-Lyapunov argument can fail to be in the Lie algebra $\mathfrak{g}$ of G and then $g(t)$ need not remain in G. Despite these difficulties, some of the theory can be made to work in general, as we show in this section.

There is no advantage for our purposes in adopting the point of view of the equations (12) and (13), even when $G \subset GL(n, \mathbb{C})$. Thus we will work with the more general setup of equations (1) and (2). In fact, we only discuss (2) or the “right handed” equation, since the other case follows easily from it. The next proposition is the most obvious adaptation of the Floquet-Lyapunov theory to Lie groups.

Proposition 6.1. *Let G be a real exponential Lie group and $\gamma(t)$ a piecewise continuous map from the reals into the Lie algebra of G which is periodic, $\gamma(t + 2a) \equiv \gamma(t)$, where $a > 0$. Suppose that $g(t)$ is the solution to (2) satisfying $g(0) = \mathrm{e_G}$. Then there is a periodic function $q : \mathbb{R} \to G$, with the same period with the property that if η is an element of $\mathfrak{g}$ satisfying $\exp_{\mathrm{G}}(\frac{\eta}{2a}) = g(2a)$, then $g(t) = q(t)\exp_{\mathrm{G}}(t\eta)$ for all $t \in \mathbb{R}$.*

We omit the proof here since the proposition is can be proved by obvious changes in the classical argument and we shall observe below that it follows from Theorem 6.4. There is a converse proposition:

Proposition 6.2. *Suppose for every C^1 periodic map $\gamma(t)$ from the reals into the Lie algebra of G, the solution to (2) can be written in the form $g(t) = q(t)\exp_G(t\eta)$, where $q(t)$ is periodic of the same period and η is in the Lie algebra. Then G is exponential.*

Proof. If G is not exponential there is an element g_1 of G that is not in $E(G)$. Let $h(t)$ $(0 \le t \le 1)$ be a C^1 curve in G satisfying $h(0) = e_G$, $h(1) = g_1$. It can be arranged that $h'(1) \equiv dR_{h(1)}h'(0)$, where $h'(0)$ can be any element of the Lie algebra. (This follows from the fact that a C^1 curve with arbitrarily prescribed tangents at the endpoints can be constructed connecting any pair of points in a connected manifold. For instance, if the pair lies within a convex coordinate chart a polygonal path is easy to construct and then it can be smoothed to become C^1 without changing the tangents at the endpoints. The general result follows from this special case.) Define $\gamma(t)$ for $0 \le t \le 1$ by $h'(t) = dR_{h(t)}\gamma(t)$. With this definition (2) is automatically satisfied for $0 \le t \le 1$ (with $h(t)$ replacing $g(t)$). If $\gamma(t)$ is extended periodically, it remains C^1. Since the solution to (2) is of the form $g(t) = q(t)\exp_G(t\eta)$ with $q(t)$ periodic, $q(0) = q(1) = e_G$, so $g(1) = h(1) = g_1 = \exp(\eta)$, a contradiction. ∎

These last two propositions taken together show that a group is exponential if and only if the most obvious generalizaion of Floquet-Lyapunov theory applies to it:

Corollary 6.3. *Let G be a real connected Lie group. Then G is exponential if and only if, for every C^1 periodic map from the reals into the Lie algebra of G, $\gamma(t)$, the solution to (2) satisfying $g(0) = e_G$ is of the form $g(t) = q(t)\exp_G(t\eta)$, where $q(t)$ is periodic and η is constant.*

Some more notation is needed for the proof of our final result. Let $g(t, u, q)$ denote the solution to (2) that satisfies $g(u) = q$. The right invariance of the time dependent vector field $dR_{g(t)}\gamma(t)$ implies that the solutions are equivariant under right translation in the sense that

$$g(t, s, q) = g(t, s, e_G)q. \tag{14}$$

The group property for solutions of differential equations combined with (14) gives

$$g(t, u, q) = g(t, s, g(s, u, q)) = g(t, s, e_G)g(s, u, q), \tag{15}$$

for any s,t, and u. Setting $q = e_G$ and $u = 0$ in (15) tells us

$$g(t, 0, e_G) = g(t, s, e_G)g(s, 0, e_G). \tag{16}$$

Theorem 6.4. *Let G be a real connected Lie group and $\gamma(t)$ be a piecewise continuous map from the reals into the Lie algebra of G which is periodic, $\gamma(t + 2a) \equiv \gamma(t)$, where $a > 0$. Suppose that $g(t)$ is the solution of (2) satisfying $g(0) = e_G$. Then there is a periodic function $\zeta : \mathbb{R} \to \mathfrak{g}$, with the same period $2a$, where ζ is constant on the intervals $(0, a)$ and $(a, 2a)$ such that if h is defined by*

$$h'(t) = \mathrm{dR}_{h(t)}\zeta(t); \qquad h(0) = \mathrm{e_G} \tag{17}$$

and q by

$$q(t) = g(t)h^{-1}(t), \tag{18}$$

$q(t) \equiv q(t+2a)$ *is periodic and g has the form* $g(t) = q(t)h(t)$.

Remark. It will be clear from the proof that the point a in the interval (0,2a) can be replaced by any other point of the interval in the statement of the theorem.

Proof. To simplify the notation here set $\mathrm{e} = \mathrm{e_G}$. Note that by Theorem 5.6

$$g(2a) = g(2a, 0, e) = \exp_{\mathrm{G}}(\eta')\exp_{\mathrm{G}}(\eta)$$

for some η and $\eta' \in \mathfrak{g}$. Let $\zeta(t) = \frac{\eta}{a}$ for $0 \leq t < a$ and $\zeta(t) = \frac{\eta'}{a}$ for $a < t < 2a$. Continue $\zeta(t)$ periodically and let $h(t)$ be defined by (17). Clearly $h(a) = h(a, 0, e) = \exp(\eta)$ and $h(2a, a, e) = \exp(\eta')$. Then by (16) applied to h,

$$h(2a, 0, e) = h(2a, a, e)h(a, 0, e) = \exp(\eta')\exp(\eta) = g(2a, 0, e). \tag{19}$$

Defining $q(t)$ by (18) it remains to verify that $q(t) \equiv q(t+2a)$. The periodicity $\zeta(t+2a) \equiv \zeta(t)$ gives

$$g(t+2a, 2a, e) \equiv g(t, 0, e)$$

and similarly

$$h(t+2a, 2a, e) \equiv h(t, 0, e).$$

Then (16) combined with these relations gives

$$g(t+2a, 0, e) = g(t+2a, 2a, e)g(2a, 0, e) = g(t, 0, e)g(2a, 0, e)$$

and similarly

$$h(t+2a, 0, e) = h(t, 0, e)h(2a, 0, e).$$

These relations along with the definition (18) of $q(t)$ and (19) show

$$q(t+2a) \equiv g(t+2a)h^{-1}(t+2a) \equiv g(t)g(2a)h^{-1}(2a)h^{-1}(t) \equiv g(t)h^{-1}(t) \equiv q(t),$$

completing the proof. ∎

The solution $h(t) = h(t, 0, e)$ can be written more explicitly. For $0 \leq t < a$, $h(t) = \exp(\frac{t\eta}{a})$ and for $a \leq t < 2a$, $h(t) = \exp(\frac{t\eta'}{a})\exp(\eta)$. More generally, if n is any integer and $0 \leq t < a$ we have

$$h(t+2na) = \exp(\frac{t\eta}{a})(\exp(\eta')\exp(\eta))^n$$

and

$$h(t+a+2na) = \exp(\frac{t\eta'}{a})\exp(\eta)(\exp(\eta')\exp(\eta))^n.$$

Consequently,

$$g(t+2na) = q(t)\exp(\frac{t\eta}{a})(\exp(\eta')\exp(\eta))^n, \quad \text{and}$$

$$g(t+a+2na) = q(t)\exp(\frac{t\eta'}{a})\exp(\eta)(\exp(\eta')\exp(\eta))^n \quad \text{for} \quad 0 \le t \le a.$$

Proposition 6.1 is a special case of Theorem 6.4 as one sees by defining η by $\exp(2\eta) = g(2a)$ and taking $\eta' = \eta$. Then $h(t) = \exp(\frac{t\eta}{a})$ and one obtains the conclusion of Proposition 6.1. In the general case where G is not exponential, the conclusion of the proposition is weakened—η is not strictly constant, but rather piecewise constant, assuming just two values.

The exposition here has benefited from many helpful suggestions by the referees. One of them has called our attention to the fact many of the results of this paper can be interpreted in terms of mathematical control theory. For instance, the switching between η and η' in Theorem 6.4 can be regarded as changing a control parameter that can take on two values. Some references that are relevant to this point of view are: [1], [12], [13], and [20].

References

[1] Brockett, R. W., *System theory on group manifolds and coset spaces*, SIAM J. on Control, **10** (1972), 265–284.

[2] Dixmier, J., *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Math. Soc. France **85** (1957), 113–121.

[3] Djokovic D., and Nguyễñ Thǎńg, *On the exponential map of almost simple real algebraic groups*, J. of Lie Theory, **5** (1996), 275–291.

[4] Djokovic D., and K. H. Hofmann, *The exponential in real Lie groups: A status report*, J. of Lie Theory, **7** (1997), 171–199.

[5] Dunford, N., and J. Schwartz, "Linear Operators I," John Wiley and Sons 1957.

[6] Grosser, S., and M. Moskowitz, *On central topological groups*, Trans. Amer. Math. Soc. **127** (1967), 317–340.

[7] Harish-Chandra, *On a Lemma of F. Bruhat*, Jour. de Math. Pures et Appliques, **35** (1956), 203–210.

[8] Hartman, P., "Ordinary Differential Equations," John Wiley and Sons 1964.

[9] Helgason, S., "Differential Geometry and Symmetric Spaces, Acad. Press 1962.

[10] Hochschild, G., "The Structure of Lie Groups," Holden-Day Inc. 1965.

[11] Hopf, H., *Über den Rang geschossener Liescher Gruppen*, Com. Math. Helvetici **13** (1940/41), 119–143.

[12] Jurdjevic, V., "Geometric Control Theory," Cambridge Studies in Advanced Mathematics **51** (1997).

[13] Jurdjevic V., and H. J. Sussmann, *Control Systems on Lie Groups*, Journal of Differential Equations **12** (1972), 313–329.

[14] Moskowitz, M., *On the surjectivity of the exponential map in certain Lie groups*, Annali di Matematica Pura ed Applicata **146** (1994), 129–143.

[15] —, *Correction and addenda to: On the surjectivity of the exponential map in certain Lie groups*, Annali di Matematica Pura ed Applicata **173** (1997), 351–358.

[16] —, *Some Remarks on Automorphisms of Bounded Displacement and Bounded Cocycles*, Monatshefte für Math. **85** (1978), 323–336.

[17] Moskowitz, M., and M. Wüstner, *Exponentiality of certain real solvable Lie groups*, Canad. Math. Bull. **41** (1998), 368–373.

[18] —, *Contributions to real exponential Lie groups*, Monatshefte für Math. **130** (2000), 29–47.

[19] Saito, M., *Sur certains groupes de Lie résolubles I*, Sci. Papers College General Educ. Univ. of Tokyo **7** (1957), 1–11.

[20] Sontag, E. D., "Mathematical Control Theory," Texts in Applied Mathematics **6**, Springer-Verlag, New York etc., 1990.

[21] Varadarajian, V. S., "Lie groups, Lie algebras, and their representations, " Prentice Hall, 1974.

[22] Wallach, N., "Real Reductive Groups, Vol. I," Acad. Press Series in Pure and applied Math. **132**, 1988.

[23] —, "Harmonic Analysis on Homogeneous Spaces, " Marcel Decker Inc., 1973.

[24] Wüstner, M., "Lie Groups with surjective exponential function," Shaker-Verlag, Aachen, 2001.

Martin Moskowitz
Department of Mathematics
CUNY Graduate Center
365 Fifth Ave.
New York NY 10016, USA
mmoskowi@axp.mat.uniroma2.it

Richard Sacksteder
Department of Mathematics
CUNY Graduate Center
365 Fifth Ave.
New York NY 10016, USA
sackbix@prodigy.net

Received May 18, 2001
and in final form October 19, 2002