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On the Sum of Iterations of the Euler Function

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Abstract

We study the sum

$$F(n) = \sum_{k=1}^{\kappa(n)} \varphi^{(k)}(n).$$

of consecutive iterations of the Euler function $\varphi(n)$ (where the last iteration satisfies $\varphi^{(\kappa(n))}(n) = 1$). We show that for almost all n, the difference $|F(n) - n|$ is not too small, and the ratio $n/F(n)$ is not an integer. The latter result is related to a question about the so-called *perfect totient numbers*, for which $F(n) = n$.

1 Introduction

Let φ denote the *Euler function*, which, for an integer $n \geq 1$, is defined as usual by

$$\varphi(n) = \#\{j \in \mathbb{Z} \mid 1 \leq j \leq n,\ \gcd(j, n) = 1\}.$$

Moreover, for an integer $k \geq 1$, we use $\varphi^{(k)}$ to denote the kth iteration of φ, that is, the function defined recursively as $\varphi^{(1)}(n) = \varphi(n)$ and $\varphi^{(k+1)}(n) = \varphi^k(\varphi(n))$, $k = 1, 2, \ldots$.

Clearly for every n there exists a uniquely defined integer $\kappa(n)$ such that $\varphi^{(\kappa(n))}(n) = 1$. Accordingly, we define the function

$$F(n) = \sum_{k=1}^{\kappa(n)} \varphi^{(k)}(n),$$

which is an additive analogue of the function

$$G(n) = \prod_{k=1}^{\kappa(n)} \varphi^{(k)}(n)$$

considered in the paper [1]. (In fact, the results of [1] are formulated for $nG(n)$ but one can easily reformulate them for $G(n)$.)

The function $F(n)$ also appears in the definition of *perfect totient numbers*, which are the integers $n \geq 2$ with $F(n) = n$; see [4, 6] and references therein. Here we use some very elementary arguments to establish several properties of this function, which seem to be new.

Let

$$\mathcal{V}(x) = \{\varphi(n) \leq x \mid n = 1, 2, \ldots\}, \qquad \mathcal{U}(x) = \{F(n) \leq x \mid n = 1, 2, \ldots\}.$$

We start with the observation that

$$\mathcal{U}(x) \subseteq \{v + F(v) \mid v \in \mathcal{V}(x)\},$$

therefore

$$\#\mathcal{U}(x) \leq \#\mathcal{V}(x).$$

There are several very tight bounds on the value set of the Euler function (see [2, 5]). These immediately imply that

$$\#\mathcal{U}(x) \leq \frac{x}{\log x} \exp\left((C + o(1))(\log\log\log x)^2\right), \qquad x \to \infty, \tag{1}$$

for some absolute constant $C = 0.8178\cdots$. This, in turn, implies that the set of perfect totient numbers is of density zero. On the other other hand, it is easy to check that $F(3^s) = 3^s$, $s = 1, 2, \ldots$. Thus there are infinitely many perfect totient numbers, and in fact $\#\mathcal{U}(x) \geq \log x / \log 3 - 1$. Here we show that in fact one can get a better bound by considering integers of the form $2^r 3^s$, $r, s = 1, 2, \ldots$.

As in the case of the classical *perfect numbers*, see [3], one can consider *multiply perfect totient numbers* for which $F(n)|n$. We show that multiply perfect totient numbers form a zero density set.

We also show that

$$|F(n) - n| \geq (\log n)^{\log 2 + o(1)}$$

for almost all n. Hence such "approximately" perfect totient numbers form a set of density zero, too.

Throughout the paper, the implied constants in the symbols "O", "$\gg$" and "$\ll$" are absolute. (We recall that the notation $U \ll V$ and $V \gg U$ is equivalent to the statement that $U = O(V)$ for positive functions U and V). We also use the symbol "o" with its usual meaning: the statement $U = o(V)$ is equivalent to $U/V \to 0$.

Finally, for any real number $z > 0$ and any integer $\nu \geq 1$, we write $\log_\nu z$ for the function defined inductively by $\log_1 z = \max\{\log z, 1\}$, where $\log z$ is the natural logarithm of z and $\log_\nu z = \log_1(\log_{\nu-1} z)$ for $\nu > 1$. When $\nu = 1$, we omit the subscript in order to simplify the notation; however, we continue to assume that $\log z \geq 1$ for any $z > 0$.

2 Main Results

Theorem 2.1. *The following bound holds:*

$$\#\mathcal{U}(x) \gg (\log x)^2.$$

Proof. For positive integer r and s, we have

$$\begin{aligned} F(2^{2r}3^{2s}) &= \sum_{i=1}^{2s} 3^{2s-i}2^{2r} + \sum_{j=0}^{2r} 2^{2r-j} \\ &= 2^{2r-1}(3^{2s}-1) + 2^{2r} - 1 = 2^{2r-1}3^{2s} + 2^{2r-1} - 1. \end{aligned}$$

Assume that

$$2^{2r-1}3^{2s} + 2^{2r-1} - 1 = 2^{2u-1}3^{2v} + 2^{2u-1} - 1$$

for some positive integers u and v. Then

$$(3^{2s}+1) = 2^{2u-2r}(3^{2v}+1)$$

which is impossible unless $u = r$, $v = s$, since

$$3^{2s} + 1 \equiv 3^{2v} + 1 \equiv 2 \pmod 4.$$

This means that the values of $F(2^{2r}3^{2s})$ are pairwise distinct and the result follows. □

Denoting by $M(x)$ the number of multiply perfect totient numbers $n \le x$, we have the following result.

Theorem 2.2. *For all positive integers $n \le x$ except possibly $o(x)$ of them, the bound*

$$M(x) \ll \frac{x}{\log x} \exp\left((C + o(1))(\log_3 x)^2\right)$$

holds.

Proof. Let

$$\Delta = \max_{n \le x} \frac{n}{\varphi(n)}.$$

Since $\varphi(n) \gg n/\log_2 n$ (see [7, Theorem 4, Chapter I.5]) we conclude that $\Delta = O(\log_2 x)$. Clearly, every n with $F(n)|n$ must be of the form $n = du$ with an positive integer $d \le \Delta$ and $u \in \mathcal{U}(x/d)$. Therefore, using (1), we deduce,

$$\begin{aligned} M(x) &\le \sum_{1 \le d \le \Delta} \#\mathcal{U}(x/d) \\ &\le x \sum_{1 \le d \le \Delta} \frac{1}{d \log(x/d)} \exp\left((C + o(1))(\log_3 (x/d))^2\right) \\ &\le \frac{x}{\log x} \exp\left((C + o(1))(\log_3 x)^2\right) \sum_{1 \le d \le \Delta} \frac{1}{d} \\ &\ll \frac{x}{\log x} \exp\left((C + o(1))(\log_3 x)^2\right) \log \Delta \\ &= \frac{x}{\log x} \exp\left((C + o(1))(\log_3 x)^2\right), \end{aligned}$$

which finishes the proof. □

Theorem 2.3. *For all positive integers $n \le x$, except possibly $o(x)$ of them, the bound*

$$|F(n) - n| \ge (\log x)^{\log 2 + o(1)}$$

holds.

Proof. Let $\nu(m)$ denote the largest power of 2 that divides m. We start with an observation that if m is not a power of 2 itself, we have $\nu(\varphi(m)) \ge \nu(m)$. It is also clear that $\nu(\varphi(m)) \ge \omega(m) - 1$ where $\omega(m)$ is the number of distinct prime divisors of n. This implies that

$$F(n) \equiv 2^{\omega(n)-1} + \cdots + 1 \equiv -1 \pmod{2^{\omega(n)-1}}.$$

From the classical *Hardy-Ramanujan* inequality, for any $y \ge 1$,

$$|\omega(n) - \log_2 x| \le y\sqrt{\log_2 x}$$

for at most $O(xy^{-2})$ positive integers $n \le x$ (see [7, Theorem 4, Chapter III.3]). Take $y = (\log_2 x)^{1/6}$ and put

$$r = \left\lfloor \log_2 x - y\sqrt{\log_2 x} - 1 \right\rfloor, \qquad s = \left\lfloor \log_2 x - 2y\sqrt{\log_2 x} \right\rfloor.$$

We see that

$$F(n) \equiv -1 \pmod{2^r}$$

for all but $O(x(\log_2 x)^{-1/3})$ positive integers $n \le x$. Therefore, for every of the remaining integers n we see that if $|F(n) - n| < 2^s$ then n belongs to one of the $O(2^s)$ residue classes modulo 2^r. Thus this is possible for at most $O\left(2^s(x/2^r + 1)\right) = O(x2^{s-r})$ positive integers $n \le x$, which finishes the proof. □

3 Open Questions

It seems quite plausible that considering integers n composed out of more fixed primes, for example, $n = 2^r 3^s 5^t$ one can improve the lower bound of Theorem 2.1. We however do not see how to create a more generic argument, which would lead to, say, the estimate

$$\frac{\log \#\mathcal{U}(x)}{\log_2 x} \to \infty, \qquad x \to \infty,$$

which, no doubt, is correct.

It is also natural to expect that the bounds of Theorems 2.2 and 2.3 are not tight and can be improved.

One can easily derive from [1, Theorem 4.2] that

$$\frac{1}{x}\sum_{n \le x} F(n) \sim \frac{3}{\pi^2}x.$$

In fact, using the full strength of [1, Theorem 4.2], one can obtain a more precise asymptotic expansion for the average value of $F(n)$.

Finally, one can also ask similar questions for the sums of iterations of other number theoretic functions, such as the the sum of divisors function $\sigma(n)$ or the Carmichael functions $\lambda(n)$.

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