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Kaprekar Triples

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Abstract

We say that 45 is a Kaprekar triple because $45^3 = 91125$ and $9 + 11 + 25 = 45$. We find a necessary condition for the existence of Kaprekar triples which makes it quite easy to search for them. We also investigate some Kaprekar triples of special forms.

1 Introduction

Kaprekar triples (sequence A006887, Sloan [5]) are numbers with a property which is easily demonstrated by example. Observe that

$$\begin{aligned}
8^3 &= 512, & 5+1+2 &= 8,\\
45^3 &= 91125, & 9+11+25 &= 45,\\
297^3 &= 26198073, & 26+198+073 &= 297,\\
4949^3 &= 121213882349, & 1212+1388+2349 &= 4949,\\
44443^3 &= 87782935806307, & 8778+29358+06307 &= 44443,\\
565137^3 &= 180493358291026353, & 180493+358291+026353 &= 565137.
\end{aligned}$$

Therefore 8, 45, 297, 4949, 44443, and 565137 are all examples of Kaprekar triples. Kaprekar triples generalize the Kaprekar numbers (sequence A006886, Sloan [5]), which were introduced by Kaprekar [4], discussed by Charosh[2], and completely characterized by Iannucci [3]. Kaprekar triples are mentioned in Wells's *Dictionary of Curious and Interesting Numbers* [6].

Formally, an n-Kaprekar triple k (where n is a natural number) satisfies the pair of equations

$$\begin{aligned} k^3 &= p \cdot 10^{2n} + q \cdot 10^n + r\,, \\ k &= p + q + r\,, \end{aligned}$$

where $0 \le r < 10^n$, $0 \le q < 10^n$, and $p > 0$ are integers. As the 3-Kaprekar triple 297 shows, p may have fewer than n digits, and so may q or r (note the leading zero in $r = 073$). The stipulation that $p > 0$ precludes many otherwise trivial examples such as

$$\begin{aligned} 100^3 &= 0 \cdot 10^8 + 100 \cdot 10^4 + 0\,, \\ 100 &= 0 + 100 + 0\,, \end{aligned}$$

i.e., 100 as a 4-Kaprekar triple. Having $p > 0$ also precludes 1 as a Kaprekar triple, in spite of its inclusion in sequence A006887 by Sloan [5].

2 The Set $\mathcal{K}(N)$

Let N be a natural number such that $N \not\equiv 1 \pmod 4$. We define the set $\mathcal{K}(N)$ of positive integers as follows: We say $k \in \mathcal{K}(N)$ if there exist nonnegative integers $r < N$, $q < N$, and a positive integer p, such that

$$k^3 = pN^2 + qN + r \tag{1}$$

and such that

$$k = p + q + r\,. \tag{2}$$

Although N satisfies (1) and (2) (with $p = N$, $q = r = 0$), we nonetheless disallow N as an element of $\mathcal{K}(N)$. Therefore, it follows that $k < N$ if $k \in \mathcal{K}(N)$. For, subtracting (2) from (1) yields

$$k(k-1)(k+1) = (N-1)(p(N+1)+q)\,, \tag{3}$$

so that $k > N$ implies

$$k < p + \frac{q}{k+1}\,.$$

Since $q/(k+1) < 1$, we have $k \le p$. Since $k < p$ contradicts (2), we have $k = p$, but this implies $q = r = 0$ and hence $k = N$ by (1). Contradiction. Therefore $k < N$ if $k \in \mathcal{K}(N)$.

Suppose $k \in \mathcal{K}(N)$. Then (3) implies $N - 1 \mid k(k-1)(k+1)$. Because $N \not\equiv 1 \pmod 4$, there exist pairwise relatively prime integers d, d_1, and d_2 such that

$$N - 1 = dd_1d_2\,, \qquad d \mid k\,, \qquad d_1 \mid k-1\,, \qquad d_2 \mid k+1\,. \tag{4}$$

Since $d \mid k$ we write

$$k = dm$$

for a positive integer m. Then $d_1 \mid dm - 1$ and $d_2 \mid dm + 1$ and so we have

$$dm \equiv 1 \pmod{d_1}\,, \qquad dm \equiv -1 \pmod{d_2}\,. \tag{5}$$

Let

$$\xi_1 \equiv d^{-1} \pmod{d_1}, \qquad \xi_2 \equiv d^{-1} \pmod{d_2},$$
$$\mu_1 \equiv d_1^{-1} \pmod{d_2}, \qquad \mu_2 \equiv d_2^{-1} \pmod{d_1}.$$

Then we have

$$m \equiv \xi_1 \pmod{d_1}, \qquad m \equiv -\xi_2 \pmod{d_2},$$

so that by the Chinese remainder theorem we have

$$m \equiv \xi_1\mu_2 d_2 - \xi_2\mu_1 d_1 \pmod{d_1 d_2}. \tag{6}$$

Moreover, m is the least positive residue such that (6) is satisfied; this is because $dm = k < N = dd_1d_2 + 1$ and thus $m \leq d_1d_2$.

For a positive integer n, we call d a *unitary divisor* of n if $d \mid n$ and $(d, \frac{n}{d}) = 1$. In this case we write $d \| n$. We have shown

Theorem 1 *If $N \not\equiv 1 \pmod 4$, then every element $k \in \mathcal{K}(N)$ is divisible by a unitary divisor d of $N-1$. If we write $k = dm$, then m satisfies (4) for some pair d_1, d_2, of unitary divisors of $N-1$ such that $d_1d_2 = (N-1)/d$.*

If $N \not\equiv 1 \pmod 4$, then Theorem 1 gives a necessary condition for finding elements k of $\mathcal{K}(N)$, and hence it may be applied to find an upper bound for $|\mathcal{K}(N)|$, the number of elements in $\mathcal{K}(N)$. For, if $N-1$ has the unique prime factorization given by $N-1 = \prod_{i=1}^{t} p_i^{a_i}$, then we call the prime powers $p_i^{a_i}$ the *components* of $N-1$. Then $d \| N-1$ if and only if d is a product of components of $N-1$ (including the empty product 1). We refer to t, the number of components of $N-1$, as $\omega(N-1)$. Thus by Theorem 1, if $N \not\equiv 1 \pmod 4$ then

$$|\mathcal{K}(N)| \leq 3^{\omega(N-1)}. \tag{7}$$

It is possible to define $\mathcal{K}(N)$ when $N \equiv 1 \pmod 4$. In this case, the factors d, d_1, and d_2 in (4) will be pairwise relatively prime if and only if d is even. If this is so, we may proceed exactly as above, so that (7) is still true.

Otherwise d is odd. Since $2^\nu \| N-1$ for some $\nu \geq 2$, we have either $2 \| d_1$, $2^{\nu-1} \| d_2$, or, $2^{\nu-1} \| d_1$, $2 \| d_2$. Note that these two cases are identical when $2^2 \| N-1$. In either case, the equations (5) still hold, and since $(d, d_1) = (d, d_2) = 1$, we see that m may be determined uniquely modulo $[d_1, d_2]$. Here, $d \| N-1$, and d_1 and d_2 are each some power of 2 multiplied by an odd unitary divisor of $N-1$. Thus (7) still holds in the case when $N \equiv 1 \pmod 4$.

3 Kaprekar Triples

In the notation of the previous section,we refer to the set $\cup_{n=1}^{\infty}\mathcal{K}(10^n)$ as the set of Kaprekar triples. If we prefer, we may refer to the set $\mathcal{K}(10^n)$, for fixed n, as the set of n-Kaprekar triples. To illustrate Theorem 1, consider the set of 6-Kaprekar triples, and note the factorization

$$10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37.$$

We may take $d = 27$, $d_1 = 259$, and $d_2 = 143$. Then

$$\xi_1 = 48\,, \qquad \xi_2 = 53\,, \qquad \mu_1 = 90\,, \qquad \mu_2 = 96\,,$$

giving

$$m \equiv 143 \cdot 96 \cdot 48 - 259 \cdot 90 \cdot 53 \equiv 20931 \pmod{37037}\,.$$

Therefore

$$m = 20931\,, \qquad d = 27\,, \qquad k = 20931 \cdot 27 = 565137\,.$$

Since

$$565137^3 = 180493358291026353\,,$$
$$565137 = 180493 + 358291 + 026353\,,$$

we have $565137 \in \mathcal{K}(10^6)$. To show that the conditions in Theorem 1 are not sufficient, consider $d = 297$, $d_1 = 37$, and $d_2 = 91$. Here,

$$\xi_1 = 1\,, \qquad \xi_2 = 19\,, \qquad \mu_1 = 32\,, \qquad \mu_2 = 24\,,$$

giving

$$m \equiv 3257\,, \qquad d = 297\,, \qquad k = 967329\,.$$

However,

$$967329^3 = 905154309885752289\,,$$

but

$$905154 + 309885 + 752289 = 1967328\,,$$

and so $967329 \notin \mathcal{K}(10^6)$. Note that $1967328 = 967329 + (10^6 - 1)$. Experimentally, we have seen that roughly one fourth of the $3^{\omega(N-1)}$ possible triples (d, d_1, d_2) of unitary divisors of $N - 1$ produce an element $k \in \mathcal{K}(N)$ when Theorem 1 is applied. The other three fourths produce k such that when p, q, and r in (1) are obtained we get

$$p + q + r = k + (N - 1)$$

instead of (2). Generally, the larger the value of $\omega(N - 1)$, the closer to 1:3 the ratio of elements of $\mathcal{K}(N)$ to non-elements becomes.

We provide some data for $N = 10^n$, for various values of n, where "ratio" refers to the ratio $|\mathcal{K}(10^n - 1)|/3^{\omega(10^n-1)}$:

n	$3^{\omega(10^n-1)}$	$\vert\mathcal{K}(10^n)\vert$	ratio
5	27	5	0.185185
6	243	37	0.152263
7	27	8	0.296296
10	243	64	0.263374
12	2187	527	0.240969
15	729	195	0.267490
19	9	1	0.111111
20	6561	1649	0.251334
21	2187	538	0.245999
23	9	1	0.111111
24	59049	14702	0.248980
30	1594323	398838	0.250161
42	4782969	1196902	0.250242
64	43046721	10759839	0.249957
80	14348907	3587901	0.250047

4 Applications

It is a simple matter to search for Kaprekar triples by applying Theorem 1. To do so, one only needs the factorizations of $10^n - 1$ for $n \geq 1$, which are easily available (for example see Brillhart et al. [1]).

In this section we will discuss Kaprekar triples of certain forms. For example, consider the set $\mathcal{K}(64M^2)$ for some positive integer M. Since

$$64M^2 - 1 = (8M-1)(8M+1)\,,$$

and since $8M-1$ and $8M+1$ are relatively prime, we can apply Theorem 1 by choosing d, d_1, and d_2 from among the unitary divisors $8M \pm 1$ and 1 of $64M^2 - 1$. If we let $d_2 = 1$, there are at least two ways to do this, one of which is to let $d = 8M - 1$ and $d_1 = 8M + 1$. In this case we have $\xi_1 = 4M$ and $\xi_2 = \mu_1 = \mu_2 = 1$, and thus

$$m = d_2\mu_2\xi_1 - d_1\mu_1\xi_2 = -4M - 1 \equiv 4M \pmod{8M+1}\,,$$

taking the least positive residue modulo $8M+1$. This gives $k = dm = 4M(8M-1)$. Similarly, taking $d = 8M+1$ and $d_1 = 8M-1$ gives $k = 4M(8M+1)$.

Thus it is possible that $4M(8M \pm 1)$ are both elements of $\mathcal{K}(64M^2)$. Indeed they are, for

$$\begin{aligned} k^3 &= 64M^3(8M \pm 1)^3 \\ &= 4096M^4(8M^2 \pm 3M) + 64M^2(24M^2 \pm M)\,, \end{aligned}$$

and,

$$(8M^2 \pm 3M) + (24M^2 \pm M) = 32M^2 \pm 4M = k\,.$$

Note that if $n \geq 3$ then 10^{2n} is of the form $64M^2$ with $M = 5^3 \cdot 10^{n-3}$. We have

Theorem 2 *For $n \geq 3$, the integers $5 \cdot 10^{n-1}(10^n \pm 1)$ are $2n$-Kaprekar triples.*

For example, 499500 and 500500 are both 6-Kaprekar triples, 49995000 and 50005000 are both 8-Kaprekar triples, and so forth.

For positive integers $r > 1$ and $n \geq 1$, we refer to an element of $\mathcal{K}(r^n)$ as a *base-r Kaprekar triple*. Note that if $p \geq 3$ then 2^{2p} has the form $64M^2$ where $M = 2^{p-3}$. Hence $2^{p-1}(2^p \pm 1)$ are binary (or base-2) Kaprekar triples. Since every even perfect number has the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime (a fact first proved by Euler), we have

Theorem 3 *Every even perfect number is a binary Kaprekar triple.*

As examples, we see that

$$\begin{aligned} 28^3 &= 5 \cdot 64^2 + 23 \cdot 64, & 5 + 23 &= 28; \\ 496^3 &= 116 \cdot 1024^2 + 380 \cdot 1024, & 116 + 380 &= 496; \\ 8128^3 &= 2000 \cdot 16384^2 + 6128 \cdot 16384, & 2000 + 6128 &= 8128. \end{aligned}$$

We can also consider the set $\mathcal{K}(4096M^4)$ for some positive integer M. Similarly as we did above, we can show that $256M^3 + 4M$ belongs to this set. Letting $M = 5^3 \cdot 10^{n-3}$ for $n \geq 3$ gives us

Theorem 4 *If $n \geq 3$ then $5 \cdot 10^{3n-1} + 5 \cdot 10^{n-1}$ is a $4n$-Kaprekar triple.*

Hence 500000500 is a 12-Kaprekar triple:

$$\begin{aligned} 500000500^3 &= 125000375000375000125000000\,, \\ 125 + 000375000375 + 000125000000 &= 500000500\,. \end{aligned}$$

Also, 500000005000 is a 16-Kaprekar triple, 500000000050000 is a 20-Kaprekar triple, and so forth.

5 Concluding Remarks

Theorem 2 shows that there always exists an n-Kaprekar triple when $n \geq 6$ is even. What about odd n? By (7), there are fewer such triples when $\omega(10^n - 1)$ is small. In fact, $\omega(10^n - 1) = 2$ when $n = 19$, 23, and 317 (see Brillhart et. al. [1]), although it is not known how long this list may be extended. The table following section 3 shows that an n-Kaprekar exists when $n = 19$ or 23. However, a simple computer search reveals that no 317-Kaprekar triples exist; thus there do not exist n-Kaprekar triples for every n.

A more general question is, are there certain forms of N for which $\mathcal{K}(N)$ is empty? For example, we can show $\mathcal{K}(N) = \emptyset$ whenever $N > 8$ is of the form $p^\alpha + 1$ for odd prime p and $\alpha \geq 1$; note that $\mathcal{K}(8)$ consists of the perfect number 6 by Theorem 3. Indeed, since $N - 1 = p^\alpha$, if $k \in \mathcal{K}(N)$ then by (4) one of three cases occur: *(i)* $p^\alpha \mid k$; *(ii)* $p^\alpha \mid k - 1$; *(iii)* $p^\alpha \mid k + 1$.

In case *(i)*, as $k < N$ we must have $k = p^\alpha$. But here,

$$\begin{aligned} k^3 &= (N-3)N^2 + 2N + (N-1), \\ (N-3) + 2 + (N-1) &= k + (N-1) \neq k. \end{aligned}$$

In case *(ii)* we have $k \equiv 1 \pmod{p^\alpha}$ by (6).

In case *(iii)*, $k \equiv -1 \pmod{p^\alpha}$ by (6), which implies $k = p^\alpha - 1$. But

$$\begin{aligned} k^3 &= (N-6)N^2 + 11N + (N-8), \\ (N-6) + 11 + (N-8) &= k + (N-1) \neq k. \end{aligned}$$

All three cases lead to contradiction (case *(ii)* contradicts $1 < k < N$).

On the other hand, there are forms of N for which $\mathcal{K}(N) \neq \emptyset$ (as we've already seen when $N = 10^{2n}$). For example, it is straightforward to check that when $N = 2^n + 1$, $n \geq 2$, we have $k = 2^{n-1} - 1 \in \mathcal{K}(N)$.

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