Journal of Integer Sequences, Vol. 6 (2003),
Article 03.3.3

Matrix Transformations of Integer Sequences

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Abstract: The integer sequences with first term 1 comprise a group $\mathcal{G}$ under convolution, namely, the Appell group, and the lower triangular infinite integer matrices with all diagonal entries 1 comprise a group $\mathbb{G}$ under matrix multiplication. If $A \in \mathcal{G}$ and $M \in \mathbb{G}$, then $MA \in \mathcal{G}$. The groups $\mathcal{G}$ and $\mathbb{G}$ and various subgroups are discussed. These include the group $\mathbb{G}^{(1)}$ of matrices whose columns are identical except for initial zeros, and also the group $\mathbb{G}^{(2)}$ of matrices in which the odd-numbered columns are identical except for initial zeros and the same is true for even-numbered columns. Conditions are determined for the product of two matrices in $\mathbb{G}^{(m)}$ to be in $\mathbb{G}^{(1)}$. Conditions are also determined for two matrices in $\mathbb{G}^{(2)}$ to commute.

1 Introduction

Let $\mathcal{G}$ be the set of integer sequences $(a_1, a_2, a_3, \ldots)$ for which $a_1 = 1$. The notations $A = (a_1, a_2, a_3, \ldots)$, $B = (b_1, b_2, b_3, \ldots)$, $C = (c_1, c_2, c_3, \ldots)$ will always refer to elements of $\mathcal{G}$. The finite sequence $(a_1, a_2, a_3, \ldots, a_n)$ will be denoted by A_n, and likewise for B_n and C_n. Let $\star$ denote convolution; i.e., if $C = A \star B$, then

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1},$$

which we shall sometimes write as $A_n \circledast B_n$, so that $A \star B$ is the sequence having $A_n \circledast B_n$ as nth term. Formally,

$$\sum_{k=1}^{\infty} c_k x^{k-1} = \left(\sum_{k=1}^{\infty} a_k x^{k-1}\right)\left(\sum_{k=1}^{\infty} b_k x^{k-1}\right).$$

In particular, if $c_1 = 1$ and $c_k = 0$ for $k \geq 2$, then the sequence B has generating function $1/(a_1 + a_2x + a_3x^2 + \cdots)$, and A and B are a pair of convolutory inverses.

Let $\mathcal{G}_n$ denote the group of finite sequences A_n under $\star$; the identity is $I_n = (1, 0, 0, \ldots, 0)$, and A_n^{-1} is the sequence B_n given inductively by $b_1 = 1$ and

$$b_n = -\sum_{k=1}^{n-1} a_{n-k+1}b_k. \tag{1}$$

for $n \geq 2$. The algebraic system $(\mathcal{G}, \star)$ is a commutative group known as the Appell subgroup of the Riordan group. Its elements, the Appell sequences, are special cases of the Sheffler sequences, which play a leading role in the umbral calculus [2, Chapter 4]; however, the umbral developments are not used in this paper. In $\mathcal{G}$, the identity and A^{-1} are the limits of I_n and A_n^{-1}. (Here, limits are of the combinatorial kind: suppose $j_1, j_2, j_3, \ldots$ is an unbounded nondecreasing sequence of positive integers and $\{a_{i,j}\}$ is a sequence of sequences such for each i,

$$(a_{k,1}, a_{k,2}, a_{k,3}, \ldots, a_{k,j_i}) = (a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,j_i})$$

for every $k > i$. Then

$$\lim_{i\to\infty} (a_{i,1}, a_{i,2}, a_{i,3}, \ldots)$$

is defined as the sequence $(a_1, a_2, a_3, \ldots)$ such that for every n there exists i_0 such that if $i > i_0$, then

$$(a_1, a_2, a_3, \ldots, a_n) = (a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}).)$$

The study of the group $(\mathcal{G}, \star)$, we shall soon see, is essentially that of a certain group of matrices. However, we shall consider first a more general group of matrices.

For any positive integer n, let $\mathbb{G}_n$ be the set of lower triangular $n \times n$ integer matrices with all diagonal entries 1, and let $\cdot$ denote matrix multiplication. Then $(\mathbb{G}_n, \cdot)$ is a noncommutative group. Now let $\mathbb{G}$ denote the set of lower triangular infinite integer matrices with all diagonal entries 1. In such a matrix, every column, excluding the zeros above the diagonal, is an element of $\mathcal{G}$, and $(\mathbb{G}, \cdot)$ is a noncommutative group. Properties of matrices in $\mathbb{G}$ arise via limits of those of matrices in $\mathbb{G}_n$. For example, if $M = (m_{ij}) \in \mathbb{G}$, then the matrix $M_n := (m_{ij})$, where $1 \leq i \leq n$ and $1 \leq j \leq n$, is an element of $\mathbb{G}_n$, and

$$M^{-1} = \lim_{n\to\infty} M_n^{-1}.$$

It is easy to check that if $A \in \mathcal{G}$ and $M \in \mathbb{G}$, then $M \cdot A \in \mathcal{G}$; here A is regarded as an infinite column vector.

Among subgroups of $\mathbb{G}$ is the Riordan group (in the case that the coefficients are all integers) introduced in [3]. Although the Riordan group will not be further discussed in this paper, the reader may wish to consult the references listed at A053121 (the Catalan triangle) in [4].

Suppose $T = (t_1, t_2, t_3, \ldots) \in \mathcal{G}$. Let $\mathbb{T}$ be the matrix in $\mathbb{G}$ whose ith row is

$$t_i, t_{i-1}, \ldots, t_1, 0, 0 \ldots,$$

so that the first column of $\mathbb{T}$ is T, and each subsequent column contains T as a subsequence. Let $\mathbb{G}^{(1)}$ be the set of all such matrices $\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{U}$ in $\mathbb{G}^{(1)}$ have first columns T and U, respectively, then the first column of $\mathbb{T}\cdot\mathbb{U}$ is the sequence $T\star U$, and $\mathbb{T}\cdot\mathbb{U}\in\mathbb{G}^{(1)}$. Clearly, $(\mathbb{G}^{(1)},\cdot)$ is isomorphic to $(\mathcal{G},\star)$. Matrices in $\mathbb{G}^{(1)}$ will be called *sequential matrices.*

One more property of the group $\mathbb{G}$, with easy and omitted proof, will be useful: if $M=(m_{ij})\in\mathbb{G}$ and $f(M):=((-1)^{i+j}m_{ij})$, then

$$(f(M))^{-1}=f(M^{-1}). \tag{2}$$

2 The Appell group $(\mathcal{G},\star)$

The first theorem in this section concerns the convolutory inverse of a linear recurrence sequence of order $m\geq 2$.

Theorem 1. *Suppose $m\geq 2$, and $a_1=1,a_2,\ldots,a_m$ are initial values of an mth order recurrence sequence given by*

$$a_n=u_1a_{n-1}+u_2a_{n-2}+\cdots+u_ma_{n-m}+r_{n-m} \tag{3}$$

for $n\geq m+1$, where $u_1,u_2,\ldots,u_m$ and $r_1,r_2,r_3,\ldots$ are integers and $u_m\neq 0$. Then the convolutory inverse, B, of A, is a sequence

$$(1,b_2,\ldots,b_m,b_{m+1},b_{m+2},\ldots)$$

for which the subsequence $(b_{m+2},b_{m+3},\ldots)$ satisfies

$$b_n=\sum_{k=1}^{m-1}b_{n-k}c_k-B_{n-m}\circledast R_{n-m},$$

where

$$c_k=-a_{k+1}+\sum_{j=1}^{k}u_ja_{k+1-j}$$

for $n\geq m+2$.

Proof: By (1), $b_1=a_1=1$. Also, $b_2=-a_2$, and

$$b_n=-a_nb_1-a_{n-1}b_2-\cdots-a_2b_{n-1}$$

for $n\geq 3$. For the rest of this proof, assume that $n\geq m+2$, and for later convenience, let

$$s_n=-a_nb_1-a_{n-1}b_2-\cdots-a_{m+2}b_{n-m-1}.$$

For $n\geq m+2$ (but not generally for $n=m+1$), the recurrence (1) gives

$$\sum_{k=1}^{m}u_kb_{n-k}=-\sum_{j=1}^{n-m-1}b_j\sum_{k=1}^{m}u_ka_{n-k-j+1}-U,$$

where

$$U = \sum_{k=1}^{m-1} u_k \sum_{j=2}^{m-k+1} a_j b_{n-k-j+1}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{m} u_k b_{n-k} &= -\sum_{j=1}^{n-m-1} b_j (a_{n+1-j} - r_{n+1-j-m}) - U \\
&= s_n + \sum_{j=1}^{n-m-1} b_j r_{n+1-j-m} - U \\
&= b_n + \sum_{j=2}^{m+1} a_j b_{n+1-j} + \sum_{j=1}^{n-m-1} b_j r_{n+1-j-m} - U,
\end{aligned}$$

so that

$$b_n = \sum_{k=1}^{m} u_k b_{n-k} - \sum_{j=2}^{m+1} a_j b_{n+1-j} - \sum_{j=1}^{n-m-1} b_j r_{n+1-j-m} + U. \tag{4}$$

Now put $n = m+1$ into (3) and substitute in (4) for a_{m+1}. The resulting coefficient of b_{n-m} is $-r_1$, and (4) simplifies to

$$\begin{aligned}
b_n &= \sum_{k=1}^{m-1} u_k b_{n-k} - \sum_{j=2}^{m} a_j b_{n+1-j} + \sum_{k=1}^{m-2} u_k \sum_{j=2}^{m-k} a_j b_{n-k-j+1} - \sum_{j=1}^{n-m} b_j r_{n+1-j-m} \\
&= \sum_{k=1}^{m-1} b_{n-k}(-a_{k+1} + \sum_{j=1}^{k} u_j a_{k+1-j}) - \sum_{j=1}^{n-m} b_j r_{n+1-j-m.} \qquad \blacksquare
\end{aligned}$$

Corollary 1. If the recurrence for A in (3) is homogeneous of order $m \geq 2$, then the recurrence for the sequence $(b_4, b_5, b_6, \ldots)$ is of order $m-1$. If $m = 2$, then the convolutory inverse of A is the sequence

$$(b_1, b_2, b_3, \ldots) = (1, \ -a_2, \ f, \ (u_1 - a_2)f, \ (u_1 - a_2)^2 f, \ (u_1 - a_2)^3 f, \ldots),$$

where $f = a_2^2 - a_3$.

Proof: Homogeneity of a means that $r_n = 0$ for $n \geq 1$, so that $b_n = \sum_{k=1}^{m-1} c_k b_{n-k}$ for $n \geq m+2$. $\blacksquare$

Example 1. The Fibonacci sequence, $A = (1, 1, 2, 3, 5, 8, \ldots)$, has inverse $(1, -1, -1, 0, 0, 0, 0, 0, \ldots)$.

Example 2. The Lucas sequence, $A = (1, 3, 4, 7, 11, 18, \ldots)$, has inverse, $(1, -3, 5, -10, 20, -40, 80, \ldots)$, recurrent with order 1 beginning at the third term.

Example 3. Let A be the 2nd-order nonhomogeneous sequence given by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2} + n - 2$ for $n \geq 3$. The inverse of A is the sequence $B = (1, -1, -2, -1, 1, 4, 6, 4, -4, -11, \ldots)$ given for $n \geq 4$ by

$$b_n = -B_{n-2} \circledast R_{n-2} = -(b_1, b_2, \cdots, b_{n-2}) \star (1, 2, 3, \ldots, n-2).$$

Example 4. Suppose that A and C are sequences in $\mathcal{G}$. Since $\mathcal{G}$ is a group, there exists B in $\mathcal{G}$ such that $A = B \star C$. For example, if A and C are the Fibonacci and Lucas sequences of Examples 1 and 2, then

$$B = A \star C^{-1} = (1, -2, 4, -8, 16, \ldots),$$

a 1st-order sequence.

Theorem 2. *Let $B = (1, b_2, b_3, \ldots)$ be the convolutory inverse of $A = (1, a_2, a_3, \ldots)$, and let $\widehat{A} = (1, -a_2, a_3, -a_4, a_5, -a_6, \ldots)$. Then the convolutory inverse of $\widehat{A}$ is the sequence $\widehat{B} = (1, -b_2, b_3, -b_4, b_5, -b_6, \ldots)$.*

Proof: Apply (2) to the subgroup $\mathbb{G}^{(1)}$ of sequential matrices. ■

Example 5. Let A be the sequence given by $a_n = \lfloor n\tau \rfloor$, where $\tau = (1 + \sqrt{5})/2$. Then

$$A = (1, 3, 4, 6, 8, 9, 11, 12, \ldots) \quad \text{and} \quad A^{-1} = (1, -3, 5, -9, 17, -30, 52, -90, \ldots).$$

Let A be the sequence given by $a_n = (-1)^{n-1}\lfloor n\tau \rfloor$. Then

$$A = (1, -3, 4, -6, 8, -9, 11, -12, \ldots) \quad \text{and} \quad A^{-1} = (1, 3, 5, 9, 17, 30, 52, 90, \ldots).$$

Example 6. Let A be the Catalan sequence, given by $a_n = \frac{1}{n}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}$. Then

$$\begin{aligned} A &= (1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots) \\ A^{-1} &= (1, -1, -1, -2, -5, -14, -42, -132, \ldots). \end{aligned}$$

Example 7. Let A be the sequence of central binomial coefficients, given by $a_n = \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}$, Then

$$A = (1, 2, 6, 20, 70, 252, 924, \ldots) \quad \text{and} \quad A^{-1} = (1, -2, -2, -4, -10, -28, -84, -264, \ldots),$$

with obvious connections to the Catalan sequence.

Certain operations on sequences in $\mathcal{G}$ are easily expressed in terms of convolution. Two of these operations are given as follows. Suppose x is an integer, and $A = (1, a_2, a_3, \ldots)$ is a sequence in $\mathcal{G}$, with inverse $B = (1, b_2, b_3, \ldots)$. Then

$$(1,\ xa_2,\ xa_3,\ xa_{4,} \ldots) = (1,\ (1-x)b_2,\ (1-x)b_3,\ (1-x)b_4, \ldots) \star A$$

and

$$(1,\ x,\ a_2,\ a_3,\ldots) = (1,\ x+b_2,\ (x-1)b_2+b_3,\ (x-1)b_3+b_4,\ldots)\star A.$$

Stated in terms of power series

$$a(t) = 1 + a_2 t + a_3 t^2 + \cdots \quad \text{and} \quad 1/a(t) = b(t) = 1 + b_2 t + b_3 t^2 + \cdots ,$$

the two operations correspond to the identities

$$\begin{aligned} xa(t) + 1 - x &= [(1-x)b(t) + x]a(t); \\ ta(t) + 1 + (x-1)t &= \{b(t) + [(x-1)b(t) + 1]t\}a(t). \end{aligned}$$

3 The group $(\mathbb{G}^{(m)}, \cdot)$

Recall that the set $\mathbb{G}$ consists of the lower triangular infinite integer matrices with all diagonal entries 1. Define $'$ on $\mathbb{G}$ as follows: if $A \in \mathbb{G}$, then A' is the matrix that remains when row 1 and column 1 of A are removed. Clearly $A' \in \mathbb{G}$. Define

$$A^{(0)} = A, \qquad\qquad A^{(n)} = (A^{(n-1)})'$$

for $n \geq 1$. Let

$$\mathbb{G}^{(m)} = \{A \in \mathbb{G} : A^{(m)} = A\}$$

for $m \geq 0$. Note that $(\mathbb{G}^{(1)}, \cdot)$ is the group of sequential matrices introduced in Section 1, and $\mathbb{G}^{(m)} \subset \mathbb{G}^{(d)}$ if and only if $d|m$.

Theorem 3. *$(G^{(m)}, \cdot)$ is a group for $m \geq 0$.*

Proof: $(\mathbb{G}^{(0)}, \cdot)$ is the group $(\mathbb{G}, \cdot)$. For $m \geq 1$, first note that $(AB)' = A'B'$, so that, inductively, $(AB)^{(q)} = A^{(q)}B^{(q)}$ for all $q \geq 1$. In particular, if A and B are in $\mathbb{G}^{(m)}$, then

$$(AB)^{(m)} = A^{(m)}B^{(m)} = AB,$$

so that $AB \in \mathbb{G}^{(m)}$. Moreover,

$$(A^{-1})^{(m)} = (A^{(m)})^{-1} = A^{-1},$$

so that $A^{-1} \in \mathbb{G}^{(m)}$. ■

4 The group $(\mathbb{G}^{(2)}, \cdot)$

Suppose that A, B, C, D are sequences in $\mathcal{G}$. Let $\langle A; B\rangle$ denote the matrix in $\mathbb{G}^{(2)}$ whose first column is $A = (a_1, a_2, \ldots)$ and whose second column is $(0, b_1, b_2, \ldots)$, where $a_1 = b_1 = 1$. We shall see that the product $\langle A; B\rangle \cdot \langle C; D\rangle$ is given by certain "mixed convolutions." Write $\langle A; B\rangle \cdot \langle C; D\rangle$ as $\langle U; V\rangle$. Then

$$u_n = \begin{cases} (a_1, b_2, a_3, \ldots, b_{n-1}, a_n) \star (c_1, c_2, \ldots, c_n), & \text{if } n \text{ is odd;} \\ (b_1, a_2, b_3, \ldots, b_{n-1}, a_n) \star (c_1, c_2, \ldots, c_n), & \text{if } n \text{ is even;} \end{cases}$$

$$v_n = \begin{cases} (b_1, a_2, b_3, \ldots, a_{n-1}, b_n) \star (d_1, d_2, \ldots, d_n), & \text{if } n \text{ is odd;} \\ (a_1, b_2, a_3, \ldots, a_{n-1}, b_n) \star (d_1, d_2, \ldots, d_n), & \text{if } n \text{ is even.} \end{cases}$$

In particular $\langle A; B\rangle \cdot \langle B; A\rangle$ is the sequential matrix of the sequence $A \star B$.

Recursive formulas for columns of $\langle A; B\rangle^{-1}$ can also be given: write $\langle A; B\rangle^{-1}$ as $\langle X; Y\rangle$, so that $\langle A; B\rangle \cdot \langle X; Y\rangle$ is the identity matrix. Each nondiagonal entry of $\langle A; B\rangle \cdot \langle X; Y\rangle$ is zero, so that, solving inductively for $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ gives

$$x_n = \begin{cases} -a_n - b_{n-1}x_2 - a_{n-2}x_3 - \cdots - b_2 x_{n-1}, & \text{if } n \text{ is odd;} \\ -a_n - b_{n-1}x_2 - a_{n-2}x_3 - \cdots - a_2 x_{n-1}, & \text{if } n \text{ is even;} \end{cases} \tag{5}$$

$$y_n = \begin{cases} -b_n - a_{n-1}y_2 - b_{n-2}y_3 - \cdots - a_2 y_{n-1}, & \text{if } n \text{ is odd;} \\ -b_n - a_{n-1}y_2 - b_{n-2}y_3 - \cdots - b_2 y_{n-1}, & \text{if } n \text{ is even.} \end{cases} \tag{6}$$

Example 8. Example 6 shows that the Catalan sequence satisfies the equation

$$(1, a_2, a_3, \ldots)^{-1} = (1, -1, -a_2, -a_3, \ldots),$$

which we abbreviate as $A^{-1} = (1, -A)$. It is natural to ask whether there are sequences A and B for which

$$\langle A; B\rangle^{-1} = \langle 1, -A; B\rangle . \tag{7}$$

This problem is solved as follows. Write the first and second columns of $\langle 1, -A; B\rangle$ as $(1, x_2, x_3, \ldots)$ and $(0, 1, y_2, y_3, \ldots)$, respectively. Equation (7) implies $x_n = -a_{n-1}$ and $y_n = b_n$ for $n \geq 2$. Thus, $b_2 = y_2$, but also, by (6), $y_2 = -b_2$, so that $b_2 = 0$. Inductively, (6) and (7) imply $b_n = 0$ for all $n \geq 3$, so that B is the convolutory identity sequence: $B = (1, 0, 0, 0, \ldots)$. Using this fact together with (5) gives

$$x_n = \begin{cases} -a_n - a_{n-2}x_3 - a_{n-4}x_5 - \cdots - a_2 x_{n-1}, & \text{if } n \text{ is even;} \\ -a_n - a_{n-2}x_3 - a_{n-4}x_5 - \cdots - a_3 x_{n-2}, & \text{if } n \text{ is odd;} \end{cases}$$

so that, substituting $x_k = -a_{k-1}$, we have a recurrence for A:

$$a_n = \begin{cases} a_{n-1} + a_{n-2}a_2 + a_{n-4}a_4 + \cdots + a_2 a_{n-2} & \text{if } n \text{ is even;} \\ a_{n-1} + a_{n-2}a_2 + a_{n-4}a_4 + \cdots + a_3 a_{n-3} & \text{if } n \text{ is odd;} \end{cases}$$

with intial values $a_1 = 1$, $a_2 = 1$. This sequence, listed as A047749 in [4], is given by

$$a_n = \begin{cases} \frac{1}{2m+1}\binom{3m}{m}, & \text{if } n = 2m; \\ \frac{1}{2m+1}\binom{3m+1}{m+1}, & \text{if } n = 2m+1. \end{cases}$$

Example 9. Let $a_n = 1$ and $b_n = F_n$ for $n \geq 1$, where F_n denotes the Fibonacci sequence in Example 1. Let C be the sequence given by $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $c_4 = 1$, and $c_n = 2^{\lfloor (n-5)/2 \rfloor}$ for $n \geq 5$. Let D be the sequence given by $d_1 = 1$, $d_2 = -1$, $d_3 = -1$, and $d_n = -c_{n+1}$ for $n \geq d_4$. Then $\langle A; B\rangle^{-1} = \langle C; D\rangle$.

Theorem 4. *If any three of four sequences A, B, C, D in G are given, then the fourth sequence is uniquely determined by the condition that $\langle A; B\rangle \cdot \langle C; D\rangle$ be a sequential matrix.*

Proof: The requirement that $\langle A; B\rangle \cdot \langle C; D\rangle$ be a sequential matrix is equivalent to an infinite system of equations, beginning with

$$\begin{aligned} d_1 &= 1 \\ b_2 + d_2 &= a_2 + c_2 \\ b_3 + a_2 d_2 + d_3 &= a_3 + b_2 c_2 + c_3. \end{aligned}$$

For $n \geq 3$, the system can be expressed as follows:

$$\begin{aligned} & b_n + a_{n-1} d_2 + b_{n-2} d_3 + \cdots + h_2 d_{n-1} + d_n \\ = \ & a_n + b_{n-1} c_2 + a_{n-2} c_3 + \cdots + h_2' c_{n-1} + c_n, \end{aligned} \tag{8}$$

where $h_2 = a_2$ if n is odd, $h_2 = b_2$ if n is even; and $h_2' = b_2$ if n is odd, $h_2' = a_2$ if n is even.

Equations (8) show that each of the four sequences is determined by the other three. ■

Example 10. By (8), D is determined by A, B, C in accord with the recurrence

$$\begin{aligned} d_n = \ & a_n + c_2 b_{n-1} + c_3 a_{n-2} + c_4 b_{n-3} + \cdots + c_{n-1} h_2' + c_n \\ & - b_n - d_2 a_{n-1} - d_3 b_{n-2} - d_4 a_{n-3} \cdots - d_{n-1} h_2. \end{aligned} \tag{9}$$

Suppose $a_n = b_n = c_{n-2} = 0$ for $n \geq 3$. Then by (9),

$$d_n = \begin{cases} -b_2 d_{n-1} - a_3 d_{n-2}, & \text{if } n \text{ is even;} \\ -a_2 d_{n-1} - b_3 d_{n-2}, & \text{if } n \text{ is odd;} \end{cases}$$

for $n \geq 4$, with $d_1 = 1$, $d_2 = a_2 - b_2$, $d_3 = a_3 - a_2 d_2 - b_3 d_1$. If $(a_1, a_2, a_3) = (1, -1, -1)$ and $(b_1, b_2, b_3) = (1, -2, -1)$ and $c_1 = 1$, then

$$D = (1, 1, 1, 3, 4, 11, 15, 41, 56, 153, \ldots),$$

which, except for the initial 1, is the sequence of denominators of the convergents to $\sqrt{3}$, indexed in [4] as A002530. In this example, $\langle A; B\rangle \cdot \langle C; D\rangle$ is the sequential matrix with first three terms $1, -1, -1$ and all others zero.

Theorem 5. *If A, B, C in G are given and $|a_2| = 1$, then there exists a unique sequence D in G such that $\langle A; B\rangle \cdot \langle C; D\rangle = \langle C; D\rangle \cdot \langle A; B\rangle$.*

Proof: Write $\langle A; B\rangle \cdot \langle C; D\rangle$ as (s_{ij}) and $\langle C; D\rangle \cdot \langle A; B\rangle$ as (t_{ij}). Equating $s_{n+1,1}$ and $t_{n+1,1}$ and solving for d_n give

$$d_n = \frac{1}{a_2}(u_n - v_n) \tag{10}$$

for $n \geq 3$, where

$$u_n = \begin{cases} c_2b_n + c_3a_{n-1} + c_4b_{n-2} + \cdots + c_na_2, & \text{if } n \text{ is odd;} \\ c_2b_n + c_3a_{n-1} + c_4b_{n-2} + \cdots + c_nb_2, & \text{if } n \text{ is even;} \end{cases}$$

$$v_n = \begin{cases} a_3c_{n-1} + a_4d_{n-2} + \cdots + a_nc_2, & \text{if } n \text{ is odd;} \\ a_3c_{n-1} + a_4d_{n-2} + \cdots + a_nd_2, & \text{if } n \text{ is even;} \end{cases}$$

with $d_1 = 1$, $d_2 = b_2c_2/a_2$. A sequence D is now determined by (10); we shall refer to the foregoing as part 1.

It is necessary to check that the equations $s_{n+1,2} = t_{n+1,2}$ implied by

$$\langle A; B\rangle \cdot \langle C; D\rangle = \langle C; D\rangle \cdot \langle A; B\rangle$$

do not impose requirements on the sequence D that are not implied by those already shown to determine D. In fact, the equations $s_{n+1,2} = t_{n+1,2}$ with initial value $d_1 = 1$ determine exactly the same sequence D. To see that this is so, consider the mapping $\langle A; B\rangle' = \langle B; A\rangle$. It is easy to prove the following lemma:

$$(\langle A; B\rangle \cdot \langle C; D\rangle)' = \langle B; A\rangle \cdot \langle D; C\rangle .$$

By part 1 applied to $\langle B; A\rangle \cdot \langle D; C\rangle$ and $\langle D; C\rangle \cdot \langle B; A\rangle$, the first column of $\langle B; A\rangle \cdot \langle D; C\rangle$ equals the first column of $\langle D; C\rangle \cdot \langle B; A\rangle$. Therefore, by the lemma, the second column of $\langle A; B\rangle \cdot \langle C; D\rangle$ equals the second column of $\langle C; D\rangle \cdot \langle A; B\rangle$, which is to say that the equations $s_{n+1,2} = t_{n+1,2}$ hold. ■

Example 11. Let $a_1 = 1$, $a_2 = 1$, and $a_n = 0$ for $n \geq 3$. Let B be the Fibonacci sequence. Let $C = (1, 1, 0, 1, 0, 0, \ldots)$, with $c_n = 0$ for $n \geq 5$. Then D is given by $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, and $d_n = L_{n-1}$ for $n \geq 4$, where (L_n) is the Lucas sequence, as in Example 1. Writing $\langle A; B\rangle \cdot \langle C; D\rangle$ as $\langle U, V\rangle$, we have $\langle U, V\rangle = \langle C; D\rangle \cdot \langle A; B\rangle$, where $U = (1, 2, 1, 3, 4, 7, 11, 18, \ldots)$ and $V = (1, 2, 5, 9, 20, 32, 66, 105, 207, \ldots)$.

5 Generalization of Theorem 4

It is natural to ask what sort of generalization Theorem 4 has for $m \geq 3$. The notation $\langle A; B\rangle$ used for matrices in $\mathbb{G}^{(2)}$ is now generalized in the obvious manner to $\langle A_1, A_2, \ldots, A_m\rangle$ in $\mathbb{G}^{(m)}$, where A_i is a sequence $(a_{i1}, a_{i2}, \ldots)$ having $a_{i1} = 1$, for $i = 1, 2, \ldots, m$.

Theorem 4A. *Suppose $A_1, A_2, \ldots, A_m$ and B_i for some i satisfying $1 \leq i \leq m$ are given. Then sequences B_j for $j \neq i$ are uniquely determined by the condition that $\langle A_1, A_2, \ldots, A_m\rangle \cdot \langle B_1, B_2, \ldots, B_m\rangle$ be a sequential matrix. Conversely, suppose $B_1, B_2, \ldots, B_m$ and A_i for some i satisfying $1 \leq i \leq m$ are given. Then sequences A_j for $j \neq i$ are uniquely determined by the condition that $\langle A_1, A_2, \ldots, A_m\rangle \cdot \langle B_1, B_2, \ldots, B_m\rangle$ be a sequential matrix.*

Proof: Let $U = AB$. For given A, each column of B uniquely determines the corresponding column of U, and each column of U determines the corresponding column of B. Thus, under the hypothesis that a particular column B_i of B is given, the equation $U = AB$ determines the corresponding column of U. Consequently, as U is a sequential matrix, every column of U is determined, and this implies that every column of B is determined.

For the converse, suppose B, together with just one column A_i of A, are given, and that the product $U = AB$ is sequential. As a first induction step,

$$a_{21}b_{11} + a_{22}b_{21} = a_{32}b_{22} + a_{33}b_{32} = \cdots \ . \tag{11}$$

As $a_{i+1,i}$ is given, equations (11) show that $a_{h+1,h}$ is determined for all $h \geq 1$. Assume for arbitrary $k \geq 1$ that $a_{h+j,h}$ is determined for all j satisfying $1 \leq j \leq k$, for all $h \geq 1$. As U is sequential,

$$\begin{aligned} & a_{k+1,1}b_{11} + a_{k+1,2}b_{21} + \cdots + a_{k+1,k+1}b_{k+1,1} \\ = \ & a_{k+2,2}b_{22} + a_{k+2,3}b_{32} + \cdots + a_{k+2,k+2}b_{k+2,2} \\ = \ & \cdots \ . \end{aligned} \tag{12}$$

As $a_{k+i,i}$ is given, equations (12) and the induction hypothesis show that $a_{k+h,h}$ is determined for all $h \geq 1$. Thus, by induction, A is determined. ■

Theorem 4A shows that Theorem 4 extends to $\mathbb{G}^{(m)}$. The method of proof of Theorem 4A clearly applies to $\mathbb{G}$, so that Theorem 4A extends to $\mathbb{G}$.

6 Transformations involving divisors

We return to the general group $(\mathbb{G}, \cdot)$ for a discussion of several specific matrix transformations involving divisors of integers. The first is given by the left summatory matrix,

$$T(n,k) = \begin{cases} 1, & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

The inverse of T is the left Möbius transformation matrix. The matrices T and T^{-1} are indexed as A077049 and A077050 in [4], where transformations by T and T^{-1} of selected sequences in $\mathcal{G}$ are referenced. In general, if A is a sequence written as an infinite column vector, then

$$T \cdot A = \{\sum_{k|n} a_k\} \qquad \text{and} \qquad T^{-1} \cdot A = \{\sum_{k|n} \mu(k) a_k\},$$

that is, the summatory sequence of A and the Möbius transform of A, respectively.

Next, define the *left summing matrix* $S = \{s(n,k)\}$ and the *left differencing matrix* $D = \{d(n,k)\}$ by

$$\begin{aligned} s(n,k) &= \begin{cases} 1, & \text{if } k \leq n; \\ 0, & \text{otherwise.} \end{cases} \\ d(n,k) &= \begin{cases} (-1)^{n+k}, & \text{if } k = n \text{ or } k = n-1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $D = S^{-1}$.

Example 12. Suppose that a sequence $C = (1, c_2, c_3, \ldots)$ in $\mathcal{G}$ is transformed to a sequence $A = (1, a_2, a_3, \ldots)$ by the sums $a_n = \sum_{k=1}^{n} c_k \lfloor n/k \rfloor$. In order to solve this system of equations, let $U(n,k) = \lfloor n/k \rfloor$ for $k \geq 1$, $n \geq 1$. Then $U = S \cdot T$, so that $U^{-1} = T^{-1} \cdot D$, which means that

$$c_n = \sum_{d|n} \mu(d)(a_{n/d} - a_{n/d-1}),$$

where $a_0 := 0$. If $a_n = 1$ for every $n \geq 1$, then $c_n = \mu(n)$. If $a_n = n$, then C is the convolutory identity, $(1, 0, 0, 0, \ldots)$. If $a_n = \binom{n+1}{2}$, then $c_n = \varphi(n)$. If $a_n = \binom{n+2}{3}$, then C is the sequence indexed as A000741 in [4] and discussed in [1] in connection with compositions of integers with relatively prime summands.

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2000 *Mathematics Subject Classification*: Primary 11A25.
Keywords: Appell sequence, convolution, Fibonacci sequence, linear recurrence, Riordan group, sequential matrix.

(Concerned with sequences A000045, A000108 A000142 A000201 A000204 A000741 A000984 A002530 A047749 A077049 A077050 A077605 A077606.)

Received November 13, 2002; revised versions received January 28, 2003; September 2, 2003. Published in *Journal of Integer Sequences*, September 8, 2003.

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