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ON SHANKS' ALGORITHM FOR COMPUTING THE CONTINUED FRACTION OF $\log_b a$

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Abstract. We give a more practical variant of Shanks' 1954 algorithm for computing the continued fraction of $\log_b a$, for integers $a > b > 1$, using the floor and ceiling functions and an integer parameter $c > 1$. The variant, when repeated for a few values of $c = 10^r$, enables one to guess if $\log_b a$ is rational and to find approximately r partial quotients.

1. Shanks' algorithm

In his article [1], Shanks gave an algorithm for computing the partial quotients of $\log_b a$, where $a > b$ are positive integers greater than 1. Construct two sequences $a_0 = a, a_1 = b, a_2, \ldots$ and $n_0, n_1, n_2, \ldots$, where the a_i are positive rationals and the n_i are positive integers, by the following rule: If $i \geq 1$ and $a_{i-1} > a_i > 1$, then

$$a_i^{n_{i-1}} \leq a_{i-1} < a_i^{n_{i-1}+1} \tag{1.1}$$
$$a_{i+1} = a_{i-1}/a_i^{n_{i-1}}. \tag{1.2}$$

Clearly (1.1) and (1.2) imply $a_i > a_{i+1} \geq 1$. Also (1.1) implies $a_i \leq a_{i-1}^{1/n_{i-1}}$ for $i \geq 1$ and hence by induction on $i \geq 0$,

$$a_{i+1} \leq a_0^{1/n_0 \cdots n_i}. \tag{1.3}$$

Also by induction on $j \geq 0$, we get

$$a_{2j} = a_0^r/a_1^s, \quad a_{2j+1} = a_1^u/a_0^v, \tag{1.4}$$

where r and u are positive integers and s and v are non–negative integers.

Two possibilities arise:

(i) $a_{r+1} = 1$ for some $r \geq 1$. Then equations (1.4) imply a relation $a_0^q = a_1^p$ for positive integers p and q and so $\log_{a_1} a_0 = p/q$.

(ii) $a_{i+1} > 1$ for all i. In this case the decreasing sequence $\{a_i\}$ tends to $a \geq 1$. Also (1.3) implies $a = 1$, unless perhaps $n_i = 1$ for all sufficiently large i; but then (1.2) becomes $a_{i+1} = a_{i-1}/a_i$ and hence $a = a/a = 1$.

If $a_{i-1} > a_i > 1$, then from (1.1) we have

$$n_{i-1} = \left\lfloor \frac{\log a_{i-1}}{\log a_i} \right\rfloor. \tag{1.5}$$

Let $x_i = \log_{a_{i+1}} a_i$ if $a_{i+1} > 1$. Then we have

Lemma 1. *If $a_{i+2} > 1$, then*

$$x_i = n_i + 1/x_{i+1}. \tag{1.6}$$

Proof. From (1.2), we have

$$\log a_{i+2} = \log a_i - n_i \log a_{i+1} \tag{1.7}$$

$$1 = \frac{\log a_i}{\log a_{i+1}} \cdot \frac{\log a_{i+1}}{\log a_{i+2}} - n_i \cdot \frac{\log a_{i+1}}{\log a_{i+2}} \tag{1.8}$$

$$= x_i x_{i+1} - n_i x_{i+1}, \tag{1.9}$$

from which (1.6) follows. □

From Lemma 1.1 and (1.5), we deduce

Lemma 2. (a) *If $\log_{a_1} a_0$ is irrational, then*

$$x_i = n_i + 1/x_{i+1} \text{ for all } i \geq 0.$$

(b) *If $\log_{a_1} a_0$ is rational, with $a_{r+1} = 1$, then*

$$x_i = \begin{cases} n_i + 1/x_{i+1}, & \text{if } 0 \leq i < r-1; \\ n_{r-1}, & \text{if } i = r-1. \end{cases}$$

In view of the equation $\log_{a_1} a_0 = x_0$, Lemma 2 leads immediately to

Corollary 1.

$$\log_{a_1} a_0 = \begin{cases} [n_0, n_1, \ldots], & \text{if } \log_{a_1} a_0 \text{ is irrational;} \\ [n_0, n_1, \ldots, n_{r-1}], & \text{if } \log_{a_1} a_0 \text{ is rational and } a_{r+1} = 1. \end{cases} \tag{1.10}$$

Remark. It is an easy exercise to show that for $j \geq 0$,

$$a_{2j} = a_0^{q_{2j-2}}/a_1^{p_{2j-2}}, \quad a_{2j+1} = a_1^{p_{2j-1}} a_0^{q_{2j-1}} \tag{1.11}$$

where p_k/q_k is the k–th convergent to $\log_{a_1} a_0$.

Example 1. $\log_2 10$: Here $a_0 = 10,\ a_1 = 2$. Then $2^3 < 10 < 2^4$, so $n_0 = 3$ and $a_2 = 10/2^3 = 1.25$.

Further, $1.25^3 < 2 < 1.25^4$, so $n_1 = 3$ and $a_3 = 2/1.25^3 = 1.024$.

Also, $1.024^9 < 1.25 < 1.024^{10}$, so $n_2 = 9$ and

$$\begin{aligned} a_4 &= 1.25/1.024^9 \\ &= 1250000000000000000000000000/1237940039285380274899124224 \\ &= 1.0097419586\cdots \end{aligned}$$

Continuing in this fashion, we obtain Table 1 and $\log_2 10 = [3, 3, 9, 2, 2, 4, 6, 2, 1, 1, \ldots]$.

i	n_i	a_i	p_i/q_i
0	3	10	3/1
1	3	2	10/3
2	9	1.25	93/28
3	2	1.024	196/59
4	2	$1.0097419586\cdots$	485/146
5	4	$1.0043362776\cdots$	2136/643
6	6	$1.0010415475\cdots$	13301/4004
7	2	$1.0001628941\cdots$	28738/8651
8	1	$1.0000637223\cdots$	42039/12655
9	1	$1.0000354408\cdots$	70777/21306
10		$1.0000282805\cdots$	
11		$1.0000071601\cdots$	

TABLE 1.

2. Some Pseudocode

In Table 2 we present pseudocode for the Shanks algorithm.

It soon becomes impractical to perform the calculations in multiprecision arithmetic, as the numerators and denominators a_i grow rapidly. If we truncate the decimal expansions of the `a[i]` to r places and represent a positive rational a as $g(a)/10^r$, where $g(a) = \lfloor 10^r a \rfloor$, the ratio `aa/bb` will be calculated as $\lfloor 10^r g(aa)/g(bb) \rfloor$. Working explicitly in integers, using the $g(a)$, then results in algorithm 1, also depicted in Table 2, with $c = 10^r$, where `int(x,y)` equals $\lfloor x/y \rfloor$, when x and y are integers.

As shown in the next section, the `A[i]` decrease strictly until they reach `c`. Also `m[0]=n[0]` and we can expect a number of the initial `m[i]` will be partial quotients. Naturally, the larger we take c, the more partial quotients will be produced.

Shanks' algorithm	algorithm 1

<pre>input: integers a>b>1
output: n[0],n[1],...
s:= 0
a[0]:= a; a[1]:= b
aa:= a[0]; bb:= a[1]
while(bb > 1){
  i:=0
  while(aa ≥ bb){
    aa:= aa/bb
    i:= i+1
  }
  a[s+2]:= aa
  n[s]:= i
  t:= bb
  bb:= aa
  aa:= t
  s:= s+1
}</pre>
	<pre>input: integers a>b>1, c> 1
	output: m[0],m[1],...
	s:= 0
	A[0]:= a*c; A[1]:= b*c
	aa:= A[0]; bb:= A[1]
	while(bb > c){
	  i:=0
	  while(aa ≥ bb){
	    aa:= int(aa*c,bb)
	    i:= i+1
	  }
	  A[s+2]:= aa
	  m[s]:= i
	  t:= bb
	  bb:= aa
	  aa:= t
	  s:= s+1
	}</pre>

Table 2.

3. Formal description of algorithm 1

We show in Theorem 2.1 below, that algorithm 1 will give the correct partial quotients when $\log_{a_1} a_0$ is rational and otherwise gives a parameterised sequence of integers which tend to the correct partial quotients when $\log_{a_1} a_0$ is irrational.

Algorithm 1 is now explicitly described. We define two integer sequences $\{A_{i,c}\}$, $i = 0, \ldots, l(c)$ and $\{m_{j,c}\}$, $j = 0, \ldots, l(c) - 2$, as follows.

Let $A_{0,c} = c \cdot a_0, A_{1,c} = c \cdot a_1$. Then if $i \geq 1$ and $A_{i-1,c} > A_{i,c} > c$, we define $m_{i-1,c}$ and $A_{i+1,c}$ by means of an intermediate sequence $\{B_{i,r,c}\}$, defined for $r \geq 0$, by $B_{i,0,c} = A_{i-1,c}$ and

$$B_{i,r+1,c} = \left\lfloor \frac{cB_{i,r,c}}{A_{i,c}} \right\rfloor, r \geq 0. \tag{3.1}$$

Then $c \leq B_{i,r+1,c} < B_{i,r,c}$, if $B_{i,r,c} \geq A_{i,c} > c$ and hence there is a unique integer $m = m_{i-1,c} \geq 1$ such that

$$B_{i,m,c} < A_{i,c} \leq B_{i,m-1,c}.$$

Then we define $A_{i+1,c} = B_{i,m,c}$. Hence $A_{i+1,c} \geq c$ and the sequence $\{A_{i,c}\}$ decreases strictly until $A_{l(c),c} = c$.

There are two possible outcomes, depending on whether or not $\log_b(a)$ is rational:

Theorem 2. (1) *If* $\log_{a_1} a_0$ *is a rational number* p/q *with* $p > q \geq 1$ *and* $\gcd(p, q) = 1$, *then*

(a) $a_0 = d^p$, $a_1 = d^q$ *for some positive integer* d*;*

(b) *if* $p/q = [n_0, \ldots, n_{r-1}]$*, where* $n_{r-1} > 1$ *if* $r > 1$*, then*
  (i) $A_{r+1,c} = c, a_{r+1} = 1$;
  (ii) $A_{i,c} = c \cdot a_i$ *for* $0 \le i \le r+1$;
  (iii) $m_{i,c} = n_i$ *for* $0 \le i \le r-1$.

(2) *If* $\log_{a_1} a_0$ *is irrational, then*
  (a) $m_{0,c} = n_0$;
  (b) $l(c) \to \infty$ *and for fixed* i*,* $A_{i,c}/c \to a_i$ *as* $c \to \infty$ *and* $m_{i,c} = n_i$ *for all large* c.

Proof. 1(a) follows from the equation $a_1^p = a_0^q$.

1(b) is also straightforward on noticing that a_i is a power of d and that we are implicitly performing Euclid's algorithm on the pair (p, q).

For 2(a), we have

$$a_1^{n_0} < a_0 < a_1^{n_0+1} \tag{3.2}$$

and $A_{0,c} = c \cdot a_0$, $A_{1,c} = c \cdot a_1$. Also by induction on $0 \le r \le n_0$,

$$B_{1,r,c} \geq ca_1^{n_0-r}, \tag{3.3}$$

$$B_{1,r,c} \leq \frac{ca_0}{a_1^r}. \tag{3.4}$$

Inequality (3.3) with $r \le n_0 - 1$ gives $B_{1,r,c} \ge A_{1,c}$, while inequality (3.4) with $r = n_0$ gives

$$B_{1,n_0,c} \le \frac{ca_0}{a_1^{n_0}} < ca_1 = A_{1,c},$$

by inequality (3.2). Hence $m_{0,c} = n_0$.

For 2(b), we use induction on $i \ge 1$ and assume $l(c) \ge i$ holds for all large c and that $A_{i-1,c}/c \to a_{i-1}$ and $A_{i,c}/c \to a_i$ as $c \to \infty$. This is clearly true when $i = 1$.

By properties of the integer part symbol, equation (3.1) gives

$$\frac{c^r A_{i-1,c}}{A_{i,c}^r} - \frac{(1 - \frac{c^r}{A_{i,c}^r})}{1 - \frac{c}{A_{i,c}}} < B_{i,r,c} \le \frac{c^r A_{i-1,c}}{A_{i,c}^r}. \tag{3.5}$$

for $r \ge 0$.

Hence for $r < n_{i-1}$, inequalities (3.5) give

$$B_{i,r,c}/c \to a_{i-1}/a_i^r \ge a_{i-1}/a_i^{n_{i-1}-1} > a_i.$$

Then, because $A_{i,c}/c \to a_i$, it follows that $B_{i,r,c} > A_{i,c}$ for all large c.

Also $B_{i,n_{i-1},c}/c \to a_{i-1}/a_i^{n_{i-1}} < a_i$, so $B_{i,n_{i-1},c} < A_{i,c}$ for all large c. Hence $m_{i-1,c} = n_{i-1}$ for all large c. Also $A_{i+1,c} = B_{i,n_{i-1},c} > c$, so $l(c) > i+1$ for all large c. Moreover $A_{i+1,c}/c \to a_{i-1}/a_i^{n_{i-1}} = a_{i+1}$ and the induction goes through. □

Example 3. Table 3 lists the sequences $m_{0,c}, \ldots, m_{l(c)-2,c}$ for $c = 2^u, u = 1, \ldots, 30$, when $a_0 = 3, a_1 = 2$.

```
1,1,
1,1,1,
1,1,1,1,
1,1,1,2,
1,1,1,2,
1,1,1,2,3,
1,1,1,2,2,2,
1,1,1,2,2,2,1,
1,1,1,2,2,2,1,2,
1,1,1,2,2,3,2,3,
1,1,1,2,2,3,2,
1,1,1,2,2,3,1,2, 1, 1,1, 2,
1,1,1,2,2,3,1,3, 1, 1,3, 1,
1,1,1,2,2,3,1,4, 3, 1,
1,1,1,2,2,3,1,4, 1, 9,1,
1,1,1,2,2,3,1,5,24, 1,2,
1,1,1,2,2,3,1,5, 3, 1,1, 2,7,
1,1,1,2,2,3,1,5, 2, 1,1, 5,3, 1,
1,1,1,2,2,3,1,5, 2, 2,1, 3,1,16,
1,1,1,2,2,3,1,5, 2,15,1, 6,2
1,1,1,2,2,3,1,5, 2, 9,5, 1,2,
1,1,1,2,2,3,1,5, 2,13,1, 1,1, 6,  1, 2, 2,
1,1,1,2,2,3,1,5, 2,17,2, 7,8,
1,1,1,2,2,3,1,5, 2,19,1,49,2, 1,
1,1,1,2,2,3,1,5, 2,22,4, 8,3, 4,  1,
1,1,1,2,2,3,1,5, 2,22,2, 1,3, 1,  3, 8,
1,1,1,2,2,3,1,5, 2,22,1, 6,3, 1,  1, 3, 4,  2,
1,1,1,2,2,3,1,5, 2,23,2, 1,1, 2,  1,12,17,
1,1,1,2,2,3,1,5, 2,23,3, 2,2, 2,  2, 1, 3,  2,
1,1,1,2,2,3,1,5, 2,23,2, 1,7, 2,  2,14, 1,  1, 6,
```

Table 3.

In fact $\log_2 3 = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, \ldots]$.

4. A heuristic algorithm

We can replace the $\lfloor x \rfloor$ function in equation (3.1) by $\lceil x \rceil$, the least integer exceeding x.

This produces an algorithm with similar properties to algorithm 1, with integer sequences $\{A'_{i,c}\}$, $i = 0, \ldots, l'(c)$ and $\{m'_{j,c}\}$, $j = 0, \ldots, l'(c)-2$. Here $A_{0,c} = A'_{0,c} = a_0 \cdot c$, $A_{1,c} = A'_{1,c} = a_1 \cdot c$ and $m_{0,c} = m'_{0,c} = n_0$. Then if $i \geq 1$ and $A'_{i-1,c} > A'_{i,c} > c$, we define $m'_{i-1,c}$ and $A'_{i+1,c}$ by means of an intermediate sequence $\{B'_{i,r,c}\}$, defined for $r \geq 0$, by $B'_{i,0,c} = A'_{i-1,c}$ and

$$B'_{i,r+1,c} = \left\lceil \frac{cB'_{i,r,c}}{A'_{i,c}} \right\rceil, r \geq 0. \tag{4.1}$$

Then $c \leq B'_{i,r+1,c} < B'_{i,r,c}$, if $B'_{i,r,c} \geq A'_{i,c} > c$.

For

$$B'_{i,r+1,c} \leq \frac{cB'_{i,r,c}}{A'_{i,c}} + 1$$

and

$$\frac{cB'_{i,r,c}}{A'_{i,c}} + 1 \leq B'_{i,r,c} \quad \Leftrightarrow \quad cB'_{i,r,c} + A'_{i,c} \leq A'_{i,c}B'_{i,r,c}$$

$$\Leftrightarrow \quad \frac{A'_{i,c}}{A'_{i,c} - c} \leq B'_{i,r,c}.$$

The last inequality is certainly true if $B'_{i,r,c} \geq A'_{i,c} > c$.

Hence there is a unique integer $m' = m'_{i-1,c} \geq 1$ such that

$$B'_{i,m',c} < A'_{i,c} \leq B'_{i,m'-1,c}.$$

Then we define $A'_{i+1,c} = B'_{i,m',c}$. Hence $A'_{i+1,c} \geq c$ and the sequence $\{A'_{i,c}\}$ decreases strictly until $A'_{l'(c),c} = c$.

If we perform the two computations simultaneously, the common initial elements of the sequences $\{m_{j,c}\}$ and $\{m'_{k,c}\}$ are likely to be partial quotients of $\log_b(a)$. With $c = 10^r$ we expect roughly r partial quotients to be produced.

If $l(c) = l'(c)$ and $A_{j,c} = A'_{j,c}$ and $m_{j,c} = m'_{j,c}$ for $j = 0, \ldots, l(c) - 2$, then $\log_b a$ is likely to be rational.

In practice, to get a feeling of certainty regarding the output when $c = 10^r$, we also run the algorithm for $c = 10^t, r - 5 \leq t \leq r + 5$.

Example 4. Table 4 lists the common values of $m_{i,c}$ and $m'_{i,c}$, when $a = 3, b = 2$ and $c = 2^r, 1 \leq r \leq 31$. It seems likely that only partial quotients are produced for all $r \geq 1$.

```
 1: 1
 2: 1
 3: 1,1,1
 4: 1,1,1
 5: 1,1,1,2
 6: 1,1,1,2
 7: 1,1,1,2,2
 8: 1,1,1,2,2
 9: 1,1,1,2,2
10: 1,1,1,2,2
11: 1,1,1,2,2
12: 1,1,1,2,2
13: 1,1,1,2,2,3,1
14: 1,1,1,2,2,3,1
15: 1,1,1,2,2,3,1
16: 1,1,1,2,2,3,1,5
17: 1,1,1,2,2,3,1,5
18: 1,1,1,2,2,3,1,5
19: 1,1,1,2,2,3,1,5,2
20: 1,1,1,2,2,3,1,5
21: 1,1,1,2,2,3,1,5,2
22: 1,1,1,2,2,3,1,5,2
23: 1,1,1,2,2,3,1,5,2
24: 1,1,1,2,2,3,1,5,2
25: 1,1,1,2,2,3,1,5,2
26: 1,1,1,2,2,3,1,5,2
27: 1,1,1,2,2,3,1,5,2
28: 1,1,1,2,2,3,1,5,2,23
29: 1,1,1,2,2,3,1,5,2,23
30: 1,1,1,2,2,3,1,5,2,23,2
31: 1,1,1,2,2,3,1,5,2,23,2
```

TABLE 4. $a = 3, b = 2, c = 2^r, 1 \leq r \leq 31$.

Example 5. Table 5 lists the common values of $m_{i,c}$ and $m'_{i,c}$, when $a = 34, b = 2$ and $c = 10^r, 1 \leq r \leq 20$. Partial quotients are not always produced, as is seen from lines 9,14 and 17.

```
1: 1,2,2
2: 1,2,2,1,1
3: 1,2,2,1,1,2
4: 1,2,2,1,1,2
5: 1,2,2,1,1,2,3,1
6: 1,2,2,1,1,2,3,1,8,1
7: 1,2,2,1,1,2,3,1,8,1,1
8: 1,2,2,1,1,2,3,1,8,1,1,2
9: 1,2,2,1,1,2,3,1,8,1,1,2,2,1,13,3,2,32,7
10:1,2,2,1,1,2,3,1,8,1,1,2,2,1
11:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
12:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1
13:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13
14:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,3
15:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2
16:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2
17:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,18,1,1,1,1,1
18:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
19:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
20:1,2,2,1,1,2,3,1,8,1,1,2,2,1,12,1,13,3,2,2,17,1
```

TABLE 5. $a = 34, b = 12, c = 10^r, r = 1, \ldots, 20$.

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