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Stability for Take-Away Games

Simon Rubinstein-Salzedo
Euler Circle
Palo Alto, CA 94306
USA
simon@eulercircle.com

Sherry Sarkar
College of Computing
Georgia Institute of Technology
Atlanta, GA 30332
USA
sherrys@gatech.edu

Abstract

In this paper, we study a family of take-away games called α-TAG, parametrized by a real number $\alpha \geq 1$. We show that for any given α, there is a half-open interval I_α containing α such that the set of losing positions for α-TAG is the same as the set of losing positions for β-TAG if and only if $\beta \in I_\alpha$. We then end with some results and conjectures on the nature of these intervals.

1 Introduction

In this paper, we study the losing positions of a certain family of games, known as *take-away games*. In our study, the games are indexed by a single parameter α, which is a real number greater than or equal to 1. It is also possible to study more general families of take-away games, as has been done by Zieve [14].

Here are the rules for the games we study. Let $\alpha \geq 1$ be a real number. We define α-TAG (short for α-TAKE-AWAY GAME) to be the two-player game played with following rules:

1. The game begins with n stones in one pile, for some nonnegative integer n. A *move* in this game consists of removing at least one stone from the pile.

2. The two players alternate making moves.
3. The first player may take up to $n-1$ stones.
4. After the first turn, a player can take up to α times the number of stones taken by the previous player on the last turn.

The winner of this game is the player who removes the last stone, or, more precisely, the loser is the player who is not able to remove a stone. (For instance, if $n=0$ or $n=1$, then the first player is not able to remove a stone, but the winner did not necessarily remove the last stone.)

Example 1. Below is an example of play in 3-TAG. Red positions and arrows denote the first player's moves, whereas blue positions and arrows denote the second player's moves. As we shall see, the first player plays correctly in this game.

$$38 \to 37 \to 36 \to 35 \to 32 \to 29 \to 23 \to 21 \to 18 \to 15 \to 11 \to 0$$

Since the game is symmetric in the two players, there are only two possible outcomes for α-TAG, assuming optimal play: either the first player has a winning strategy, or the second player has a winning strategy. In accordance with standard combinatorial game theory parlance, we call a position in which the first player has a winning strategy an $\mathcal{N}$ position, and a position in which the second player has a winning strategy a $\mathcal{P}$ position.

There is a useful recursive way of determining which positions are $\mathcal{N}$ positions and which are $\mathcal{P}$ positions, thanks to the following lemma:

Lemma 2 ([1, Theorem 2.13])**.** *A position is an $\mathcal{N}$ position iff there exists a move to a $\mathcal{P}$ position. A position is a $\mathcal{P}$ position iff all moves lead to $\mathcal{N}$ positions.*

Studying any impartial combinatorial game like α-TAG means determining which positions are the $\mathcal{P}$ positions and which are the $\mathcal{N}$ positions. Since in a typical game most positions are $\mathcal{N}$ positions, it is customary to focus on determining the smaller set of $\mathcal{P}$ positions. Formally, a position in α-TAG consists of two pieces of information: the *pile size* (i.e., the number of stones remaining), and the *move dynamic* (i.e., the maximum number of stones that may be removed on the next turn). However, in the current work, we are solely interested in determining the outcome class ($\mathcal{N}$ or $\mathcal{P}$) of the initial position, so we will be able to simplify our analysis by working only with the pile size, with a bit of care.

Definition 3. Let $T(\alpha)$ be the sequence of pile sizes n such that the only move a player can make to win α-TAG in a pile of size n with optimal play is to remove all remaining stones.

We note, of course, that during game play, it may not be possible to remove all the stones from a pile of size n; whether that move is allowable or not depends on the last move played. We also note that $T(\alpha)$ consists of exactly those n such that the initial position of α-TAG with n stones is a $\mathcal{P}$ position.

Example 4. For $\alpha = 2$, we have $3 \in T(2)$, since if the first player tries removing only 1 or 2 stones, the next player wins by removing the remaining stones. The only winning strategy for the first player would be to remove all the stones. If we consider the game of 4 stones, the first player could remove 1 stone leaving the second player with 3 stones. As noted previously, the only way to win a game of 3 stones is to remove all 3, and the second player is restricted to removing at most 2. Therefore, we see that $4 \notin T(2)$ since the first player can win with optimal play by playing some move other than removing all the stones.

Schwenk [8] showed that the sequence $T(\alpha)$ can be enumerated by a sequence which eventually satisfies a simple recurrence of the form $P_n = P_{n-1} + P_{n-k}$ for some k, for sufficiently large values of n; see Theorem 12.

The main result in this paper is Theorem 23, which says that the sequences $T(\alpha)$ change in discrete intervals based on α. For instance, if $1 \leq \alpha < 2$, then $T(\alpha) = (0, 1, 2, 4, 8, 16, \ldots)$ consists of 0 together with the powers of 2. Similarly, when $2 \leq \alpha < \frac{5}{2}$, then $T(\alpha) = (0, 1, 2, 3, 5, 8, 13, 21, \ldots)$ consists of the Fibonacci numbers. We think of this as a stability theorem for take-away games: even though the rules and allowable moves in the game differ whenever we change α even slightly (for sufficiently large n), these extra options do not change the optimal outcomes of the game. Most of the paper is devoted to proving this theorem, and then we end with some further results and questions about the nature of these stable intervals.

2 History

One commonly studied game, first introduced by Whinihan [12], is the $\alpha = 2$ version of the game described above, or better known as FIBONACCI NIM. The $T(\alpha)$ positions for this game are the Fibonacci numbers. FIBONACCI NIM is interesting because its winning strategy relies on the following theorem:

Theorem 5 (Zeckendorf, [6, 13])**.** *Every positive integer can be uniquely expressed as the sum of pairwise nonconsecutive Fibonacci numbers.*

Zeckendorf's theorem together with the following lemma provides us with a winning strategy for FIBONACCI NIM.

Lemma 6. *For $i \geq 2$, we have $F_{i+1} \leq 2F_i < F_{i+2}$.*

One can construct a winning strategy for any positive non-Fibonacci integer by combining Zeckendorf's theorem with Lemma 6. Suppose that there are n stones. We look at the Zeckendorf representation of n, say

$$n = F_{i_k} + F_{i_{k-1}} + \cdots + F_{i_1},$$

where for each j with $1 \leq j \leq k-1$ we have $i_{j+1} - i_j \geq 2$. If $k \geq 2$, then a winning strategy for the first player is to remove the smallest part of the Zeckendorf representation, i.e., F_{i_1}.

Due to Lemma 6, the second player will not be able to remove the entire next Zeckendorf part. Since all Fibonacci numbers are $T(\alpha)$ positions, the second player is forced to play essentially in the next term F_{i_2}, and lose in that part. We will see this line of reasoning again when we study the $T(\alpha)$ positions of the general α-TAG.

Example 7. We illustrate an example of Player 1 executing the winning strategy with 12 stones. The Zeckendorf decomposition of 12 stones is

$$12 = 8 + 3 + 1.$$

Therefore, the winning play looks like the following

$$12 \to 11 \to 9 \to 8 \to \cdots$$

Note that the first player removes the smallest Zeckendort part, therefore forcing the second player to play (and lose) the game of 3 stones, the next smallest Zeckendorf part. This forces the second player to begin the game of 8 stones which is another $T(\alpha)$ position.

The nature of our results are similar to those of Fraenkel [4] on Wythoff's game. Fraenkel also characterized the $\mathcal{P}$ positions of a parameter-based variant of Wythoff's game with a recurrence and with an algebraic formula. More generally, the questions we answer here are reminiscent of those asked by Duchêne, Fraenkel, Nowakowski, Rigo, and Ho [3, 5]. The authors of those two papers study the modifications that can be made to the set of Wythoff's game rules to keep the set of $\mathcal{P}$ positions constant.

3 $\mathcal{P}$ Positions of α-tag

In the previous section, we computed the sequence $T(2)$ and showed that it is the sequence of Fibonacci numbers. Next, we consider the sequence $T(\alpha)$ for an arbitrary real number $\alpha \geq 1$. The computation of the sequence $T(\alpha)$ relies on a generalization of Zeckendorf's theorem, first introduced by Schwenk [8]. Following Schwenk [8], we generate a sequence P^α as follows. Let the first two terms of P^α be $P^\alpha_0 = 0$, $P^\alpha_1 = 1$. Then define

$$P^\alpha_{k+1} = P^\alpha_k + P^\alpha_j,$$

where j is the the unique index such that

$$\alpha \cdot P^\alpha_j \geq P^\alpha_k > \alpha \cdot P^\alpha_{j-1}.$$

Example 8. The sequence P^2 is the Fibonacci sequence, as is $P^{2.4}$. On the other hand, $P^{2.5}$ is the sequence

$$1, 2, 3, 5, 7, 10, 15, 22, \ldots$$

There is a generalization of Zeckendorf's theorem based on the sequence P^α.

Theorem 9 (Generalized Zeckendorf theorem, [8]). *Any positive integer n can be uniquely expressed as a sum of terms of the sequence P with the following condition*

$$n = P^{\alpha}_{i_k} + P^{\alpha}_{i_{k-1}} + \cdots + P^{\alpha}_{i_1} \quad \textit{where} \quad \alpha \cdot P^{\alpha}_{i_j} < P^{\alpha}_{i_{j+1}} \quad \textit{for all } j < k.$$

The proof is very similar to that of the classical Zeckendorf theorem.

Theorem 10 ([8]). *For any $\alpha \geq 1$, the sequence $T(\alpha)$ is equal to the sequence (P^{α}_i).*

The details of the proof can be found in Schwenk's paper. The intuition, as described earlier, is that the winning strategy for α-TAG is to remove the smallest generalized Zeckendorf part. From now on, we will refer to P^{α}_i instead of $T(\alpha)$ for this sequence. When α is fixed or clear from context, we shall simply write P_i instead of P^{α}_i.

Definition 11. The *window* $W_\alpha(P^{\alpha}_i)$ of a term P^{α}_i is

$$W_\alpha(P^{\alpha}_i) = \{P^{\alpha}_j \in T(\alpha) : \alpha \cdot P^{\alpha}_{i-1} < P^{\alpha}_j \leq \alpha \cdot P^{\alpha}_i\}.$$

For some $P = P^{\alpha}_i \in T(\alpha)$, the window $W_\alpha(P)$ is the set of $Q = Q^{\alpha}_j \in T(\alpha)$ such that $P + Q = Q^{\alpha}_{j+1}$ is the next term in $T(\alpha)$. For P occurring early in the sequence $T(\alpha)$, $W_\alpha(P)$ may contain several elements. However, for sufficiently large values of $P \in T(\alpha)$, the $W_\alpha(P)$ consists of just a single element, and this is what causes the sequence of $T(\alpha)$ positions to satisfy a simple recurrence:

Theorem 12 ([8]). *Fix $\alpha \geq 1$. Then there exists an integer k such that, for sufficiently large values of n, we have $P^{\alpha}_n = P^{\alpha}_{n-1} + P^{\alpha}_{n-k}$.*

Corollary 13. *For n sufficiently large, $W_\alpha(P^{\alpha}_n)$ is a set of size 1.*

4 Lemmas about linear recurrences

In this section, we present some general lemmas about linear recurrences, as well as some about the specific family that are relevant to α-TAG; we provide references to the literature when we were able to find other sources for them.

Definition 14. Let $c_0, c_1, \ldots, c_{k-1} \in \mathbb{C}$. We say that a sequence of complex numbers $a_0, a_1, \ldots$ satisfies the *eventual linear recurrence relation* $a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$ if the relation holds for all sufficiently large n.

Lemma 15. *Let k be a positive integer, and let $a_0, a_1, \ldots$ be a sequence of complex numbers satisfying the eventual linear recurrence relation $a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$ for all sufficiently large n. Let $\chi(x) = x^k - c_{k-1}x^{k-1} - c_{k-2}x^{k-2} - \cdots - c_0$ be the characteristic polynomial of the eventual recurrence, and let $r_1, \ldots, r_k$ be its complex zeros, repeated with multiplicity. If all the r_i's are distinct, then there exist $\beta_1, \ldots, \beta_k \in \mathbb{C}$ such that*

$$a_n = \beta_1 r_1^n + \beta_2 r_2^n + \cdots + \beta_k r_k^n$$

for all sufficiently large n.

See [11, Theorem 4.1.1] for a proof.

From now on, we shall arrange the r_i's in decreasing order of magnitude: $|r_1| \geq |r_2| \geq \cdots \geq |r_k|$.

Lemma 16. *With the notation of Lemma 15, suppose that all the r_i's are distinct. Suppose furthermore that all the β_i's are nonzero. If $a_n > 0$ for all sufficiently large n, then r_1 is positive and real, $r_1 > |r_2|$, and $\beta_1 > 0$. We call r_1 the* positive dominant root.

See [2, Theorem 1] for a proof.

Lemma 17. *With the notation of Lemma 15, suppose that all the r_i's are distinct. Suppose also that the a_i's are all integers. Suppose that $\chi(x)$ factors over $\mathbb{Q}$ as*

$$\chi(x) = \chi_1(x)\chi_2(x)\cdots\chi_j(x),$$

where each $\chi_i(x)$ is irreducible over $\mathbb{Q}$. If $r_{i_1}, \ldots, r_{i_d}$ are the zeros of $\chi_1(x)$, then either $\beta_{i_1} = \beta_{i_2} = \cdots = \beta_{i_d} = 0$, or else all of $\beta_{i_1}, \ldots, \beta_{i_d}$ are nonzero.

Proof. By [11, Proposition 4.2.2], the generating function for a_n has the form

$$\sum_{n=0}^{\infty} a_n x^n = R(x) + \frac{\beta_{i_1}}{1 - r_{i_1}x} + \cdots + \frac{\beta_{i_k}}{1 - r_{i_k}x},$$

where $R(x) \in \mathbb{Q}(x)$. Let K be the Galois closure of $\mathbb{Q}(\beta_{i_1}, \ldots, \beta_{i_k}, r_{i_1}, \ldots, r_{i_k})(x)$ over $\mathbb{Q}(x)$, and let $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(x))$ be an arbitrary element. Then σ permutes $r_{i_1}, \ldots, r_{i_d}$, and since $\sum_{n=0}^{\infty} a_n x^n$ is fixed by σ, we must have

$$\sigma\left(\frac{\beta_{i_1}}{1 - r_{i_1}x}\right) = \frac{\beta_{i_j}}{1 - r_{i_j}x}$$

for some j with $1 \leq j \leq d$. Furthermore, $\mathrm{Gal}(K/\mathbb{Q}(x))$ acts transitively on the terms $\frac{\beta_{i_j}}{1-r_{i_j}x}$, so for each j with $1 \leq j \leq d$, there is some $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(x))$ that sends $\frac{\beta_{i_1}}{1-r_{i_1}x}$ to $\frac{\beta_{i_j}}{1-r_{i_j}x}$. Thus if $\beta_{i_1} \neq 0$, then $\beta_{i_k} \neq 0$ for $1 \leq j \leq d$, and vice versa. □

Lemma 18. *For all $k \geq 2$, $k \not\equiv 5 \pmod 6$ the polynomial $x^k - x^{k-1} - 1$ is irreducible over $\mathbb{Q}$. When $k \equiv 5 \pmod 6$, then $x^k - x^{k-1} - 1$ factors as $x^2 - x + 1$ times an irreducible factor.*

Remark 19. Note that $x^2 - x + 1 = \Phi_6(x)$ is the sixth cyclotomic polynomial, so its zeros are the primitive sixth roots of unity.

Proof. Selmer [9] shows that the polynomial $f(x) = x^k - x - 1$ is irreducible for all $k \geq 2$, and that $g(x) = x^k + x + 1$ is irreducible when $k \not\equiv 2 \pmod 3$, and factors as $x^2 + x + 1$ times an irreducible factor when $k \equiv 2 \pmod 3$. When k is even, we have $x^k - x^{k-1} - 1 = -x^k f(-\frac{1}{x})$, so it is irreducible. When k is odd, we have $x^k - x^{k-1} - 1 = x^k g(-\frac{1}{x})$, so it is irreducible when $k \not\equiv 5 \pmod 6$ and factors as $x^2 - x + 1$ times an irreducible factor when $k \equiv 5 \pmod 6$. □

Lemma 20. *If $k \geq 2$, then the polynomial $x^k - x^{k-1} - 1$ contains at most two zeros of any given absolute value.*

Proof. Selmer [9] shows that on any circle $|x| = r$ in the complex plane, the polynomials $x^k \pm (x+1)$ have only at most two zeros. Since the zeros of $x^k - x^{k-1} - 1$ are the negative reciprocals of the zeros of $x^k \pm (x+1)$ (depending on the parity of k), it follows that these polynomials also have at most two zeros on any given circle $|x| = r$. □

Lemma 21. *Let $k \geq 6$. With notation as in Lemma 15, if $a_n = a_{n-1} + a_{n-k}$ for all sufficiently large n, then $|r_2| > 1$, and r_2 is nonreal.*

Proof. First, note that $r_1 > 1$, because the product of the zeros is equal to ± 1, so some zero (and in particular the largest in absolute value) must have absolute value at least 1. Now suppose for some $k \geq 6$, we have that $|r_2| \leq 1$. We consider two cases: $|r_2| = 1$ and $|r_2| < 1$. Suppose first that $|r_2| < 1$. Then r_1 is a Pisot number, i.e., a real algebraic integer greater than 1, all of whose Galois conjugates have absolute value less than 1. As shown in [10], the smallest Pisot number is the positive zero of $x^3 - x - 1$, or $1.3247\ldots$. However, for every $k \geq 6$, $1.3^k - 1.3^{k-1} - 1 > 0$ whereas $1^k - 1^{k-1} - 1 = -1 < 0$, so $1 < r_1 < 1.3$. Thus r_1 cannot be a Pisot number.

Suppose now that $|r_2| = 1$. If $k \equiv 5 \pmod 6$, then Lemmas 18 and 20 imply that r_2 and r_3 are the primitive sixth roots of unity, and that $|r_4| < 1$. This means that r_1 is again a Pisot number. However, this cannot be the case for the same reason as before, as r_1 is smaller than the smallest Pisot number. On the other hand, if $k \not\equiv 5 \pmod 6$ and $|r_2| = 1$, then r_2 is a Galois conjugate of r_1, so r_1 is a Salem number, i.e., an algebraic integer greater than 1 all of whose conjugates have absolute values at most 1, with at least one of the conjugates having an absolute value equal to 1. As shown in [7, §6], the minimal polynomial of any Salem number is a reciprocal polynomial, i.e., a polynomial $p(x)$ such that $p(x) = x^{\deg(p)} p(\frac{1}{x})$. Since $x^k - x^{k-1} - 1$ is not a reciprocal polynomial, r_1 cannot be a Salem number. Thus $|r_2| > 1$ for all $k \geq 6$.

Finally, we must show that r_2 is nonreal. When k is odd, r_1 is the only real zero of $x^k - x^{k-1} - 1$, so clearly r_2 is nonreal. When k is even, $x^k - x^{k-1} - 1$ has two real zeros: the positive zero r_1 and a negative zero. However, the negative zero lies between -1 and 0 and is thus not r_2 for $k \geq 6$, since $|r_2| > 1$. □

Lemma 22. *Let $k \geq 6$. If $a_0, a_1, a_2, \ldots$ is a sequence of positive integers satisfying $a_n = a_{n-1} + a_{n-k}$ for all sufficiently large n, then, with notation as in Lemma 15, r_1 is real, $\beta_1 > 1$, $|r_1| > |r_2| = |r_3| > |r_4|$, and $\beta_2, \beta_3 \neq 0$. Furthermore, $\beta_3 = \bar{\beta}_2$, where the bar denotes complex conjugation.*

Proof. By Lemma 16 and the assumption that a_n is positive and satisfies the recurrence $a_n = a_{n-1} + a_{n-k}$ for all sufficiently large n, it follows immediately r_1 is real, $\beta_1 > 1$, and $|r_1| > |r_2|$. Furthermore, r_2 is nonreal by Lemma 21. Since nonreal zeros of polynomials with real coefficients come in complex conjugate pairs, it follows that the complex conjugate $\bar{r}_2$ of r_2 is also a zero of $x^k - x^{k-1} - 1$. Thus $|r_2| = |r_3|$. By Lemma 20, $|r_2| > |r_4|$.

To see that $\beta_2, \beta_3 \neq 0$, note that all the zeros of $\chi(x)$, except possibly the two sixth roots of unity satisfying $x^2 - x + 1$, have the same minimal polynomial over $\mathbb{Q}$ by Lemma 18. Since $|r_2| > 1$, r_2 is not one of those roots of unity. Thus r_1, r_2, r_3 are all zeros of the same irreducible factor of χ, and since $\beta_1 \neq 0$, Lemma 17 implies that $\beta_2, \beta_3 \neq 0$ as well.

To see that $\beta_3 = \bar{\beta}_2$, note that since $\mathrm{Gal}(\mathbb{C}(x)/\mathbb{R}(x)) = \{1, z \mapsto \bar{z}\}$ acts on the $\frac{\beta_i}{1-r_i x}$'s in the partial fraction decomposition of $\sum_{n=0}^{\infty} a_n x^n$ and complex conjugation sends r_2 to r_3, it must send $\frac{\beta_2}{1-r_2 x}$ to $\frac{\beta_3}{1-r_3 x}$. Thus $\bar{\beta}_2 = \beta_3$. □

5 Stability

We now come to the main result of the paper.

Theorem 23. *For any $\alpha \geq 1$, there exists a half-open interval $I_\alpha = [\alpha_0, \alpha_1)$ containing α such that for any $\beta \in I_\alpha$, the sequence P_i^β is the same as the sequence P_i^α, and for all $\beta \notin I_\alpha$, the two sequences are not the same, in that there is some integer i for which $P_i^\alpha \neq P_i^\beta$.*

Before we prove Theorem 23, let us take a look at why it ought to be true, by means of a typical example. Let us suppose that $\alpha = 3$ and look at the sequence P_n^3. This sequence begins

$$P_n^3 : 0, 1, 2, 3, 4, 6, 8, 11, 15, 21, 29, 40, 55, \ldots,$$

with $P_n^3 = P_{n-1}^3 + P_{n-4}^3$ for sufficiently large n. For example, $P_8^3 = 15$. To compute P_9^3, we must add to $P_8^3 = 15$ the unique P_i^3 for which

$$3P_{i-1}^3 < P_8^3 \leq 3P_i^3, \tag{1}$$

which is 6. Thus $P_9^3 = 15 + 6 = 21$. If we were to increase 3 to $\frac{15}{4}$ and all the previous $\mathcal{P}$-positions in $\frac{15}{4}$-TAG agreed with those of 3-TAG, then the left inequality in (1) with 3 replaced with $\frac{15}{4}$, namely

$$\frac{15}{4} P_{i-1}^{\frac{15}{4}} < P_8^{\frac{15}{4}},$$

would fail since $P_{i-1}^{\frac{15}{4}} = 4$. Note that if we replace 3 with $\frac{15}{4} - \varepsilon$ for any $\varepsilon > 0$ in the left inequality of (1), the inequality would still hold. Thus *if* all the $\mathcal{P}$-positions of 3-TAG and $\frac{15}{4}$-TAG agree up to 15, then the next term is definitely different.

We can perform analogous calculations starting from any term of the sequence P_n^3. If $\alpha > 3$, the only way that the sequence P_n^α could differ from P_n^3 is if α is greater than the analogous ratio, starting with some term of P_n^3. In fact, one of these ratios is $\frac{21}{6} = \frac{7}{2}$, so the $\mathcal{P}$-positions of α-TAG are only equal to those of 3-TAG when $3 \leq \alpha < \frac{7}{2}$.

The proof of Theorem 23 is now reduced to showing that, for any α, the infimum of the sequence of such ratios is achieved. In particular, since all the ratios are greater than α, it follows that this infimum is strictly greater than α.

To this end, we introduce some notation for these ratios. Fix an α, and define a sequence $Q_k = Q_k^\alpha$ by setting

$$Q_k^\alpha = \frac{\widehat{P}_k^\alpha}{P_k^\alpha},$$

where

$$\widehat{P}_k^\alpha = \min\{P_i^\alpha \in T(\alpha) : P_i^\alpha > \max(W_\alpha(P_k^\alpha))\}$$

is the smallest term in the sequence P_i^α greater than all the elements of the window of P_k^α. Alternatively, $\widehat{P}_k^\alpha = P_{j+1}^\alpha$, where $P_j^\alpha = \max(W_\alpha(P_k))$. As discussed above, the next $\beta > \alpha$ for which there exists an i such that $P_i^\beta \neq P_i^\alpha$ is $\inf_k\{Q_k^\alpha\}$.

The following lemma will be key to proving Theorem 23.

Lemma 24. *Let $\alpha \geq 2$ be a real number. The sequence $Q_n = Q_n^\alpha$ converges to some real number $r_1 > 1$, and Q oscillates around the point of its convergence, in the sense that there are arbitrarily large integers n such that $Q_n > r_1$, as well as arbitrarily large integers n such that $Q_n < r_1$.*

Proof. There is some positive integer $k \geq 2$ such that the sequence P^α satisfies the linear recurrence of $P_n = P_{n-1} + P_{n-k}$ for all sufficiently large k. When $k \leq 5$, the remainder of the proof requires minor modifications since we cannot quite use Lemma 22, but most of it still works in that case as well. The cases $k \leq 5$ can also be checked by hand if desired. When $k = 2$, r_2 is real, so a slightly different argument must be made, but again, most of the proof still works. From now on, we will assume that $k \geq 6$.

Since we are interested in the limiting or tail behavior of the sequence Q_n, we may ignore the initial terms, where P_n does not satisfy the eventual recurrence $P_n = P_{n-1} + P_{n-k}$. Let us consider the characteristic polynomial of the recurrence

$$\chi(x) = x^k - x^{k-1} - 1,$$

and let the zeros of χ be $r_1, r_2, r_3, \ldots, r_k$, where $|r_1| > |r_2| \geq \cdots \geq |r_k|$. By Lemma 15, we know that there exist $\beta_1, \ldots, \beta_k \in \mathbb{C}$ such that

$$P_n = \beta_1 r_1^n + \beta_2 r_2^n + \beta_3 r_3^n + \cdots + \beta_k r_k^n$$

for all sufficiently large values of n. The sequence of ratios eventually converges to r_1^k. We want to know if the sequence of ratios oscillate below and above r_1^k. Thus, we study the sequence

$$\frac{P_{n+k}}{P_n} - r_1^k.$$

We have

$$\begin{aligned}\frac{P_{n+k}}{P_n} - r_1^k &= \frac{r_1^{n+k} + \frac{\beta_2}{\beta_1} r_2^{n+k} + \frac{\beta_3}{\beta_1} r_3^{n+k} + \cdots + \frac{\beta_k}{\beta_1} r_k^{n+k}}{r_1^n + \frac{\beta_2}{\beta_1} r_2^n + \frac{\beta_3}{\beta_1} r_3^n + \cdots + \frac{\beta_k}{\beta_1} r_k^n} - r_1^k \\ &= \frac{\beta_2 r_2^n (r_2^k - r_1^k) + \beta_3 r_3^n (r_3^k - r_1^k) + \cdots + \beta_k r_k^n (r_k^k - r_1^k)}{\beta_1 r_1^n + \beta_2 r_2^n + \beta_3 r_3^n + \cdots + \beta_k r_k^n}.\end{aligned}$$

The denominator is positive since it is just equal to P_n. We must show that the numerator is positive for infinitely many n and negative for infinitely many n. Note that $\beta_i, (r_i^k - r_1^k)$ are all constants; only $r_2^n, r_3^n, \ldots, r_k^n$ change as a function of n.

At this point, we are trying to determine if

$$\beta_2 r_2^n (r_2^k - r_1^k) + \beta_3 r_3^n (r_3^k - r_1^k) + \cdots + \beta_k r_k^n (r_k^k - r_1^k) \tag{2}$$

displays oscillatory behavior as a function of n. By Lemma 22, $\beta_2, \beta_3 \neq 0$ and $|r_3| > |r_4|$, so for sufficiently large values of n, r_2^n and r_3^n will dominate the rest of the terms, so for sufficiently large values of n, the sign of (2) will be the same as the sign of $\beta_2 r_2^n (r_2^k - r_1^k) + \beta_3 r_3^n (r_3^k - r_1^k)$. Note that the behavior of terms r_2 and r_3, the next zeros of largest magnitude, are what really determine the behavior of the entire sequence for sufficiently large n. Let us write

$$r_2^k - r_1^k = \rho e^{i\phi}, \qquad r_3^k - r_1^k = \rho e^{-i\phi}$$

and

$$r_2 = r e^{i\theta}, \qquad r_3 = r e^{-i\theta}.$$

Then we have

$$r_2^n = r^n e^{in\theta}, \qquad r_3^n = r^n e^{-in\theta}.$$

Thus we have

$$\beta_2 r_2^n (r_2^k - r_1^k) + \beta_3 r_3^n (r_3^k - r_1^k) = \beta_2 r^n e^{in\theta} \rho e^{i\phi} + \beta_3 r^n e^{-in\theta} \rho e^{-i\phi}.$$

Since $\bar{\beta}_2 = \beta_3$ by Lemma 22, we may also write

$$\beta_2 = s e^{i\psi}, \qquad \beta_3 = s e^{-i\psi},$$

so that we have

$$\beta_2 r_2^n (r_2^k - r_1^k) + \beta_3 r_3^n (r_3^k - r_1^k) = 2 s r^n \rho \cos(\psi + n\theta + \phi).$$

Since ϕ, ψ, and θ are fixed and $\theta \not\equiv 0 \pmod{\pi}$, we know that $\cos(\psi + n\theta + \phi)$ is positive for infinitely many values of n and negative for infinitely many values of n. Thus there are infinitely many values of n for which $Q_n > r_1$, and infinitely many values of n for which $Q_n < r_1$, as desired. $\square$

Using Lemma 24, we can now prove Theorem 23.

Proof of Theorem 23. Define Q_i^α by

$$Q_i^\alpha = \frac{P_{j+1}^\alpha}{P_i^\alpha}$$

where $P_j = \max(W_\alpha(P_i))$. We established in Lemma 24 that Q^α has a minimum. Say we have some $\alpha < \beta < \min(Q^\alpha)$. We will show that $P^\beta = P^\alpha$. Say $P^\beta \neq P^\alpha$. A sequence of $T(\alpha)$ positions is determined by

$$P_{i+1} = P_i + P_j \quad \text{if} \quad P_j \in W_\alpha(P_i).$$

If $P^\beta \neq P^\alpha$, this implies there is a first occurrence of i such that $W_\beta(P_i^\beta) \neq W_\alpha(P_i^\alpha)$. Since $\beta > \alpha$, this means that $\max(W_\beta(P_i^\beta)) > \max(W_\alpha(P_i^\alpha))$. Say $P_j = \max(W_\alpha(P_i^\alpha))$. Then $\max(W_\beta(P_i^\beta)) \geq P_{j+1}$, which means

$$P_{j+1} \leq \beta \cdot P_i,$$

or

$$Q_i = \frac{P_{j+1}}{P_i} \leq \beta,$$

contrary to our assumption. Next, we show that if $P^\beta = P^\alpha$, then $\beta < \min(Q^\alpha)$. Say $\beta \geq \min(Q^\alpha)$. Let the index at which Q^α reaches its minimum be k. The sequence $T(\alpha)$ is determined by

$$P_{i+1} = P_i + P_j \quad \text{if} \quad P_j \in W_\alpha(P_i).$$

We will show that $\max(W_\beta(P_k^\beta)) > \max(W_\alpha(P_k^\alpha))$. Let $\max(W_\alpha(P_k^\alpha)) = P_x$. Thus, $\min(Q^\alpha) = \frac{P_{x+1}}{P_k}$. Note that

$$\alpha P_{k-1} < P_x \leq \alpha \cdot P_k$$

and

$$P_x < P_{x+1} \leq \beta \cdot P_k.$$

Therefore, $\max(W_\beta(P_k^\beta)) \geq P_{x+1} > P_x = \max(W_\alpha(P_k^\alpha))$. But since $W_\beta(P_k^\beta) \neq W_\alpha(P_k^\alpha)$, $P^\beta \neq P^\alpha$, which is a contradiction. □

In short, the $T(\alpha)$ positions remain the same in certain intervals as α changes. Table 1 shows the first several stable intervals. Note that the same eventual recurrence can describe more than one set of $T(\alpha)$ positions, as seen with the recurrence $P_n = P_{n-1} + P_{n-5}$. This is because it takes longer for the recurrence to start holding when $\frac{7}{2} \leq \alpha < \frac{11}{3}$ than it does when $\frac{11}{3} \leq \alpha < \frac{43}{11}$.

6 Cutoffs

Definition 25. A *cutoff* is some number $\alpha \geq 1$ such that, for any $\beta < \alpha$, the sequences P_n^α and P_n^β are not identical.

In other words, the cutoffs are the endpoints of the stable intervals of Theorem 23. The first few cutoffs are $1, 2, \frac{5}{2}, 3, \frac{7}{2}, \frac{11}{3}, \frac{43}{11}, 4, \frac{13}{3}$.

Range	Eventual recurrence	Initial conditions
$1 \leq \alpha < 2$	$P_n = P_{n-1} + P_{n-1}$	0,1
$2 \leq \alpha < \frac{5}{2}$	$P_n = P_{n-1} + P_{n-2}$	0,1,2
$\frac{5}{2} \leq \alpha < 3$	$P_n = P_{n-1} + P_{n-3}$	0,1,2,3,5
$3 \leq \alpha < \frac{7}{2}$	$P_n = P_{n-1} + P_{n-4}$	0,1,2,3,4,6
$\frac{7}{2} \leq \alpha < \frac{11}{3}$	$P_n = P_{n-1} + P_{n-5}$	0,1,2,3,4,6,8,11,15,21
$\frac{11}{3} \leq \alpha < \frac{43}{11}$	$P_n = P_{n-1} + P_{n-5}$	0,1,2,3,4,6,8,11
$\frac{43}{11} \leq \alpha < 4$	$P_n = P_{n-1} + P_{n-6}$	0,1,2,3,4,6,8,11,14,18,24,32,43
$4 \leq \alpha < \frac{13}{3}$	$P_n = P_{n-1} + P_{n-6}$	0,1,2,3,4,5,7,9,12
$\frac{13}{3} \leq \alpha < \frac{31}{7}$	$P_n = P_{n-1} + P_{n-7}$	0,1,2,3,4,5,7,9,12,15,19,24,31,40,52
$\frac{31}{7} \leq \alpha < \frac{9}{2}$	$P_n = P_{n-1} + P_{n-7}$	0,1,2,3,4,5,7,9,12,15,19,24,31
$\frac{9}{2} \leq \alpha < \frac{14}{3}$	$P_n = P_{n-1} + P_{n-7}$	0,1,2,3,4,5,7,9,11,14,18

Table 1: Stable intervals for α-TAG

Remark 26. Before proving Theorem 23, it might be more natural to define a cutoff to be a number $\alpha \geq 1$ such that for any $\beta < \alpha$ and any $\gamma > \alpha$, the sequences P_n^β and P_n^γ are not identical. Theorem 23 implies that these two definitions coincide, but later in this section we will see that it is possible to prove parts of the Theorem 23 in a simpler way but that does not guarantee that the two definitions match.

Corollary 27. *All cutoffs are rational numbers.*

Proof. The cutoffs are infima of sequences of rational numbers, and these infima are always achieved and hence rational. □

In order to investigate the cutoffs more thoroughly, we consider a new sequence generated from the sequence P_i^α.

Definition 28. The *sequence of indices of recurrence* S_i^α is defined by

$$S_i^\alpha = \max\{j : P_i^\alpha + P_{i+j-1}^\alpha = P_{i+j}^\alpha\}.$$

Example 29. Let $\alpha = \frac{7}{2}$. Then we have the following initial values of P_i and S_i:

P_i	1	2	3	4	6	8	11	15	21	27	35	46	61
S_i	3	4	4	4	5	5	5	5	5	5	5	5	5

Lemma 30 ([8])**.** *For some α-TAG, with $T(\alpha)$ positions P_t, if $\alpha \cdot P_{i-1} < P_j \leq \alpha \cdot P_i$, then $\alpha \cdot P_{i+1} \geq P_{j+1}$.*

Lemma 31. *For every i, we have $S_i^\alpha \leq S_{i+1}^\alpha$.*

Proof. Recall that the window $W_\alpha(P_i^\alpha)$ of $P_i^\alpha \in T(\alpha)$ is

$$W_\alpha(P_i^\alpha) = \{P_j^\alpha \in T(\alpha) : P_i^\alpha + P_j^\alpha = P_{j+1}^\alpha \in T(\alpha)\}.$$

We proved previously that

$$W_\alpha(P_i^\alpha) = \{P_j \in T(\alpha) : \alpha \cdot P_{i-1} < P_j \leq \alpha \cdot P_i\}.$$

Say $P_j = \max\{W_\alpha(P_i^\alpha)\}$. Since

$$\alpha \cdot P_{i-1} < P_j \leq \alpha \cdot P_i,$$

Lemma 30 implies that $P_{j+1} \leq \alpha \cdot P_{i+1}$. Next, we prove that $\alpha \cdot P_i < P_{j+1}$. We prove this with contradiction. Assume $P_{j+1} \leq \alpha \cdot P_i$. This would imply

$$\alpha \cdot P_{i-1} < P_j < P_{j+1} \leq \alpha \cdot P_i.$$

This means $P_{j+1} \in W_\alpha(P_i^\alpha)$. However, we said $P_j = \max\{W_\alpha(P_i^\alpha)\}$ so this is a contradiction. Therefore, we have shown that

$$\alpha \cdot P_i < P_{j+1} \leq \alpha \cdot P_{i+1}.$$

So, from assumption, $P_j = \max\{W_\alpha(P_i^\alpha)\}$, and Lemma 30 implies $P_{j+1} \in W_\alpha(P_{i+1}^\alpha)$. Thus $S_i^\alpha = j - i - 1$ and $S_{i+1}^\alpha \geq j + 1 - (i+1) - 1 = j - i - 1$. Therefore, Lemma 30 implies that S^α is a monotonically increasing sequence. □

Lemma 32 ([8])**.** *Suppose there exists a j such that*

$$P_{j+i+1} = P_{j+i} + P_{j+i-k} \tag{3}$$

for all $i \in \{0, 1, \ldots, k+1\}$. Then (3) holds for every nonnegative integer i.

Lemma 33. *The number of cutoffs in any closed interval $[a, b]$ is finite.*

Proof. We first prove that the number of eventual recurrences in the interval is finite. There are at least two ways of doing this. One way would be to prove that the degree k of the eventual recurrence increases with α; this is true, but we have not proven it. An alternative approach is to use a result of Zieve [14]. Zieve proves that

$$\frac{\log(\alpha - 1)}{\log(\alpha) - \log(\alpha - 1)} \leq k \leq \frac{\log(\alpha)}{\log(\alpha + 1) - \log(\alpha)}.$$

It follows that for all $\alpha \in [a, b]$, we have

$$\frac{\log(a - 1)}{\log a - \log(a - 1)} \leq k \leq \frac{\log(b)}{\log(b + 1) - \log b}.$$

Since k is an integer, there are only finitely many eventual recurrences in a closed interval. Thus it remains to show that there are only finitely many sequences with the eventual recurrence $P_n = P_{n-1} + P_{n-k}$. From Lemma 31, we know that S^α is an increasing sequence. By Lemma 32, any positive integer $m \le k$ can appear at most $m+1$ times in the sequence S^α. Thus there are only finitely many possible initial strings of the sequence S^α before the sequence stabilizes at k. It follows that there are only finitely many cutoffs in any closed interval $[a, b]$. □

Remark 34. Lemma 33 almost gives us another proof of Theorem 23: it shows that the $T(\alpha)$ positions remain constant on intervals, except for a discrete set of exceptional points. However, we were not able to see how to use Lemma 33 to show that there are no exceptional points. Note that Definition 28 through Lemma 33 do not rely on the proof of Theorem 23.

Theorem 35. *Every integer $n \ge 2$ is a cutoff.*

Proof. Let $n \ge 2$ be an integer, let α be the last cutoff before n. The largest element of $W_\alpha(1)$ is $\lfloor \alpha \rfloor < n$. Consider the sequence $T(n)$ and $W_n(1)$. The largest element of $W_n(1)$ is n. Therefore, $W_n(1) \ne W_\alpha(1)$. We assumed α to be the last cutoff before n. Thus, n is the next cutoff. □

We can also prove a generalization of this theorem.

Theorem 36. *Let $x \equiv 0 \pmod{n!}$ and $x > 0$. Then $x + \frac{1}{n}$ is a cutoff.*

Before we prove Theorem 36, let us explain the intuitive reason behind it, which we make precise using windows. Let α be the largest cutoff before $x + \frac{1}{n}$. Since all integers are cutoffs, we have $\alpha \ge x$. The sequence $T(\alpha)$ begins as an arithmetic progression with difference 1, then becomes an arithmetic progression with difference 2, then by 3, and so forth until it becomes an arithmetic progression with difference n:

$$0, 1, \ldots, x, x+1, x+3, x+5, \ldots, 2x+1, 2x+4, \ldots, 3x+1, 3x+5, \ldots, nx+1, nx+n+2.$$

Recall that the next cutoff after α can be thought of as the minimum of Q_i^α. One of the elements of Q_i^α (which therefore upper bounds the next cutoff after α) is $Q_n = \frac{nx+1}{n}$.

Proof of Theorem 36. We proceed by induction on n, proving the given statement together with an auxiliary result that aids in the inductive step. The auxiliary result is that if α is the largest cutoff less than $x + \frac{1}{n}$, then $\max(W_\alpha(n)) = nx - n + 1$. For the original statement, the base case, $n = 1$, is simply Theorem 35. For the auxiliary statement, the largest cutoff less than $x + 1$ is simply x because the sequence $T(\alpha)$ begins $0, 1, 2, \ldots, x+1$. The next term is $x + 3$. Thus $\max(W_\alpha(1)) = x$, as claimed.

Now suppose that the result is true for n, and we will prove it for $n+1$. Let $x \equiv 0 \pmod{(n+1)!}$, and let α be the last cutoff before $x + \frac{1}{n+1}$. We consider the sequence $T(\alpha)$. Since $x \equiv 0 \pmod{(n+1)!}$, we also have $x \equiv 0 \pmod{n!}$, so $\max(W_\alpha(n)) = nx - n + 1$. Since $n + 1 \in T(\alpha)$, the next term in $T(\alpha)$ after $nx - n + 1$ is in $W_\alpha(n+1)$, and that

n	2.5	3	3.5	4	4.5	5	5.5	6	10	20	30	40	75
$\gamma(n)$	3	4	5	8	11	14	18	21	74	424	1144	2100	9084

Table 2: Number of Cutoffs from 1 to n

next term is $\max(W_\alpha(n)) + n = nx + 1$. Let us now compute $W_\alpha(n+1)$. It begins with $nx+1$, and it is an arithmetic progression with common difference $n+1$, so its elements are of the form $nx+1+k(n+1)$, where $nx+1+k(n+1) \in W_\alpha(n+1)$ if and only if $nx+1+k(n+1) \le \alpha(n+1)$. Since $x \le \alpha < x + \frac{1}{n+1}$, we have $nx+1+k(n+1) \in W_\alpha(n+1)$ if and only if $k < \frac{x}{n+1}$, so

$$\max(W_\alpha(n+1)) = nx + 1 + \left(\frac{x}{n+1} - 1\right)(n+1) = (n+1)x - (n+1) + 1,$$

completing the induction. □

Theorem 36 show that for all integers d, there exists a cutoff whose denominator in lowest terms is d. In fact, it is quite common for rational numbers with small denominators to appear as cutoffs, even when they are not guaranteed by Theorem 36. For instance, all half-integers from $\frac{5}{2}$ to $\frac{29}{2}$ are cutoffs, but $\frac{31}{2}$ is not. The next few half-integers that are not cutoffs are $\frac{43}{2}$, $\frac{75}{2}$, $\frac{79}{2}$, and $\frac{95}{2}$. It would be interesting to investigate the nature of the cutoffs with a given denominator. For example, for those arithmetic progressions of rational numbers such that Theorem 36 does not guarantee that all are cutoffs, is it true that infinitely many are cutoffs and infinitely many are not cutoffs? Or are there other arithmetic progressions containing only cutoffs or only noncutoffs (or all but finitely many cutoffs or noncutoffs)?

We have written a number of computer programs to aid the calculations of the sequences $T(\alpha)$ and the generation of cutoffs[1]. Based on the data displayed in Table 2 and Figure 1, we make the following conjecture:

Conjecture 37. Let $\gamma(n)$ be the number of cutoffs up to n. Then $\lim_{n\to\infty} \frac{\gamma(n)}{n^2}$ exists and is nonzero.

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[1]Computer programs as well as cutoff data can be found at `https://github.com/sherrysarkar`.

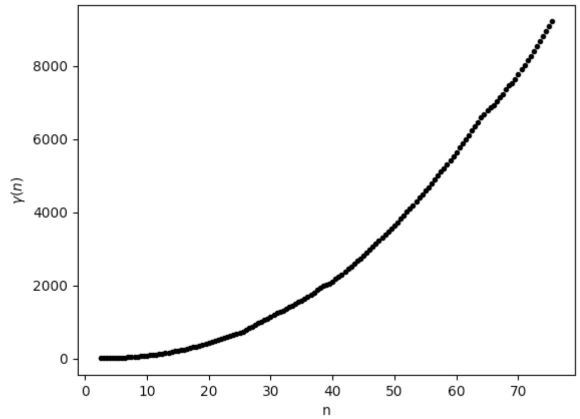

Figure 1: $\gamma(n)$ versus n

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