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Repdigits as Sums of Two Padovan Numbers

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Abstract

In this paper, we determine all repdigits in base 10 that can be written as a sum of two Padovan numbers.

1 Introduction

We recall that a *repdigit* is a positive integer r that has only one distinct digit when written in base 10. That is, r is of the form $d\left(\frac{10^\ell-1}{9}\right)$ for some positive integers d, ℓ with $1 \leq d \leq 9$.

Recently, the problem of looking for repdigits in a linear recurrence sequence has been studied. For example, Luca [10] and Marques [12] prove that 55 and 44 are the largest repdigits in the Fibonacci and Tribonacci sequences, respectively. More generally, let $k \geq 2$ be an integer. Let $\mathcal{F}^{(k)} := (F_n^{(k)})_{n\geq 0}$ be the k-*Fibonacci sequence* given by

$$F_0^{(k)} = \cdots = F_{k-2}^{(k)} = 0, \quad F_{k-1}^{(k)} = 1$$

and the recurrence formula

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)} \quad \text{for all} \quad n \geq 0.$$

Observe that the cases $k = 2$, 3 are the Fibonacci and Tribonacci sequences, respectively. Bravo and Luca [3] extended the results mentioned above by showing that the only repdigits with at least two digits in a k-Fibonacci sequence are, for $k = 2, 3$ the aforementioned ones, while for $k > 3$ there are no such repdigits belonging to $\mathcal{F}^{(k)}$. Actually, Bravo and Luca [4] considered and solved the more general Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = d\left(\frac{10^\ell - 1}{9}\right), \tag{1}$$

in positive integers $k \geq 2$, $n \geq m$, $\ell \geq 2$, and $1 \leq d \leq 9$. From their result we see that, for all k there are at most 4 solutions and the largest one is 777, which is found in $\mathcal{F}^{(4)}$. Motivated by these results, in this note we study the similar problem with the Padovan sequence.

The *Padovan sequence* $(P_n)_{n\geq 0}$, named after the architect R. Padovan, is the ternary recurrence sequence given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \quad \text{for all} \quad n \geq 0. \tag{2}$$

This is the sequence A000931 in *Sloane's Encyclopedia* [14] and it was introduced by Stewart [15], where he relates it to the Fibonacci sequence. Its first few terms are

$$0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, \ldots$$

In this note we study the problem of writing repdigits as sums of two Padovan numbers. More precisely, we solve the Diophantine equation

$$P_n + P_m = d\left(\frac{10^\ell - 1}{9}\right), \tag{3}$$

in non-negative integers n, m, ℓ, d with $n \geq m$, $\ell \geq 2$ and $1 \leq d \leq 9$. To establish our result we make the following conventions. Since $P_1 = P_2 = P_3 = 1$ and $P_4 = P_5 = 2$ we will assume that $n, m \neq 1, 2, 4$. That is, whenever we think of 1 and 2 as members of the Padovan sequence, we think of them as being P_3 and P_5, respectively. Thus, our result is

Theorem 1. *All non-negative integer solutions (n, m, ℓ, d) of equation (3) with $n \geq m$, $\ell \geq 2$ and $1 \leq d \leq 9$ belong to the set*

$$\left\{\begin{array}{cccc} (10,5,2,1), & (9,7,2,1), & (13,3,2,2), & (14,8,2,3) \\ (13,11,2,3), & (15,9,2,4), & (14,12,2,4), & (17,3,2,6) \\ (17,11,2,7), & (16,14,2,7), & (18,5,2,8), & (31,7,4,3) \end{array}\right\}.$$

The set of repdigits r that can be written as sums of two Padovan numbers is

$$\{11, 22, 33, 44, 66, 77, 88, 3333\},$$

and their respective representations are

$$\begin{aligned}
11 &= P_{10} + P_5 = P_9 + P_7;\\
22 &= P_{13} + P_3;\\
33 &= P_{14} + P_8 = P_{13} + P_{11};\\
44 &= P_{15} + P_9 = P_{14} + P_{12};\\
66 &= P_{17} + P_3;\\
77 &= P_{17} + P_{11} = P_{16} + P_{14};\\
88 &= P_{18} + P_5;\\
3333 &= P_{31} + P_7.
\end{aligned}$$

In particular, we obtain that in the Padovan sequence there are no repdigits with more than one digit.

We also note that the problem of writing repdigits as sum of at most three Fibonacci numbers has been solved by Luca [11]. The problem of writing Fibonacci numbers as sum of two repdigits has been studied by Alvarado and Luca [6].

The method of proof of our result is the classical one with linear forms in logarithms and the reduction method of Baker-Davenport as used in [2, 4, 11], for example.

2 Tools

In this section, we gather the tools we need to prove Theorem 1. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined by

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\left\{|\alpha^{(i)}|, 1\right\}\right).$$

This height has the following basic properties. For α, β algebraic numbers and $m \in \mathbb{Z}$ we have

- $h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log 2$;
- $h(\alpha\beta) \leq h(\alpha) + h(\beta)$;
- $h(\alpha^m) = |m| h(\alpha)$.

Now let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1, \ldots, \alpha_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_s|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}}\, h(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad i = 1, 2, \ldots, s.$$

The first tool we need is the following result due to Matveev [13]; also see Bugeaud, Mignotte and Siksek [5, Theorem 9.4].

Theorem 2. *Assume that* $\Lambda \neq 0$. *Then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1+\log d_{\mathbb{L}}) \cdot (1+\log B) A_1 \cdots A_s.$$

In this note, we always use $s = 3$. Further, $\mathbb{L} = \mathbb{Q}(\gamma)$, where γ is defined at the beginning of Section 3, has degree $d_{\mathbb{L}} = 3$. Thus, once and for all we fix the constant

$$C := 2.70444 \times 10^{12} > 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 \cdot (1+\log 3).$$

Our second tool is a version of the reduction method of Baker and Davenport [1] based on their Lemma. We use the one given by Bravo, Gómez and Luca [2]. Also see Dujella and Pethő [7]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 3. *Let M be a positive integer. Let $\tau, \mu, A > 0, B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|\mu\, q\| - M\|\tau\, q\| > 0$. If (n, m, w) is a positive solution to the inequality*

$$0 < |n\,\tau - m + \mu| < \frac{A}{B^w}$$

with $n \leq M$, then

$$w < \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following result of Guzmán and Luca [8] will be very useful.

Lemma 4. *If $m \geq 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$. Then*

$$x < 2^m T (\log T)^m.$$

3 Proof of Theorem 1

To start with, we recall some properties of the Padovan sequence. For a complex number z we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Let us consider the $\mathbb{Q}$-irreducible polynomial $X^3 - X - 1$. It is clear that its zeros are $\gamma, \delta, \overline{\delta}$ where

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

It can be proved, by induction for example, the Binet formula

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n \quad \text{for all} \quad n \geq 0, \tag{4}$$

where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \qquad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \qquad c_3 = \overline{c_2}.$$

Formula (4) follows from the general theorem on linear recurrence sequences since the above polynomial is the characteristic polynomial of the Padovan sequence. We note that

$$\gamma = 1.32471\ldots,\ |\delta| = 0.86883\ldots,\ c_1 = 0.54511\ldots,\ |c_2| = 0.28241\ldots$$

Further, the inequality

$$\gamma^{n-3} \le P_n \le \gamma^{n-1} \tag{5}$$

holds for all $n \ge 1$. This can also be proved by induction.

Now we start the study of equation (3) in non-negative integers (n, m, ℓ, d) where, as we have said, $n, m \ne 1, 2, 4$ and $\ell \ge 2$. Assume that $n \ge m$. Since $\ell \ge 2$ we assume that $n \ge 1$. From (5) and (3), we obtain

$$\gamma^{n-3} \le P_n \le P_n + P_m = d\left(\frac{10^\ell - 1}{9}\right) \le 10^\ell,$$

and

$$\gamma^{n+2} \ge 2P_n \ge P_n + P_m = d\left(\frac{10^\ell - 1}{9}\right) \ge 10^{\ell-1},$$

where we use $\gamma^3 > 2$. Thus,

$$(n-3)\frac{\log\gamma}{\log 10} \le \ell \quad \text{and} \quad (n+2)\frac{\log\gamma}{\log 10} \ge \ell - 1. \tag{6}$$

Since $\log\gamma/\log 10 = 0.122123\ldots$ we have, from (6), that if $n \le 360$ then $\ell \le 45$. Running a *Mathematica* program in the range $0 \le m \le n \le 360$, $1 \le d \le 9$ and $2 \le \ell \le 45$ we obtain, with our conventions, all solutions listed in Theorem 1. We will prove that these are all of them.

From now on, we assume $n > 360$. From (6) we get that $\ell > 43$ and also that $n > \ell$. From the Binet formula (4), we rewrite our equation (3) and obtain

$$\left| c_1\gamma^n - \frac{d10^\ell}{9} \right| < 2|c_2| + \gamma^{m-1} + 1 < \gamma^2 + \gamma^{m-1} < \gamma^{m+5}.$$

Dividing through by $d10^\ell/9$ we get

$$\left| \frac{9c_1}{d}\gamma^n 10^{-\ell} - 1 \right| < \frac{9\gamma^{m+5}}{d10^\ell} < \frac{1}{\gamma^{n-m-16}}, \tag{7}$$

where we use $\gamma^{n-3} \le 10^\ell$ and $9 < \gamma^8$. Let Λ be the expression inside the absolute value in the left-hand side of (7). Observe that $\Lambda \ne 0$. To see this, we consider the $\mathbb{Q}$-automorphism

σ of the Galois extension $\mathbb{Q}(\gamma, \delta)$ over $\mathbb{Q}$ given by $\sigma(\gamma) := \delta$ and $\sigma(\delta) := \gamma$. Now, if $\Lambda = 0$, then $\sigma(\Lambda) = 0$. Thus,

$$10^{\ell-1} \leq \frac{d10^{\ell}}{9} = |\sigma\left(c_1\gamma^n\right)| = |c_2|\,|\delta|^n < |c_2| < \frac{1}{3},$$

which is a contradiction since $\ell > 43$. Hence, $\Lambda \neq 0$ and we apply Matveev's inequality to it. To do this we consider,

$$\alpha_1 = \frac{9c_1}{d},\ \alpha_2 = \gamma,\ \alpha_3 = 10, \quad b_1 = 1,\ b_2 = n,\ b_3 = -\ell.$$

Thus, $B = n$. We have that $h(\alpha_2) = \log\gamma/3$ and $h(\alpha_3) = \log 10$. For α_1 we use the height properties to conclude that

$$h(\alpha_1) \leq \log\gamma + 13\log 2.$$

So, we choose $A_1 = 27.9$, $A_2 = 0.3$ and $A_3 = 7$. From Matveev's inequality we obtain

$$\log|\Lambda| > -C(1 + \log n) \cdot 27.9 \cdot 0.3 \cdot 7,$$

which, combined with (7) gives us

$$(n - m)\log\gamma < 1.58454 \times 10^{14}(1 + \log n). \tag{8}$$

Now we find an upper bound on n. To do this, from the Binet formula (4) we rewrite our equation (3) as

$$\left|c_1\left(\gamma^{n-m} + 1\right)\gamma^m - \frac{d10^{\ell}}{9}\right| \leq 4|c_2| + 1 < \gamma^3.$$

Dividing through by $d10^{\ell}/9$ we obtain

$$\left|\frac{9c_1(\gamma^{n-m} + 1)}{d}\gamma^m 10^{-\ell} - 1\right| < \frac{9\gamma^3}{d10^{\ell}} < \frac{1}{\gamma^{n-14}}. \tag{9}$$

Let Λ_1 be the expression inside the absolute value in the left-hand side of (9). Note that with a similar argument as the above one, it can be proved that $\Lambda_1 \neq 0$. So, we apply Matveev's inequality to it. We consider

$$\alpha_1 = \frac{9c_1(\gamma^{n-m} + 1)}{d},\ \alpha_2 = \gamma,\ \alpha_3 = 10, \quad b_1 = 1,\ b_2 = m,\ b_3 = -\ell.$$

Thus, $B = n$. The heights of α_2 and α_3 have already been calculated. From the properties of the heights we get that

$$h(\alpha_1) \leq \log\gamma + 14\log 2 + (n - m)\frac{\log\gamma}{3} < \frac{1.58455 \times 10^{14}(1 + \log n)}{3},$$

where we have used (8). We choose $A_1 = 1.58455 \times 10^{14}(1 + \log n)$, A_2 and A_3 as above. Therefore, from Theorem 2 we obtain

$$\log|\Lambda| > -C(1 + \log n) \cdot \left(1.58455 \times 10^{14}(1 + \log n)\right) \cdot 0.3 \cdot 7,$$

which, combined with (9) gives us $n < 1.28011 \times 10^{28}(\log n)^2$. Now, from Lemma 4 we get

$$n < 2.14474 \times 10^{32}. \tag{10}$$

Now we will reduce the above bound on n. To do this, we first consider

$$\Gamma = \ell \log 10 - n \log \gamma + \log\left(\frac{d}{9c_1}\right), \quad 1 \leq d \leq 9.$$

Assume $n - m \geq 20$ and go to (7). Note that $e^{-\Gamma} - 1 = \Lambda \neq 0$. Thus, $\Gamma \neq 0$. If $\Gamma < 0$ then

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = |\Lambda| < \frac{1}{\gamma^{n-m-16}}.$$

If $\Gamma > 0$ we have that $1 - e^{-\Gamma} = |e^{-\Gamma} - 1| < 1/2$. Hence, $e^{\Gamma} < 2$. Thus, we get

$$0 < \Gamma < e^{\Gamma} - 1 = e^{\Gamma}|\Lambda| < \frac{2}{\gamma^{n-m-16}}.$$

So, in both cases we have

$$0 < |\Gamma| < \frac{2}{\gamma^{n-m-16}}.$$

Dividing through by $\log\gamma$ we get

$$|\ell\,\tau - n + \mu_d| < \frac{640}{\gamma^{n-m}}, \tag{11}$$

where

$$\tau := \frac{\log 10}{\log\gamma} \quad \text{and} \quad \mu_d := \frac{\log(d/9c_1)}{\log\gamma}, \quad 1 \leq d \leq 9.$$

Now we apply Lemma 3. Put $M := 2.14474 \times 10^{32}$. Since $\ell < n$, from (10) we have that M is the upper bound on ℓ. A quick computation with *Mathematica* reveals that the convergent

$$\frac{p_{81}}{q_{81}} = \frac{7680712497306530625416042950269134710}{9379949046448702810198451347586799 13}$$

of τ is such that $q_{81} > 6M$ and $\varepsilon_d \geq 0.0440913 > 0$. Thus, with $A := 640$, $B := \gamma$ we calculated each value $\log(q_{81}640/\varepsilon_d)/\log\gamma$ and found that all of them are at most 328. Thus we have that

$$n - m \leq 328.$$

Next, we consider

$$\Gamma_1 = \ell \log 10 - m \log \gamma + \log\left(\frac{d}{9c_1(\gamma^{n-m}+1)}\right), \quad 1 \le d \le 9,$$

and go to inequality (9). Note that $e^{-\Gamma_1} - 1 = \Lambda_1 \neq 0$. Thus, $\Gamma_1 \neq 0$. Since $n > 360$, with an argument similar to the above one we get

$$0 < |\Gamma_1| < \frac{2}{\gamma^{n-14}}.$$

Dividing through by $\log \gamma$ the we obtain

$$|\ell\,\tau - m + \mu_d| < \frac{365}{\gamma^n}, \tag{12}$$

where τ is as above, and

$$\mu_d := \frac{\log(d/9c_1(\gamma^{n-m}+1))}{\log \gamma} \quad \text{for} \quad 1 \le d \le 9.$$

Consider

$$\mu_{d,k} = \frac{\log(d/9c_1(\gamma^k+1))}{\log \gamma}, \qquad d = 1, \ldots, 9, \qquad k = 1, \ldots, 328.$$

Note that $m \neq 0$, since if it is not the case we would have $n \le 328$, which contradicts $n > 360$. Thus, we apply Lemma 3 again. A quick computation with *Mathematica* gives that the 81-th convergent of τ again well works, that is, it is such that $q_{81} > 6M$ and $\varepsilon_{d,k} \ge 0.0000152967 > 0$ for all $d = 1, \ldots, 9$, $k = 1, \ldots, 328$ except for the case $\varepsilon_{9,11}$, which is always negative. Thus, for all of these values we calculated $\log(q_{81}\,365/\varepsilon_{d,k})$ and we find that all of them are at most 354. Thus, $n \le 354$ for all of these values.

The problem in case of the pair $(9, 11)$ is that we have the identity

$$\frac{2\gamma + 3}{\gamma(\gamma+1)(\gamma^{11}+1)} = \frac{1}{\gamma^9}.$$

Thus, inequality (12) is

$$|\ell\,\tau - (m+9)| < \frac{365}{\gamma^n}$$

and we use the theory of continued fractions to study it. Recall, for example, that if

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$$

then p/q is a convergent of α. See [9, Thm. 184, p. 153]. This result is known as Legendre's theorem. Since $n > 360$ and $M > \ell$, we have $730\ell < \gamma^n$. From Legendre's theorem, this implies that $(m+9)/\ell$ is a i-th convergent of τ. Let $[a_0, a_1, a_2, \ldots] = [8, 5, 3, \ldots]$ be the continued fraction expression of τ. A quick computation with *Mathematica* gives that

$$q_{71} \leq M < q_{72} \quad \text{and} \quad b := \max\{a_j : j = 1, 2, \ldots, 72\} = 49.$$

In particular, we have that $b \geq a_{i+1}$. Then, from the properties of the convergents of a continued fraction we obtain

$$\frac{1}{(b+2)\ell} < |\ell\,\tau - (m+9)| < \frac{365}{\gamma^n},$$

which yields

$$\gamma^n < 365 \cdot 51 \cdot M.$$

Thus $n \leq 299$. So, together with the results above, this shows that in all cases we have $n \leq 354$, which contradicts what we assumed about n. This finishes the proof of Theorem 1.

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