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The γ-Vectors of Pascal-like Triangles Defined by Riordan Arrays

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We define and characterize the γ-matrix associated with Pascal-like matrices that are defined by ordinary and exponential Riordan arrays. We also define and characterize the γ-matrix of the reversions of these triangles, in the case of ordinary Riordan arrays. We are led to the γ-matrices of a one-parameter family of generalized Narayana triangles. Thus these matrices generalize the matrix of γ-vectors of the associahedron. The principal tools used are the bivariate generating functions of the triangles and Jacobi continued fractions.

1 Introduction

A polynomial $P_n(x)=\sum_{k=0}^n a_{n,k}x^k$ of degree n is said to be *reciprocal* if

$$P_n(x)=x^nP_n(1/x).$$

Thus we have

$$[x^k]P_n(x)=a_{n,k}=[x^k]x^nP_n(1/x).$$

Now

$$\begin{aligned}[x^k]x^nP_n(1/x) &= [x^{k-n}]\sum_{i=0} a_{n,i}\frac{1}{x^i}\\ &= [x^{k-n}]\sum_{i=0} a_{n,i}x^{-i}\\ &= a_{n,n-k}.\end{aligned}$$

Thus $P_n(x) = \sum_{k=0}^n a_{n,k} x^k$ defines a family of reciprocal polynomials if and only if $a_{n,k} = a_{n,n-k}$. We shall call a lower-triangular matrix $(a_{n,k})$ *Pascal-like* if

1. $a_{n,k} = a_{n,n-k}$
2. $a_{n,0} = a_{n,n} = 1$.

Such a matrix will then be the coefficient array of a family of monic reciprocal polynomials.

We have the following well-known result [7]

Proposition 1. *Let $P_n(x)$ be a reciprocal polynomial of degree n. Then there exists a unique polynomial γ_n of degree $\lfloor \frac{n}{2} \rfloor$ with the property*

$$P_n(x) = (1+x)^n \gamma_n \left(\frac{x}{(1+x)^2} \right).$$

If $P_n(x)$ has integer coefficients then so does $\gamma_n(x)$.

By this means, we can associate with every Pascal-like matrix $(a_{n,k})$ a matrix $(\gamma_{n,k})$ so that for all n, we have

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-2k}.$$

We shall call this matrix the γ-matrix associated with the coefficient array $(a_{n,k})$ of the family of polynomials $P_n(x)$.

We can characterize the matrix $(a_{n,k})$ in terms of the γ-matrix $(\gamma_{n,k})$ as follows. Before we do this, we shall change our notation somewhat. In algebraic topology, it is customary to use the notation $h(x)$ for palindromic (reciprocal) polynomials [9, 15]. Thus we shall set $h_n(x) = \sum_{k=0}^n h_{n,k} x^k$, where $(h_{n,k})$ now denotes a Pascal-like matrix. We shall denote by $h(x,y)$ the bivariate generating function of this matrix.

Proposition 2. *For a Pascal-like matrix $(h_{n,k})$ we have*

$$h_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i} \gamma_{n,i}.$$

Proof. We have

$$\begin{aligned} h_{n,k} &= [x^k]\sum_{i=0}^{n} h_{n,i}x^i \\ &= [x^k]\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i}x^i(1+x)^{n-2i} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i}[x^k]x^i(1+x)^{n-2i} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i}[x^{k-i}]\sum_{j=0}^{n-2i}\binom{n-2i}{j}x^j \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i}\binom{n-2i}{k-i}. \end{aligned}$$

□

Example 3. The identity

$$\binom{n}{k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i}\delta_{i,0}$$

shows that the matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the γ-matrix for the binomial matrix $\mathbf{B} = (\binom{n}{k})$ A007318. Here, we have used the *Annnnnn* number of the On-Line Encyclopedia of Integer Sequences [13, 14] for the binomial matrix (Pascal's triangle).

When $(\gamma_{n,k})$ is the γ-matrix for $(h_{n,k})$, we shall say the $(\gamma_{n,k})$ *generates*, or *is the generator of*, the matrix $(h_{n,k})$.

Example 4. The matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

with $\gamma_{n,0} = 1$, $\gamma_{n,\lfloor \frac{n}{2} \rfloor} = 1$, and 0 otherwise, generates the matrix $(h_{n,k})$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 7 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

2 Pascal-like matrices defined by Riordan arrays

We now wish to characterize the γ-matrices that are generators for the family of Pascal-like matrices that are determined by the one-parameter family of Riordan arrays

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x} \right).$$

We shall also determine the (generalized) γ-matrices associated with the reversion of these triangles. We recall that an ordinary Riordan array $(g(x), f(x))$ is defined [1, 10, 11] by two power series

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots,$$
$$f(x) = x + f_2 x^2 + f_3 x^3 + \cdots,,$$

where the (n,k)-th element of the resulting lower-triangular matrix is given by

$$a_{n,k} = [x^n] g(x) f(x)^k.$$

Such matrices are invertible. When they have integer entries, the inverse again is an integer matrix (note that we have $a_{n,n} = 1$ in our case because $g_0 = 1$ and $f_1 = 1$). The bivariate generating function of the Riordan array (g, f) is given by

$$\frac{g(x)}{1 - y f(x)}.$$

Matrices defined in a similar manner but with $f(x)$ replaced by $\phi(x) = x^2 + \phi_3 x^3 + \ldots$ are called "stretched" Riordan arrays [5]. They are not invertible but they do possess left inverses.

Example 5. The stretched Riordan array $\left(\frac{1}{1-x}, x^2\right)$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

It is the γ-matrix for the Pascal-like triangle that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 9 & 5 & 1 & 0 & 0 \\ 1 & 6 & 14 & 14 & 6 & 1 & 0 \\ 1 & 7 & 20 & 29 & 20 & 7 & 1 \end{array}\right).$$

Example 6. The matrix $\binom{n-k}{k}$ is the stretched Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$ that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 \end{array}\right).$$

It generates the Pascal-like matrix that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{array}\right).$$

We shall see that this is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$, which is A008288, the triangle of Delannoy numbers.

The bivariate generating function of the stretched Riordan array $(g(x), \phi(x))$ is given by

$$\frac{g(x)}{1-y\phi(x)}.$$

We have the following proposition [4].

Proposition 7. *The Riordan array* $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ *is Pascal-like (for any* $r \in \mathbb{Z}$*).*

This is clear since in this case we have

$$h_{n,k} = \sum_{j=0}^{k} \binom{k}{j}\binom{n-j}{n-k-j} r^j = \sum_{j=0}^{k} \binom{k}{j}\binom{n-k}{n-k-j}(r+1)^j.$$

We can now characterize the γ-matrices that generate these Pascal-like matrices.

Proposition 8. *The* γ*-matrices that generate the Pascal-like matrices* $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ *defined by ordinary Riordan arrays are given by the stretched Riordan arrays*

$$\left(\frac{1}{1-x}, \frac{rx^2}{1-x}\right),$$

with (n,k)*-th term*

$$\gamma_{n,k} = \binom{n-k}{k} r^k.$$

Proof. The generating function of the Pascal-like matrix $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is given by

$$h(x,y) = \frac{1}{1-x}\frac{1}{1-y\frac{x(1+rx)}{1-x}} = \frac{1}{1-(1+y)x-rx^2y}.$$

Similarly, the generating function of the matrix $(\binom{n-k}{k}r^k)$ is given by

$$\gamma(x,y) = \frac{1}{1-x}\frac{1}{1-y\frac{rx^2}{1-x}} = \frac{1}{1-x-rx^2y}.$$

We now have

$$h(x,y) = \gamma\left((1+y)x, \frac{y}{(1+y)^2}\right).$$

□

We recall that for a generating function $f(x)$, its INVERT(α) transform is the generating function

$$\frac{f(x)}{1+\alpha x f(x)}.$$

Note that

$$\frac{\frac{v}{1+\alpha x v}}{1-\alpha x \frac{v}{1+\alpha x v}} = v,$$

and thus the inverse of the INVERT(α) transform is the INVERT($-\alpha$) transform.

Corollary 9. *The generating function $h(x,y)$ of the Pascal-like matrix $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is the INVERT(y) transform of the generating function $\gamma(x,y)$ of the corresponding γ-matrix.*

Proof. A direct calculation shows that for $\gamma(x,y)=\frac{1}{1-x-rx^2y}$ we have

$$\frac{\gamma(x,y)}{1-yx\gamma(x,y)} = \frac{1}{1-(y+1)x-rx^2y} = h(x,y).$$

□

Equivalently, we can say that the generating function of the γ-matrix is the INVERT($-y$) transform of the generating function of the corresponding Pascal-like matrix.

We make the following observation, which will be relevant when we discuss a family of generalized Narayana triangles. The γ-matrix corresponding to the signed Pascal-like matrix

$$\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x}\right)$$

has generating function

$$\frac{1}{1+x+rx^2y}.$$

This is the matrix with general term $(-1)^{n-k}r^k\binom{n-k}{k}$. By a signed Pascal-like matrix in this case we mean that $a_{n,k}=a_{n,n-k}$ but we now have $a_{n,0}=a_{n,n}=(-1)^n$.

We close this section by recalling the formula

$$\gamma_n = (1+x)^n \gamma_n\left(\frac{x}{(1+x)^2}\right).$$

We now note that the inverse of the Riordan array

$$\left(1, \frac{x}{(1+x)^2}\right)$$

is given by

$$\left(1, xc(x)^2\right),$$

where

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}$$

is the generating function of the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ A000108. In fact, we have the following result [9].

Proposition 10. ***(Zeilberger's Lemma).***

$$\gamma_{n,k}=[x^k]\frac{h_n(xc(x)^2)}{c(x)^n}.$$

We can use this result to find an explicit formula for $\gamma_{n,k}$ in terms of $h_{n,k}$. We let $\alpha_{n,k}$ be the general (n,k)-th element of the Riordan array $(1, xc(x)^2)$ [8]. We have

$$\alpha_{n,k}=\begin{cases}1, & \text{if } n=0 \text{ and } k=0;\\ \binom{2n-1}{n-k}\frac{2k}{n+k}, & \text{otherwise;}\end{cases}$$

or, equivalently,

$$\alpha_{n,k}=\binom{2n-1}{n-k}\frac{2k+0^{n+k}}{n+k+0^{n+k}}=\binom{2n-2}{n-k}-\binom{2n-2}{n-k-2}.$$

We let $\beta_{n,k}$ be the general (n,k)-th term of the Riordan array $\left(1,\frac{x}{c(x)}\right)$. We have $\beta_{n,n}=1$, and

$$\beta_{n,k}=\sum_{j=0}^{n-k}\frac{(-1)^j}{n-k}\binom{k+j-1}{j}\binom{2(n-k)}{n-k-j},$$

otherwise. This is essentially A271875. Then we have the following result.

Corollary 11. *We have*

$$\gamma_{n,k}=\sum_{i=0}^{k}\left(\sum_{j=0}^{n}h_{n,j}\alpha_{i,j}\right)\beta_{n+k-i,n}.$$

Proof. We have

$$\begin{aligned}[x^k][x^k]\frac{h_n(xc(x)^2)}{c(x)^n} &= \sum_{i=0}^{n}[x^i]\sum_{j=0}^{n}h_{n,j}(xc(x^2))^j[x^{k-i}]\frac{1}{c(x)^n}\\ &= \sum_{i=0}^{k}\left(\sum_{j=0}^{n}h_{n,j}[x^i](xc(x)^2)^j\right)[x^{k-1+n}]\frac{x^n}{c(x)^n}\\ &= \sum_{i=0}^{k}\left(\sum_{j=0}^{n}h_{n,j}\alpha_{i,j}\right)\beta_{n+k-i,n}.\end{aligned}$$

□

This gives us the following formula:

$$\gamma_{n,k}=\sum_{i=0}^{k}\sum_{j=0}^{n} h_{n,j}\binom{2i-1}{i-j}\frac{2j+0^{i+j}}{i+j+0^{i+j}}\cdot\begin{cases}1, & \text{if } i=k;\\ \sum_{m=0}^{k-i}\frac{m(-1)^m}{k-i}\binom{n-1+m}{m}\binom{2(k-i)}{k-i-m}, & \text{otherwise;}\end{cases}$$

which we can also write as

$$\gamma_{n,k}=\sum_{i=0}^{k}\sum_{j=0}^{n} h_{n,j}\binom{2i-1}{i-j}\frac{2j+0^{i+j}}{i+j+0^{i+j}}\,\mathrm{If}\left[i=k,1,\sum_{m=0}^{k-i}\frac{m(-1)^m}{k-i}\binom{n-1+m}{m}\binom{2(k-i)}{k-i-m}\right].$$

Example 12. If we take $(h_{n,k})$ to be the triangle of Eulerian numbers A008292 that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & 0 & 0 & 0\\ 1 & 11 & 11 & 1 & 0 & 0 & 0\\ 1 & 26 & 66 & 26 & 1 & 0 & 0\\ 1 & 57 & 302 & 302 & 57 & 1 & 0\\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \end{pmatrix}$$

we find that the γ-matrix $(\gamma_{n,k})$ is the triangle A101280 that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 0 & 0\\ 1 & 8 & 0 & 0 & 0 & 0 & 0\\ 1 & 22 & 16 & 0 & 0 & 0 & 0\\ 1 & 52 & 136 & 0 & 0 & 0 & 0\\ 1 & 114 & 720 & 272 & 0 & 0 & 0 \end{pmatrix}.$$

This is the triangle of γ-vectors for the permutahedra (of type A). It also gives the number of permutations of n objects with k descents such that every descent is a peak [12].

Example 13. We consider the Pascal-like matrix $(h_{n,k})=\left(\frac{1}{1-x},x\right)$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We note that the row elements are constant. We have that

$$\gamma_{n,k}=\sum_{i=0}^k \sum_{j=0}^n \binom{2i-1}{i-j}\frac{2j+0^{i+j}}{i+j+0^{i+j}} \text{ If }\left[i=k,1,\sum_{m=0}^{k-i}\frac{m(-1)^m}{k-i}\binom{n-1+m}{m}\binom{2(k-i)}{k-i-m}\right].$$

We find that the γ-matrix in this case begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 & 0 \\ 1 & -5 & 6 & -1 & 0 & 0 & 0 \end{array}\right).$$

This is the matrix $\left(\binom{n-k}{k}(-1)^k\right)$. Thus

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-2i}{k-i}\binom{n-i}{i}(-1)^i=\text{If}[k \le n,1,0].$$

3 Stretched Riordan arrays as γ-matrices

Every stretched Riordan array of the form

$$\left(\frac{1}{1-x}, x^2 g(x)\right),$$

where

$$g(x)=1+g_1x+g_2x^2+\cdots$$

can be used to generate a Pascal-like matrix. Thus with each power series $g(x)$ above we can associate a Pascal-like matrix whose γ-matrix is given by this stretched Riordan array.

In this section, we shall concentrate on the case when $g(x)=\frac{1+rx}{1-x}$.

Example 14. For $r=1$, we obtain the γ-matrix that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 5 & 0 & 0 & 0 & 0 \\ 1 & 9 & 13 & 1 & 0 & 0 & 0 \end{array}\right).$$

The corresponding Pascal-like matrix then begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 9 & 17 & 9 & 1 & 0 & 0 \\ 1 & 12 & 36 & 36 & 12 & 1 & 0 \\ 1 & 15 & 64 & 101 & 64 & 15 & 1 \end{array}\right).$$

The row sums of this matrix, which begin

$$1, 2, 5, 14, 37, 98, 261, \ldots$$

give A077938, with generating function

$$\frac{1}{1-2x-x^2-2x^3}.$$

The diagonal sums, which begin

$$1, 1, 2, 4, 8, 16, 31, \ldots$$

are the Pentanacci numbers A001591 with generating function

$$\frac{1}{1-x-x^2-x^3-x^4-x^5}.$$

We have the following proposition.

Proposition 15. *The Pascal-like triangle that begins*

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & r+5 & r+5 & 1 & 0 & 0 & 0 \\ 1 & 2r+7 & 4r+13 & 2r+7 & 1 & 0 & 0 \\ 1 & 3r+9 & 11r+25 & 11r+25 & 3r+9 & 1 & 0 \\ 1 & 4r+11 & r^2+22r+41 & 2r^2+36r+63 & r^2+22r+41 & 4r+11 & 1 \end{array}\right)$$

with γ-matrix given by the stretched Riordan array $\left(\frac{1}{1-x}, \frac{x^2(1+rx)}{1-x}\right)$, has row sums with generating function

$$\frac{1}{1-2x-x^2-2rx^3},$$

and diagonal sums given by the generalized Pentanacci numbers with generating function

$$\frac{1}{1-x-x^2-x^3-rx^4-rx^5}.$$

4 Reverting triangles

Let $h(x,y)$ be the generating function of the lower-triangular matrix $h_{n,k}$, with $h_{0,0}=1$. By the reversion of this triangle, we shall mean the triangle whose generating function $h^*(x,y)$ is given by

$$h^*(x,y)=\frac{1}{x}\text{Rev}_x(xh(x,y)).$$

Procedurally, this means that we solve the equation

$$uh(u,y)=x$$

and then we divide the solution $u(x,y)$ that satisfies $u(0,y)=0$ by x.

Proposition 16. *The generating function of the reversion of the Pascal-like matrix defined by the Riordan array* $\left(\frac{1}{1-x},\frac{x(1+rx)}{1-x}\right)$ *is given by*

$$h^*(x,y)=\frac{1}{1+x(y+1)}c\left(\frac{-rx^2y}{(1+x(y+1))^2}\right),$$

where

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}$$

is the generating function of the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$. *(A000108).*

Proof. Solving the equation

$$\frac{u}{1-u(y+1)-ru^2y}=x$$

gives us

$$h^*(x,y)=\frac{-1-x(y+1)+\sqrt{1+2x(y+1)+x^2(1+2y(2r+1)+y^2)}}{2rx^2y}.$$

Thus

$$h^*(x,y)=\frac{1}{1+x(y+1)}c\left(\frac{-rx^2y}{(1+x(y+1))^2}\right).$$

□

We note that we can now calculate an expression for the terms of the reverted triangle, since, using the language of Riordan arrays, we have

$$h^*(x,y)=\left(\frac{1}{1+y(x+1)},\frac{-rx^2y}{(1+x(y+1))^2}\right)\cdot c(x).$$

Proposition 17. *We have*

$$[x^n][y^i]h^*(x,y) = h^*_{n,i} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n(-r)^k \binom{n}{2k} C_k \binom{n-2k}{i-k}.$$

The γ-matrix of the reverted triangle $(h^*_{n,k})$ is given by

$$\gamma^*_{n,k} = (-1)^n(-r)^k \binom{n}{2k} C_k.$$

The γ-matrix $(\gamma^*_{n,k})$ of the reverted triangle $(h^*_{n,k})$ is the reversion of the triangle $\gamma_{n,k}$.

Proof. The expression for $h^*_{n,k}$ results from a direct calculation. Reverting the expression $\gamma(x,y) = \frac{1}{1-x-rx^2y}$ in the sense above gives us

$$\gamma^*(x,y) = \frac{1}{1+x} c\left(\frac{-rx^2y}{(1+x)^2}\right),$$

from which we deduce the other statements. □

Example 18. For $r = -1, 0, 1$, the triangles $(h_{n,k})$ begin, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{pmatrix}.$$

The corresponding reverted triangles $(h^*_{n,k})$, are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -6 & -6 & -1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ -1 & -15 & -50 & -50 & -15 & -1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ -1 & -5 & -10 & -10 & -5 & -1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & -4 & -2 & 1 & 0 & 0 \\ -1 & 5 & 10 & 10 & 5 & -1 & 0 \\ 1 & -9 & -15 & -15 & -15 & -9 & 1 \end{pmatrix}.$$

Note that for $r=-1$, the reverted triangle is $(-1)^n$ times the Narayana triangle A001263.

The corresponding γ-matrices $(\gamma_{n,k})$ are given by, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 & 0 \\ 1 & -5 & 6 & -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding reverted γ-matrices $(\gamma^*_{n,k})$ are then, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 & 0 & 0 & 0 \\ -1 & -10 & -10 & 0 & 0 & 0 & 0 \\ 1 & 15 & 30 & 5 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 2 & 0 & 0 & 0 & 0 \\ -1 & 10 & -10 & 0 & 0 & 0 & 0 \\ 1 & -15 & 30 & -5 & 0 & 0 & 0 \end{pmatrix}.$$

It is interesting to represent the generating functions of the $(\gamma^*_{n,k})$ and the $(h^*_{n,k})$ triangles as Jacobi continued fractions. We have

Proposition 19. *The generating function $h^*(x,y)$ can be expressed as the Jacobi continued fraction*

$$\mathcal{J}(-(y+1),-(y+1),-(y+1),\ldots;-ry,-ry,-ry,\ldots).$$

The generating function $\gamma^*(x,y)$ can be expressed as the Jacobi continued fraction

$$\mathcal{J}(-1,-1,-1,\ldots;-ry,-ry,-ry,\ldots).$$

Proof. We solve the continued fraction equation

$$u=\frac{1}{1+(y+1)x+rx^2u}$$

to retrieve the generating function $h^*(x,y)$. Similarly, we solve the continued fraction equation

$$u=\frac{1}{1+x+rx^2u}$$

to retrieve the generating function $\gamma^*(x,y)$. □

Note that we have used the notation $\mathcal{J}(a,b,c,\ldots;r,s,t,\ldots)$ to denote the Jacobi continued fraction [2, 16]

$$\cfrac{1}{1-ax-\cfrac{rx^2}{1-bx-\cfrac{sx^2}{1-cx-\cfrac{tx^2}{1-\cdots}}}}.$$

We can now express the relationship between the generating functions $h^*(x,y)$ and $\gamma^*(x,y)$ in terms of repeated binomial transforms.

Corollary 20. *The generating function $h^*(x,y)$ is the $(-y)$-th binomial transform of the γ generating function $\gamma^*(x,y)$:*

$$h^*(x,y)=\frac{1}{1+xy}\gamma^*\left(\frac{x}{1+xy},y\right).$$

Equivalently, the γ generating function $\gamma^*(x,y)$ is the y-th binomial transform of the generating function $h^*(x,y)$:

$$\gamma^*(x,y)=\frac{1}{1-xy}h^*\left(\frac{x}{1-xy},y\right).$$

This reflects the general assertion that the reversion of an INVERT transform is a binomial transform.

5 The γ-vectors of generalized Narayana numbers

The Riordan array $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x}\right)$, with bivariate generating function

$$\frac{1}{1+x(y+1)+rx^2y},$$

has a γ-matrix with generating function

$$\frac{1}{1+x+rx^2y}.$$

We shall call elements of the reversions of the Riordan array $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x}\right)$ r-Narayana numbers. The Narayana numbers $N_{n,k} = \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}$ are then the 1-Narayana numbers. The bivariate generating function for the r-Narayana numbers is given by

$$\frac{1}{1-x(y+1)}c\left(\frac{rx^2y}{(1-x(y+1))^2}\right).$$

The bivariate generating function for the γ-matrix of the r-Narayana numbers is then obtained by reverting the generating function $\frac{1}{1+x+rx^2y}$. We thus obtain the following result.

Proposition 21. *The γ-matrix for the r-Narayana numbers has generating function*

$$\frac{1}{1-x}c\left(\frac{rx^2y}{(1-x)^2}\right).$$

This is the matrix that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & r & 0 & 0 & 0 & 0 & 0 \\ 1 & 3r & 0 & 0 & 0 & 0 & 0 \\ 1 & 6r & 2r^2 & 0 & 0 & 0 & 0 \\ 1 & 10r & 10r^2 & 0 & 0 & 0 & 0 \\ 1 & 15r & 30r^2 & 5r^3 & 0 & 0 & 0 \end{array}\right),$$

with general term

$$\binom{n}{2k}r^kC_k.$$

For $r = -1, 0, 1$, the matrices $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x}\right)$ begin, respectively,

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -5 & -5 & -1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ -1 & -9 & -25 & -25 & -9 & -1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{array}\right), \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ -1 & -5 & -10 & -10 & -5 & -1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}\right),$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The corresponding matrices of r-Narayana numbers are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -2 & -4 & -2 & 1 & 0 & 0 \\ 1 & -5 & -10 & -10 & -5 & 1 & 0 \\ 1 & -9 & -15 & -15 & -15 & -9 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{pmatrix}.$$

This last matrix, as expected, is the Narayana triangle A001263. The corresponding γ-matrices for these r-Narayana triangles are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 2 & 0 & 0 & 0 & 0 \\ 1 & -10 & 10 & 0 & 0 & 0 & 0 \\ 1 & -15 & 30 & -5 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 & 0 & 0 & 0 \\ 1 & 10 & 10 & 0 & 0 & 0 & 0 \\ 1 & 15 & 30 & 5 & 0 & 0 & 0 \end{pmatrix}.$$

This last matrix is A055151. The rows of this triangle are the γ-vectors of the n-dimensional (type A) associahedra [9]. We have seen that its elements are given by

$$\gamma_{n,k}=\sum_{i=0}^{k}\sum_{j=0}^{n}N_{n,j}\binom{2i-1}{i-j}\frac{2j+0^{i+j}}{i+j+0^{i+j}}\,\text{If}\left(k=i,1,\sum_{m=0}^{k-i}\frac{m(-1)^m}{k-i}\binom{n-1+m}{m}\binom{2(k-i)}{k-i-m}\right),$$

where $N_{n,k}$ denotes the (n,k)-th Narayana number A001263.

The relationship between the γ-matrix and the r-Narayana numbers can be further clarified as follows.

Proposition 22. *The generating function of the r-Narayana numbers can be expressed as the Jacobi continued fraction*

$$\mathcal{J}((y+1),(y+1),(y+1),\ldots;ry,ry,ry,\ldots).$$

The generating function of the corresponding γ-matrix can be expressed as the Jacobi continued fraction

$$\mathcal{J}(1,1,1,\ldots;ry,ry,ry,\ldots).$$

Corollary 23. *The generating function of the r-Narayana numbers is the y-th binomial transform of the generating function of the corresponding γ-matrix.*

$$h^*(x,y)=\frac{1}{1-xy}\gamma^*\left(\frac{x}{1-xy},y\right).$$

Equivalently, the γ generating function $\gamma^*(x,y)$ is the $(-y)$-th binomial transform of the generating function $h^*(x,y)$:

$$\gamma^*(x,y)=\frac{1}{1+xy}h^*\left(\frac{x}{1+xy},y\right).$$

6 Pascal-like triangles defined by exponential Riordan arrays

We recall that an exponential Riordan array $[g(x),f(x)]$ [1, 6] is defined by two exponential generating functions

$$g(x)=1+g_1\frac{x}{1!}+g_2\frac{x}{2!}+\cdots,$$

and

$$f(x)=\frac{x}{1!}+f_2\frac{x^2}{2!}+\cdots,$$

with its (n,k)-th term $a_{n,k}$ given by

$$a_{n,k}=\frac{n!}{k!}[x^n]g(x)f(x)^k.$$

In the context of Pascal-like matrices, we have that the exponential Riordan array

$$[e^x, x(1+rx/2)],$$

with general term

$$h_{n,k} = \frac{n!}{k!} \sum_{j=0}^{k} \frac{r^j}{(n-k-j)!2^j},$$

is a Pascal-like matrix [3]. This matrix begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & r+2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3r+3 & 3r+3 & 1 & 0 & 0 & 0 \\ 1 & 6r+4 & 3r^2+12r+6 & 6r+4 & 1 & 0 & 0 \\ 1 & 10r+5 & 15r^2+30r+10 & 15r^2+30r+10 & 10r+5 & 1 & 0 \\ 1 & 15r+6 & 45r^2+60r+15 & 15r^3+90r^2+90r+20 & 45r^2+60r+15 & 15r+6 & 1 \end{array}\right).$$

We have the following result.

Proposition 24. *The γ-matrix of the Pascal-like exponential Riordan array $[e^x, x(1+rx/2)]$ is the matrix with general term*

$$\binom{n}{2k} r^k (2k-1)!!$$

In fact, the generating function of the exponential Riordan array $[e^x, x(1+rx/2)]$ is given by

$$\mathcal{J}(y+1, y+1, y+1, \ldots; ry, 2ry, 3ry, \ldots)$$

while that of its γ-matrix is given by

$$\mathcal{J}(1, 1, 1, \ldots; ry, 2ry, 3ry, \ldots).$$

Proposition 25. *The generating function of the γ-matrix of the Pascal-like exponential Riordan array $[e^x, x(1+rx/2)]$ has generating function*

$$e^{x(1+rxy/2)}.$$

Proof. By the theory of exponential Riordan arrays, the generating function of the Riordan array $[e^x, x(1+rx/2)]$ is given by

$$e^x e^{xy(1+rx/2)}.$$

Taking the $(-y)$-th binomial transform of this, we obtain

$$e^{x(1+rxy/2)}.$$

□

Example 26. For $r = 1$, we get the γ-matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 3 & 0 & 0 & 0 & 0 \\ 1 & 10 & 15 & 0 & 0 & 0 & 0 \\ 1 & 15 & 45 & 15 & 0 & 0 & 0 \end{pmatrix}.$$

This is A100861, the triangle of Bessel numbers that count the number of k-matchings of the complete graph $K(n)$. The corresponding Pascal-like matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 21 & 10 & 1 & 0 & 0 \\ 1 & 15 & 55 & 55 & 15 & 1 & 0 \\ 1 & 21 & 120 & 215 & 120 & 21 & 1 \end{pmatrix}.$$

This is A100862, which counts the number of k-matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$.

Example 27. For $r = 2$, we obtain the γ-matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 12 & 12 & 0 & 0 & 0 & 0 \\ 1 & 20 & 60 & 0 & 0 & 0 & 0 \\ 1 & 30 & 180 & 120 & 0 & 0 & 0 \end{pmatrix}.$$

This is A059344, where row n consists of the nonzero coefficients of the expansion of $2^n x^n$ in terms of Hermite polynomials with decreasing subscripts. The corresponding Pascal-like matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 9 & 1 & 0 & 0 & 0 \\ 1 & 16 & 42 & 16 & 1 & 0 & 0 \\ 1 & 25 & 130 & 130 & 25 & 1 & 0 \\ 1 & 36 & 315 & 680 & 315 & 36 & 1 \end{pmatrix}.$$

The row sums of this matrix are given by A000898, the number of symmetric involutions of $[2n]$ (Deutsch).

7 Conclusion

It is the case that the set of Pascal-like matrices defined by Riordan arrays is a restricted one. Nevertheless, we hope that this note indicates that they have interesting properties, including in particular their generating γ-matrices. In the case of Pascal-like matrices defined by ordinary Riordan arrays, we have seen that be reverting them, we find additional (signed) Pascal-like triangles, including triangles of Narayana type. The γ-matrices of these new triangles are again the reversions of the original triangles' γ-matrices.

We have also shown that stretched Riordan arrays play a useful role, and in particular can lead to further (non-Riordan) Pascal-like matrices. We have also found it useful to use Riordan array techniques to find an explicit closed form formula for the elements $\gamma_{n,k}$ of the γ-matrix in terms of $h_{n,k}$.

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