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Generalization of an Identity of Apostol

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Abstract

We extend and generalize an identity of Apostol, involving Bernoulli numbers, to every sequence of complex numbers. Moreover, our result allows us to obtain other relations involving Appell polynomial sequences and second-order linear recurrence sequences.

This paper is dedicated in memory of professor
Tom M. Apostol (*1923 – 2016*)

1 Introduction and statement of results

With each complex sequence u we associate the sequence $T(u) = u^*$ defined as follows:

$$u_m^* = \sum_{k=0}^{m} (-1)^k \binom{m}{k} u_k, \quad (m \geq 0). \tag{1}$$

The mapping T is called a *binomial transformation* [4], and the sequence u^* is called the *dual sequence* of u. It is easy to prove that $(u^*)^* = u$, and if u is linearly recurrent over $\mathbb{C}$, then u^* is also linearly recurrent over $\mathbb{C}$. Moreover, if $C(x)$ is a characteristic polynomial of u, then $C(1-x)$ is a characteristic polynomial of u^*. The sequence u is called *invariant* (resp., *inverse invariant*) under the binomial transformation if $u^* = u$ (resp., if $u^* = -u$).

Let $(r)_j$ be the real sequence defined by $(r)_j = j!\binom{r}{j}$, where r and j are positive integers, and B_m is the m'th Bernoulli number defined by:

$$\frac{z}{e^z - 1} = \sum_{m=0}^{+\infty} B_m \frac{z^m}{m!}. \tag{2}$$

In 2008, Apostol [1] proved the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+2-k} = \frac{B_{n+1}}{n+1}, \quad n \geq 1. \tag{3}$$

The main result of this paper is the following theorem, which gives a simplified expression for the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{u_k}{n+r+1-k},$$

and allows us to generalize relation (3).

Theorem 1. *For every sequence u of complex numbers, and all non-negative integers r and n, the following holds:*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{u_k}{n+r+1-k} = \sum_{j=0}^{r} (-1)^j (r)_j \frac{u^*_{n+j+1}}{(n+j+1)_{j+1}} + (-1)^n r! \frac{u_{n+r+1}}{(n+r+1)_{r+1}}. \tag{4}$$

We can easily prove, from relation (2) and the equality $\frac{-z}{e^{-z}-1} = \frac{z}{e^z-1} e^z$, that $((-1)^n B_n)_{n \geq 0}$ is invariant under the binomial transformation. Thus, we deduce, from Theorem 1, the following relation for integers $r \geq 0$ and $n \geq 1$.

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+r+1-k} = \sum_{j=0}^{r-1} (-1)^{n+1} (r)_j \frac{B_{n+j+1}}{(n+j+1)_{j+1}}. \tag{5}$$

It is clear that the relation (5) is a generalization of identity (3).

Theorem 1 can also be applied to the sequence of Bell numbers. Recall that for $n \geq 1$, the nth Bell number b_n is the number of distinct partitions of a set of n elements (sequence A000110 in the OEIS [3]). By convention, we set $b_0 = 1$. If $u_n = (-1)^n b_n$ for $n \geq 0$, the well-known relation $\sum_{k=0}^{n} \binom{n}{k} b_k = b_{n+1}$ for $n \geq 0$ yields $u_n^* = b_{n+1}$ for $n \geq 0$. By applying Theorem 1 for $r = 1$ to the sequence u, we obtain an identity similar to relation (3) for the Bell numbers.

$$\sum_{k=0}^{n} \binom{n}{k} \frac{b_k}{n+2-k} = \frac{(n+3)b_{n+2} - b_{n+3}}{(n+2)(n+1)}.$$

Theorem 1 enables us to obtain some identities such as the following:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2^k(n+k+1)} = \frac{2^n}{2n+1} \binom{2n}{n}^{-1}, \tag{6}$$

due to Sun [5, Relation (1.13)]. To do this, it is enough to apply Theorem 1 for $r = n$ to the sequence $u = (2^m)_{m \geq 0}$ for which we have $u^* = ((-1)^m)_{m \geq 0}$. One gets the following:

$$2^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2^k(n+k+1)} = -\frac{n!n!}{(2n+1)!} \sum_{j=0}^{n} \binom{2n+1}{j} + \frac{n!n!}{(2n+1)!} 2^{2n+1}.$$

From this we can deduce relation (6) by using the well-known identity $\sum_{j=0}^{n} \binom{2n+1}{j} = 2^{2n}$.

The following corollary gives us some identities when u is a second-order linear recurrent complex sequence such as the Fibonacci numbers A000045, the Lucas numbers A000032, the Pell numbers A000129, the companion Pell numbers A002203, the Jacobsthal numbers A001045, and the Jacobsthal-Lucas numbers A014551.

Corollary 2. *If u is a second-order linear recurrent sequence of complex numbers with $x^2 - ax - b$ as characteristic polynomial, then the following relations are satisfied for all non-negative integers r and n:*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{a^k b^{n+r+1-k} u_k}{n+r+1-k} = \sum_{j=0}^{r} (-1)^j (r)_j \frac{b^{r-j} u_{2n+2j+2}}{(n+j+1)_{j+1}} + (-1)^{r+1} \frac{r! a^{n+r+1} u_{n+r+1}}{(n+r+1)_{r+1}}. \tag{7}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{b^{n+r+1-k} u_{2k}}{n+r+1-k} = \sum_{j=0}^{r} (-1)^{n+1} (r)_j \frac{a^{n+j+1} b^{r-j} u_{n+j+1}}{(n+j+1)_{j+1}} + (-1)^n \frac{r! u_{2n+2r+2}}{(n+r+1)_{r+1}}. \tag{8}$$

If $u_0 = 0$, then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^{n+r-k+1} u_k}{n+r+1-k} = \sum_{j=0}^{r} (-1)^{j+1} (r)_j \frac{a^{r-j} u_{n+j+1}}{(n+j+1)_{j+1}} + (-1)^n \frac{r! u_{n+r+1}}{(n+r+1)_{r+1}}. \tag{9}$$

If $(u_0, u_1) = (2, a)$*, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a^{n+r+1-k}u_k}{n+r+1-k} = \sum_{j=0}^{r}(-1)^j(r)_j\frac{a^{r-j}u_{n+j+1}}{(n+j+1)_{j+1}} + (-1)^n\frac{r!u_{n+r+1}}{(n+r+1)_{r+1}}. \quad (10)$$

Theorem 1 enables us also to get an identity for Appell polynomial sequences. Recall that an Appell polynomial sequence [2] associated with a formal series $S(z) \in \mathbb{C}[[z]]$ is the polynomial sequence $(A_n(x))_{n\geq 0}$ of $\mathbb{C}[x]$ given by the generating relation $\sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = S(z)e^{xz}$.

Corollary 3. *If* $(A_m(x))_{m\geq 0}$ *is an Appell polynomial sequence, then for all complex numbers* λ *and all non-negative integers* r, n*, the following holds:*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda^{n+r+1-k}A_k(x)}{n+r+1-k} = \sum_{j=0}^{r}(-1)^j(r)_j\lambda^{r-j}\frac{A_{n+j+1}(x+\lambda)}{(n+j+1)_{j+1}} + (-1)^{r+1}\frac{r!\ A_{n+r+1}(x)}{(n+r+1)_{r+1}}. \quad (11)$$

We now consider some examples. Let α be a complex number. We let $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$, $H_n(x)$, and $L_n^{(\alpha)}(x)$ denote, respectively, the generalized Bernoulli polynomial, the generalized Euler polynomial, the Hermite polynomial, and the generalized Laguerre polynomial of degree n defined as follows:

$$\begin{aligned}
\left(\frac{z}{e^z-1}\right)^{\alpha} e^{zx} &= \sum_{n=0}^{+\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!},\\
\left(\frac{2}{e^z+1}\right)^{\alpha} e^{zx} &= \sum_{n=0}^{+\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!},\\
e^{2xz-z^2} &= \sum_{n=0}^{+\infty} H_n(x)\frac{z^n}{n!},\\
L_n^{(\alpha)}(x) &= \sum_{j=0}^{n}\binom{\alpha+n}{n-j}(-1)^j\frac{x^j}{j!}.
\end{aligned}$$

Define, for non-negative integers n, $K_n^{(\alpha)}(x) = \frac{n!}{(\alpha+n)_n}x^n L_n^{(\alpha)}\left(\frac{-1}{x}\right)$. It is not difficult to see that the polynomial sequences $\left(B_n^{(\alpha)}(x)\right)_{n\geq 0}$, $\left(E_n^{(\alpha)}(x)\right)_{n\geq 0}$, $\left(\frac{1}{2^n}H_n(x)\right)_{n\geq 0}$, and $\left(K_n^{(\alpha)}(x)\right)_{n\geq 0}$ are the Appell polynomial sequences associated, respectively, with the formal series $\left(\frac{z}{e^z-1}\right)^{\alpha}$, $\left(\frac{2}{e^z+1}\right)^{\alpha}$, $e^{\frac{-z^2}{4}}$, and $\sum_{n=0}^{+\infty}\frac{1}{(\alpha+n)_n}\frac{z^n}{n!}$. Applying Corollary 3, we get, for all complex numbers α

and λ and all non-negative integers r and n the following identities:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda^{n+r+1-k}B_k^{(\alpha)}(x)}{n+r+1-k}=\sum_{j=0}^{r}(-1)^j(r)_j\lambda^{r-j}\frac{B_{n+j+1}^{(\alpha)}(x+\lambda)}{(n+j+1)_{j+1}}+(-1)^{r+1}\frac{r!\ B_{n+r+1}^{(\alpha)}(x)}{(n+r+1)_{r+1}},$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda^{n+r+1-k}E_k^{(\alpha)}(x)}{n+r+1-k}=\sum_{j=0}^{r}(-1)^j(r)_j\lambda^{r-j}\frac{E_{n+j+1}^{(\alpha)}(x+\lambda)}{(n+j+1)_{j+1}}+(-1)^{r+1}\frac{r!E_{n+r+1}^{(\alpha)}(x)}{(n+r+1)_{r+1}},$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(2\lambda)^{n+r+1-k}H_k(x)}{(n+r+1-k)}=\sum_{j=0}^{r}(-1)^j(r)_j(2\lambda)^{r-j}\frac{H_{n+j+1}(x+\lambda)}{(n+j+1)_{j+1}}+(-1)^{r+1}\frac{r!\ H_{n+r+1}(x)}{(n+r+1)_{r+1}}.$$

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k}{k}^{-1}\frac{(\lambda x)^{n+r+1-k}L_k^{(\alpha)}(x)}{n!(n+r+1-k)}\\&\quad=\sum_{j=0}^{r}(-1)^j\frac{(r)_j(\lambda x)^{r-j}(1+x\lambda)^{n+j+1}L_{n+j+1}^{(\alpha)}\left(\frac{x}{1+x\lambda}\right)}{(\alpha+n+j+1)_{n+j+1}}+(-1)^{r+1}r!\frac{L_{n+r+1}^{(\alpha)}(x)}{(\alpha+n+r+1)_{n+r+1}}.\end{aligned}$$

2 Proofs

The proof of Theorem 1 is mainly based on the following lemma.

Lemma 4. *For all non-negative integers n, r, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{n+r+1-k}=\sum_{j=0}^{r}(-1)^j(r)_j\frac{(1-x)^{n+j+1}}{(n+j+1)_{j+1}}+(-1)^n r!\frac{x^{n+r+1}}{(n+r+1)_{r+1}}. \quad (12)$$

Proof. Let r, n be non-negative integers, $P(x)=x^r$, $Q(x)=\frac{(1+x)^{n+r}}{(n+r)_r}$, and

$$I(x)=\int_0^x P(t)Q^{(r)}(t)dt.$$

By a direct computation, we get

$$I(x)=\int_0^x t^r(1+t)^n dt=\sum_{k=0}^{n}\binom{n}{k}\frac{x^{n+r+1-k}}{n+r+1-k}. \quad (13)$$

Using generalized integration by parts, we have

$$\begin{aligned}I(x)&=\left[\sum_{j=0}^{r-1}(-1)^jP^{(j)}(t)Q^{(r-j-1)}(t)\right]_0^x+(-1)^r\int_0^x P^{(r)}(t)Q(t)dt\\&=\sum_{j=0}^{r-1}(-1)^j(r)_jx^{r-j}\frac{(1+x)^{n+j+1}}{(n+j+1)_{j+1}}+(-1)^r r!\frac{(1+x)^{n+r+1}}{(n+r+1)_{r+1}}+\frac{(-1)^{r+1}r!}{(n+r+1)_{r+1}}.\end{aligned}$$

Hence

$$I(x)=\sum_{j=0}^{r}(-1)^j(r)_j\frac{x^{r-j}(1+x)^{n+j+1}}{(n+j+1)_{j+1}}+\frac{(-1)^{r+1}r!}{(n+r+1)_{r+1}}. \tag{14}$$

According to relation (13), it is obvious that $I(x)$ is a polynomial of degree $n+r+1$. We let $J(x)$ denote the reciprocal polynomial of $I(x)$. We have $J(x)=x^{n+r+1}I\left(\frac{1}{x}\right)$. Using the two expressions for $I(x)$ obtained in relations (13) and (14), we get two expressions for $J(-x)$. By equating both expressions of $J(-x)$, we obtain relation (12). □

Proof of Theorem 1. Let u be a complex sequence. Consider the linear function L_u from $\mathbb{C}[x]$ to $\mathbb{C}^{\mathbb{N}}$ defined for all $n\geq 0$ by $L_u(x^n)=u_n$. Note that for all $n\geq 0$, we have $L_u\left((1-x)^n\right)=u_n^*$. Applying L_u to both sides of relation (12), we get relation (4). The proof of Theorem 1 is then complete. □

Proof of Corollary 2. It can be clearly seen that relations (7) and (8) are satisfied in the case where $b=0$. We can assume that $b\neq 0$. We note that for $n\geq 0$, $x^2-(ax+b)$ divides $x^{2n}-(ax+b)^n$. Then $x^{2n}-(ax+b)^n$ is a characteristic polynomial of u. Hence $L_u\left(x^{2n}-(ax+b)^n\right)=0$. We conclude that

$$u_{2n}=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k u_k.$$

Thus $\left(\left(-\frac{a}{b}\right)^n u_n\right)^*=\frac{u_{2n}}{b^n}$ and $\left(\frac{u_{2n}}{b^n}\right)^*=\left(-\frac{a}{b}\right)^n u_n$. By applying Theorem 1 to the sequences $\left(\left(-\frac{a}{b}\right)^n u_n\right)_{n\geq 0}$ and $\left(\frac{u_{2n}}{b^n}\right)_{n\geq 0}$ we obtain relations (7) and (8). To prove relations (9) and (10), consider the two sequences v and w defined by $v_n=\frac{u_n}{a^{n-1}}$ and $w_n=\frac{u_n}{a^n}$ $(n\geq 0)$. We have $(v_0^*,v_1^*)=(-v_0,-v_1)$ and $(w_0^*,w_1^*)=(w_0,w_1)$. Moreover, the sequences v, v^*, w, and w^* have the same characteristic polynomial, i.e., $x^2-x-\frac{b}{a^2}$. Then v is inverse invariant under the binomial transformation and w is invariant under the binomial transformation. Finally, by applying Theorem 1 to the sequences v and w we get relations (9) and (10). □

Proof of Corollary 3. Let $\left(A_n(x)\right)_n$ be an Appell polynomial sequence defined by

$$\sum_{n=0}^{\infty}A_n(x)\frac{z^n}{n!}=S(z)e^{xz}$$

and λ a complex number. Obviously, for $\lambda=0$, the relation (11) is satisfied. Suppose now that $\lambda\neq 0$, we consider the sequence u defined for a fixed complex number x by $u_n=(-1)^n\frac{A_n(x)}{\lambda^n}$. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda^n u_n^*\frac{z^n}{n!}=&\left(\sum_{n=0}^{\infty}\lambda^n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}A_n(x)\frac{z^n}{n!}\right)\\ =&S(z)e^{(\lambda+x)z}=\sum_{n=0}^{\infty}A_n(x+\lambda)\frac{z^n}{n!}.\end{aligned}$$

It follows that $u_n^* = \frac{A_n(x+\lambda)}{\lambda^n}$. By application of Theorem 1 to the sequence u, we get relation (11). $\square$

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