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Minimal Polynomials of Algebraic Cosine Values, II

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Abstract

We derive alternative proofs of two recent results due to Gürtas, and a new recursion related to the minimal polynomials of algebraic cosine values. We determine the minimal polynomials of algebraic values of the sine function evaluated at rational multiples of π. We also correct a formula from our earlier work.

1 Introduction

In 1933, Lehmer proved the following result [3]: *let* $n \in \mathbb{N}$, $n > 2$, $k \in \{1, 2, \ldots, n\}$ *with* $\gcd(k, n) = 1$. *Then the value* $2\cos(2k\pi/n)$ *is an algebraic integer of degree* $\varphi(n)/2$ (sequence A023022 in the OEIS [7]) *whose minimal polynomial is* $\psi_n(x) \in \mathbb{Z}[x]$, *where*

$$\psi_n(x + x^{-1}) = x^{-\varphi(n)/2}\Phi_n(x), \tag{1}$$

where Φ_n *is the nth cyclotomic polynomial and* φ *denotes Euler's totient function* (sequence A000010 in the OEIS).

The first and the third authors [8] derived some reduction formulae, based on the work of Surowski and McCombs, [6], that can be used to completely determine explicit forms of the polynomials $\psi_n(x)$. Almost simultaneously, Gürtas [2] considered the same problem, but used a different approach based on Ramanujan's trigonometric sum and its connection with the power sum function to derive a recursive relation among the coefficients of ψ_n, as well as certain other properties. The present work has three objectives.

1. To amend the formula in part IV of Theorem 5 of our earlier work [8].

2. To give alternative proofs of the last two results (Theorems 3.4 and 3.8) in [2] based on the results and ideas in [8].

3. To derive more useful recursive relations among the coefficients of ψ_n and to determine minimal polynomials of the sine values at rational multiples of π.

Since there is no universal agreement regarding the notation used in most of the works in this area, especially those appearing in [8] and [2], throughout this work we stick to the one employed in [8]. Throughout, let

$$X = x + x^{-1}, \quad X_s = x^s + x^{-s} \quad (s \in \mathbb{N}).$$

2 A correction

The formula stated in part IV of Theorem 5 in [8] is not correct as it was derived from the erroneous usage of (1) at $n = 2$. We now establish its correct version.

Proposition 1. *For* $e \in \mathbb{N}$, $e \geq 2$, *the minimal polynomial of* $2\cos(2k\pi/2^e)$, *where* $k \in \{1, 2, \ldots, 2^e\}$, $\gcd(k, 2) = 1$, *is*

$$\psi_{2^e}(x) = \sum_{k=0}^{\lfloor 2^{e-3} \rfloor} (-1)^k \left(\binom{2^{e-2} - k}{k} + \binom{2^{e-2} - k - 1}{k - 1} \right) x^{2^{e-2} - 2k}.$$

Proof. Following the approach in the proof of Theorem 5 part IV in [8], using (1) and the well-known relation (see e.g. [8, Lemma 3 B])

$$\Phi_{mq^e}(x) = \Phi_{mq}(x^{q^{e-1}}),$$

we have

$$\psi_{2^e}(X) = \psi_{2^e}\left(x + x^{-1}\right) = x^{-\varphi(2^e)/2}\Phi_{2^e}(x) = x^{-2^{e-2}}\Phi_2(x^{2^{e-1}}).$$

Substituting $\Phi_2(T) = 1 + T$ [4, Example 2.46, p. 65], we get

$$\begin{aligned}\psi_{2^e}(X) &= x^{-2^{e-2}}\left(1 + x^{2^{e-1}}\right) = x^{-2^{e-2}} + x^{2^{e-2}} = X_{2^{e-2}} \\ &= \sum_{k=0}^{\lfloor 2^{e-2}/2 \rfloor} (-1)^k \left(\binom{2^{e-2}-k}{k} + \binom{2^{e-2}-k-1}{k-1}\right) X^{2^{e-2}-2k},\end{aligned}$$

where the last equality comes from [8, Lemma 4]. Since the polynomial

$$\psi_{2^e}(X) - \sum_{k=0}^{\lfloor 2^{e-2}/2 \rfloor} (-1)^k \left(\binom{2^{e-2}-k}{k} + \binom{2^{e-2}-k-1}{k-1}\right) X^{2^{e-2}-2k}$$

vanishes for infinitely many real (or complex) values of $X = x + 1/x$, it is identically zero, i.e., all its coefficients are zero, and the result follows. □

3 Alternative proofs of three results

In this section, based upon our work in [8], we give new proofs of the following two results due to Gürtas [2].

Theorem 2. *([2, Theorem 3.4]) Let $q \in \mathbb{N}$ be odd ≥ 3, and let $d = \varphi(q)/2 = \varphi(2q)/2$. If*

$$\psi_q(2x) = 2^d \sum_{i=0}^{d} (-1)^i e_i x^{d-i}$$

and

$$\psi_{2q}(2x) = 2^d \sum_{i=0}^{d} (-1)^i e_i' x^{d-i},$$

then $e_i = (-1)^i e_i'$ $(i = 0, 1, \ldots, d)$.

Theorem 3. *([2, Theorem 3.8]) Let $n > 4$. If n is divisible by 4, then $\psi_n(2x)$ is a polynomial consisting of even powers of x only.*

To prove Theorem 2, we need a lemma.

Lemma 4. *Let $q \in \mathbb{N}$ be odd ≥ 3, and let $d = \varphi(q)/2$. Then*

$$\psi_{2q}(x) = (-1)^d\, \psi_q(-x).$$

Proof. Using (1) and [4, Problem 2.57 (d)], we have

$$\begin{aligned}\psi_{2q}(X) &= \psi_{2q}\left(x + x^{-1}\right) = x^{-d}\Phi_{2q}(x) = x^{-d}\Phi_q(-x) \\ &= x^{-d}(-x)^d\psi_q\left(-x - x^{-1}\right) = (-1)^d\psi_q(-X).\end{aligned}$$

Since the polynomial (in X) expression $\psi_{2q}(X) - (-1)^d\, \psi_q(-X)$ vanishes for infinitely many real (or complex) values of $X = x + 1/x$, it must vanish identically, i.e., all its coefficients are identically zero, yielding $\psi_{2q}(x) = (-1)^d\, \psi_q(-x)$. □

Proof of Theorem 2. Using Lemma 4, we get

$$2^d \sum_{i=0}^{d} (-1)^i e_i'(2x)^{d-i} = \psi_{2q}(2x) = (-1)^d\, \psi_q(-2x) = 2^d \sum_{i=0}^{d} (-1)^i e_i(-2x)^{d-i},$$

and the result follows at once from equating coefficients. □

To prove Theorem 3, we need another lemma.

Lemma 5. *Let $e \in \mathbb{N}$, let p be a prime, and let $q \in \mathbb{N}$, $q > 1$ with $\gcd(q,p) = 1$. Then*

$$\psi_{p^e q}(X) = \frac{\psi_q(X_{p^e})}{\psi_q(X_{p^{e-1}})}.$$

Proof. Using (1) and the well-known relation (see e.g. [8, Lemma 3 B]) $\Phi_{mq^e}(x) = \Phi_{mq}(x^{q^{e-1}})$, we get

$$\begin{aligned}\psi_{p^e q}(X) &= \psi_{p^e q}\left(x + x^{-1}\right) = x^{-\varphi(p^e q)/2}\Phi_{p^e q}(x) = x^{-\varphi(p^e q)/2}\Phi_{pq}(x^{p^e-1}) \\ &= x^{-\varphi(p^e q)/2}\frac{\Phi_q\left((x^{p^{e-1}})^p\right)}{\Phi_q\left(x^{p^{e-1}}\right)} = x^{-\varphi(p^e q)/2}\frac{(x^{p^e})^{\varphi(q)/2}\psi_q(x^{p^e} + x^{-p^e})}{(x^{p^e-1})^{\varphi(q)/2}\psi_q(x^{p^e-1} + x^{-p^{e-1}})} \\ &= \frac{\psi_q\left(X_{p^e}\right)}{\psi_q(X_{p^{e-1}})}.\end{aligned}$$

□

Proof of Theorem 3. Let $n = 2^e q$, where $e \geq 2$ and q is an odd positive integer chosen such that $n = 2^e q > 4$. From Lemma 5, and [8, Lemma 4], i.e.,

$$X_s = \sum_{k=0}^{\lfloor s/2 \rfloor} (-1)^k \left(\binom{s-k}{k} + \binom{s-k-1}{k-1} \right) X^{s-2k},$$

with the convention that $\binom{n}{r}=0$ for negative r, we have

$$\psi_n(2X)=\psi_{2^e q}(2X)=\frac{\psi_q\left(\sum_{k=0}^{2^e/2}(-1)^k\left(\binom{2^e-k}{k}+\binom{2^e-k-1}{k-1}\right)(2X)^{2^e-2k}\right)}{\psi_q\left(\sum_{k=0}^{2^{e-1}/2}(-1)^k\left(\binom{2^{e-1}-k}{k}+\binom{2^{e-1}-k-1}{k-1}\right)(2X)^{2^{e-1}-2k}\right)}.$$

Since the polynomial (in X)

$$\psi_{2^e q}(2X)\psi_q\left(\sum_{k=0}^{2^{e-1}/2}(-1)^k\left(\binom{2^{e-1}-k}{k}+\binom{2^{e-1}-k-1}{k-1}\right)(2X)^{2^{e-1}-2k}\right)$$
$$-\psi_q\left(\sum_{k=0}^{2^e/2}(-1)^k\left(\binom{2^e-k}{k}+\binom{2^e-k-1}{k-1}\right)(2X)^{2^e-2k}\right)$$

vanishes for infinitely many real values of $X=x+1/x$, it must vanish identically, and since $\psi_s(X)$ is a monic polynomial, we identically have

$$\psi_n(2x)\psi_q\left(\sum_{k=0}^{2^{e-1}/2}(-1)^k\left(\binom{2^{e-1}-k}{k}+\binom{2^{e-1}-k-1}{k-1}\right)(2x)^{2^{e-1}-2k}\right)$$
$$=\psi_q\left(\sum_{k=0}^{2^e/2}(-1)^k\left(\binom{2^e-k}{k}+\binom{2^e-k-1}{k-1}\right)(2x)^{2^e-2k}\right). \qquad (2)$$

Observe next that from $e\geq 2$, the arguments of ψ_q in the second factor on the left and in the right-hand expression, namely,

$$\sum_{k=0}^{2^e/2}(-1)^k\left(\binom{2^e-k}{k}+\binom{2^e-k-1}{k-1}\right)(2x)^{2^e-2k} \quad \text{and}$$
$$\sum_{k=0}^{2^{e-1}/2}(-1)^k\left(\binom{2^{e-1}-k}{k}+\binom{2^{e-1}-k-1}{k-1}\right)(2x)^{2^{e-1}-2k}$$

are polynomials with even exponents. Thus, in (2) the right-hand expression and the second polynomial factor on the left contain only even powers of x. This forces the polynomial $\psi_n(2x)$ to contain only even powers of x. $\square$

We end this section by presenting another derivation of ψ_p [6, Theorem 2.1] based on the following binomial identity which is identity (1.60) in [1, p. 8].

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}(xy)^k(x+y)^{n-2k}=\frac{x^{n+1}-y^{n+1}}{x-y}. \qquad (3)$$

Theorem 6. *Let $p = 2s+1$ be an odd prime. The minimal polynomial of $2\cos(2\pi/p)$ is*

$$\psi_p(x) = \sum_{j=0}^{\lfloor s/2 \rfloor} (-1)^j \binom{s-j}{j} x^{s-2j} - \sum_{j=1}^{\lfloor (s+1)/2 \rfloor} (-1)^j \binom{s-j}{j-1} x^{s-(2j-1)}.$$

Proof. Putting $y = 1/x$ in (3), we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \left(x + \frac{1}{x}\right)^{n-2k} = \frac{x^{n+1} - 1/x^{n+1}}{x - 1/x}. \tag{4}$$

Adopting the convention that $\binom{m}{j} = 0$ if $j < 0$, and using (4), we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k-1} \left(x + \frac{1}{x}\right)^{n-2k} = \sum_{\ell=0}^{\lfloor n/2 \rfloor - 1} (-1)^{\ell+1} \binom{n-\ell-2}{\ell} \left(x + \frac{1}{x}\right)^{n-2\ell-2}$$
$$= -\frac{x^{n-1} - 1/x^{n-1}}{x - 1/x}. \tag{5}$$

Adding (4) and (5), we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \left(x + \frac{1}{x}\right)^{n-2k} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k-1} \left(x + \frac{1}{x}\right)^{n-2k}$$
$$= \frac{(x^2-1)(x^{n-1} + 1/x^{n+1})}{x - 1/x} = x^n + 1/x^n = X_n. \tag{6}$$

We claim that the sum of the X_n's with odd and even indices are given, respectively, by

$$X_1 + X_3 + \cdots + X_{2t+1} = \sum_{j=0}^{t} (-1)^j \binom{2t+1-j}{j} X^{2t+1-2j} \tag{7}$$

$$X_2 + X_4 + \cdots + X_{2t} = \sum_{j=0}^{t-1} (-1)^j \binom{2t-j}{j} X^{2t-2j} - 2\delta_t, \tag{8}$$

where $\delta_t = 0$ for even t, and $\delta_t = 1$ for odd t. We begin with (7), which holds trivially for $t = 0$. Assume that it holds up to $t-1$, i.e., assume

$$X_1 + X_3 + \cdots + X_{2t-1} = \sum_{j=0}^{t-1} (-1)^j \binom{2t-1-j}{j} X^{2t-1-2j} = \sum_{k=1}^{t} (-1)^{k-1} \binom{2t-k}{k-1} X^{2t+1-2k}. \tag{9}$$

From (6), with $n = 2t + 1$, we get

$$\begin{aligned} X_{2t+1} &= \sum_{k=0}^{t}(-1)^k \left(\binom{2t+1-k}{k} + \binom{2t-k}{k-1}\right) X^{2t+1-2k} \\ &= X^{2t+1} + \sum_{k=1}^{t}(-1)^k \left(\binom{2t+1-k}{k} + \binom{2t-k}{k-1}\right) X^{2t+1-2k}. \end{aligned} \tag{10}$$

The identity (7) follows from induction by adding (9) and (10). We proceed now to verify (8). When $t = 1$, the right-hand expression is equal to

$$X^2 - 2 = \left(x + \frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2} = X_2,$$

and we are done in this case. Assume that it holds up to $t - 1$, i.e., assume that

$$\begin{aligned} X_2 + X_4 + \cdots + X_{2t-2} &= \sum_{j=0}^{t-2}(-1)^j \binom{2t-2-j}{j} X^{2t-2-2j} - 2\delta_{t-1} \\ &= \sum_{k=1}^{t-1}(-1)^{k-1} \binom{2t-1-k}{k-1} X^{2t-2k} - 2\delta_{t-1} \end{aligned} \tag{11}$$

From (6), with $n = 2t$, we get

$$\begin{aligned} X_{2t} &= \sum_{k=0}^{t}(-1)^k \left(\binom{2t-k}{k} + \binom{2t-k-1}{k-1}\right) X^{2t-2k} \\ &= X^{2t} + \sum_{k=1}^{t}(-1)^k \left(\binom{2t-k}{k} + \binom{2t-k-1}{k-1}\right) X^{2t-2k}. \end{aligned} \tag{12}$$

Adding (11) and (12), we get

$$X_2 + \cdots + X_{2t-2} + X_{2t} = X^{2t} + \sum_{k=1}^{t-1}(-1)^k \binom{2t-k}{k} X^{2t-2k} + (-1)^t 2 - 2\delta_{t-1},$$

and using the definition of δ_t, the identity (8) follows by induction.

From (1) and the shape of the p^{th} cyclotomic polynomial [4, Chapter 2], we get

$$\psi_p(X) = \psi_p\left(x + x^{-1}\right) = x^{-s}\Phi_p(x) = 1 + (x + x^{-1}) + \cdots + (x^s + x^{-s}) = 1 + X_1 + \cdots + X_s.$$

For odd $s = 2t+1$, using (7) and (8) we have

$$\begin{aligned}
\psi_p(X) &= (1 + X_2 + X_4 + \cdots + X_{2t}) + (X_1 + X_3 + \cdots + X_{2t+1}) \\
&= \left(1 + \sum_{j=0}^{t-1} (-1)^j \binom{2t-j}{j} X^{2t-2j} - 2\delta_t\right) + \left(\sum_{j=0}^{t} (-1)^j \binom{2t+1-j}{j} X^{2t+1-2j}\right) \\
&= \sum_{j=0}^{t} (-1)^j \binom{2t-j}{j} X^{2t-2j} + \sum_{j=0}^{t} (-1)^j \binom{2t+1-j}{j} X^{2t+1-2j} \\
&= \sum_{k=1}^{t+1} (-1)^{k-1} \binom{2t-k+1}{k-1} X^{2t+2-2k} + \sum_{j=0}^{t} (-1)^j \binom{2t+1-j}{j} X^{2t+1-2j}.
\end{aligned}$$

For even $s = 2t$, using (7) and (8) we have

$$\begin{aligned}
\psi_p(X) &= (1 + X_2 + X_4 + \cdots + X_{2t}) + (X_1 + X_3 + \cdots + X_{2t-1}) \\
&= \left(1 + \sum_{j=0}^{t-1} (-1)^j \binom{2t-j}{j} X^{2t-2j} - 2\delta_t\right) + \left(\sum_{j=0}^{t-1} (-1)^j \binom{2t-1-j}{j} X^{2t-1-2j}\right) \\
&= \sum_{j=0}^{t} (-1)^j \binom{2t-j}{j} X^{2t-2j} + \sum_{k=1}^{t} (-1)^{k-1} \binom{2t-k}{k-1} X^{2t+1-2k},
\end{aligned}$$

and the assertion of the theorem holds for $\psi_p(X)$. As argued before, this relation holds for infinitely many values of $X = x + 1/x$ yielding it to be an identity, and the desired result follows. □

4 More properties

Lemmas 4 and 5 enable us to deduce the next interesting result.

Theorem 7. *Let $q \in \mathbb{N}$ be odd ≥ 3, $d := \varphi(q)/2$.*

(a) We have $\psi_q(-X)\psi_q(X) = (-1)^d \psi_q(X^2 - 2)$;

(b) If $\psi_q(x) := \sum_{i=0}^{d} a_i x^i$, then for $\ell \in \{0, 1, \ldots, d\}$ we have

$$\sum_{\substack{0 \leq i,j \leq d \\ i+j=2\ell}} (-1)^i a_i a_j = (-1)^d \sum_{k=\ell}^{d} \binom{k}{\ell} a_k (-2)^{k-\ell}$$

$$\sum_{\substack{0 \leq i,j \leq d \\ i+j \text{ is odd}}} (-1)^i a_i a_j = 0.$$

Proof. (a) Taking $p = 2, e = 1$ in Lemma 5 and equating with the expression in Lemma 4, we get

$$\frac{\psi_q(X_2)}{\psi_q(X)} = \psi_{2q}(X) = (-1)^d \psi_q(-X).$$

The result follows at once from $X_2 = x^2 + x^{-2} = (x + x^{-1})^2 - 2 = X^2 - 2$.
(b) From part (a), arguing as before we see that the polynomial

$$\psi_q(-X)\psi_q(X) - (-1)^d \psi_q(X^2 - 2)$$

vanishes identically, and since each polynomial ψ_q is monic, we get $\psi_q(-x)\psi_q(x) = (-1)^d \psi_q(x^2 - 2)$, i.e.,

$$\sum_{i=0}^{d} a_i(-x)^i \sum_{j=0}^{d} a_j x^j = (-1)^d \sum_{k=0}^{d} a_k (x^2 - 2)^k.$$

The first and second assertions follow from equating the coefficients of the even (respectively, odd) powers of x on both sides. The second assertion can also be verified by noting that since $i + j$ is odd, the integers i and j have different parities. Therefore, $(-1)^i a_i a_j + (-1)^j a_j a_i = 0$ for every such pair. □

4.1 Minimal polynomials of sine values

Lehmer [3, Theorem 2, p. 166] also proved the result about the values of the sine function at rational multiples of π. We now use an analysis in [5, Theorem 3.9, pp. 37–39], which amends inaccuracies in the case $n \equiv 4 \pmod 8$ of [3, Theorem 2, p. 166], to determine the minimal polynomial of $\sin(2k\pi/n)$ A178182.

Theorem 8. *A. Let $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n\}$ with $n > 2$, $n \neq 4$ and $\gcd(k, n) = 1$.*

(a) If n is odd, then $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is $\psi_{4n}(x)$ of degree $\varphi(n)$.

(b) If $n \equiv 2 \pmod 4$, then $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is $\psi_{2n}(x)$ of degree $\varphi(n)$.

(c) If $n \equiv 0 \pmod 8$, then $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is $\psi_n(x)$ of degree $\varphi(n)/2$.

(d) If $n \equiv 4 \pmod 8$, $n > 4$ and $k \equiv n/4 \pmod 4$, then $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is $\psi_{n/4}(x)$ of degree $\varphi(n)/4$.

(e) If $n \equiv 4 \pmod 8$, $n > 4$ and $k \not\equiv n/4 \pmod 4$, then $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is $\psi_{n/2}(x)$ of degree $\varphi(n)/4$.

B. If $n = 1$ or $n = 2$, then $2\sin(2\pi/n) = 0$ is an algebraic integer whose minimal polynomial is x. If $n = 4$, then $2\sin(2\pi/n) = 2$ is an algebraic integer whose minimal polynomial is $x-2$, while $2\sin(2\cdot 3\pi/n) = -2$ is an algebraic integer whose minimal polynomial is $\psi_2(x) = x+2$.

Proof. A. Note that

$$2\sin\left(\frac{2k\pi}{n}\right) = 2\cos 2\pi\left(\frac{k}{n} - \frac{1}{4}\right) = 2\cos\left(\frac{2\pi(4k-n)}{4n}\right).$$

If n is odd, then the fraction $(4k-n)/4n$ is in reduced form and so $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is the same as that of $2\cos\left(\frac{2\pi(4k-n)}{4n}\right)$, i.e., $\psi_{4n}(x)$ with $\deg\psi_{4n} = \varphi(n)$.

If $n \equiv 2 \pmod 4$, then $\frac{4k-n}{4n} = \frac{2k-n/2}{2n}$, where the last fraction is in reduced form. Thus, $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is that of $2\cos\left(\frac{2\pi(2k-n/2)}{2n}\right)$, i.e., $\psi_{2n}(x)$ with $\deg\psi_{2n} = \varphi(2n)/2 = \varphi(n)$.

If $n \equiv 0 \pmod 4$, then there are two subcases.

- If $n \equiv 0 \pmod 8$, then $\frac{4k-n}{4n} = \frac{k-n/4}{n}$ is in reduced form because $\gcd(k,n) = 1$. Thus, $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is that of $2\cos\left(\frac{2\pi(k-n/4)}{n}\right)$, i.e., $\psi_n(x)$ with $\deg\psi_n = \varphi(n)/2$.

- If $n \equiv 4 \pmod 8$, then the fraction $(4k-n)/4n$ reduces to one with denominator $n/4$ in case $k \equiv n/4 \pmod 4$ and denominator $n/2$ otherwise. In the former case, $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is that of $2\cos\left(\frac{2\pi(k-n/4)/4}{n/4}\right)$, i.e., $\psi_{n/4}(x)$ with degree $\varphi(n/4)/2 = \varphi(n)/4$. In the latter case, $2\sin(2k\pi/n)$ is an algebraic integer whose minimal polynomial is that of $2\cos\left(\frac{2\pi(k-n/4)/2}{n/2}\right)$, i.e., $\psi_{n/2}(x)$ with degree $\varphi(n/2)/2 = \varphi(n)/4$.

The assertions in part B are easily checked directly. □

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