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Some Congruences for Central Binomial Sums Involving Fibonacci and Lucas Numbers

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Abstract

We present several polynomial congruences about sums with central binomial coefficients and harmonic numbers. In the final section we collect some new congruences involving Fibonacci and Lucas numbers.

1 Introduction

Recently, the following identity was proposed by Knuth in the problem section of the *American Mathematical Monthly* [3]:

$$\left(\sum_{k=1}^{\infty}\binom{2k}{k}\frac{x^k}{k}\right)^2 = 4\sum_{k=1}^{\infty}\binom{2k}{k}(H_{2k-1}-H_k)\frac{x^k}{k}, \tag{1}$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. Playing around with this formula, we realized that there is a corresponding polynomial congruence, namely, for all prime numbers p,

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right)^2 \equiv 4\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k-1}-H_k)\frac{x^k}{k} \pmod{p}. \tag{2}$$

By using this congruence together with some previous results given in [5, 6], we find that for all prime numbers $p > 3$,

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} \equiv (2x-\alpha)^p \pounds_2(-\beta/\alpha) + 2\alpha^p \pounds_2(\beta/\alpha) \pmod{p} \tag{3}$$

where $\pounds_2(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^2}$ is the *finite dilogarithm* and

$$\alpha = \frac{1}{2}\left(1+\sqrt{1-4x}\right) \qquad \text{and} \quad \beta = \frac{1}{2}\left(1-\sqrt{1-4x}\right).$$

These kind of congruences have been actively investigated and many interesting formulas have been discovered (see the references in [5, 6]). For example, by letting $x = 1$ in (3), we recover the congruence

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{k} \equiv \frac{7}{12} \left(\frac{p}{3}\right) B_{p-2}(1/3) \pmod{p} \tag{4}$$

which appeared in [4], where $\left(\frac{x}{y}\right)$ denotes the Legendre symbol, and $B_n(x)$ is the n-th Bernoulli polynomial. Moreover, we show several congruences involving Fibonacci numbers F_n and Lucas numbers L_n. Two of them are as follows: for all prime numbers $p > 5$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} F_{3k} \equiv \frac{13}{10} \left(\frac{p}{5}\right) q_L^2 \pmod{p}, \tag{5}$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} L_{3k} \equiv \frac{5}{2} q_L^2 \pmod{p}, \tag{6}$$

where $q_L = (L_p - 1)/p$ is the so-called *Lucas quotient.*

The paper is organized into four sections. The next section is devoted to a brief introduction to the finite polylogarithm. In Section 3 we present the proofs of the main theorems about the polynomial congruences and in the final section we establish various congruences involving Fibonacci numbers.

2 The finite polylogarithm

The *classical polylogarithm* function is defined for complex $|z| < 1$ and all positive integers d by the power series

$$\mathrm{Li}_d(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^d}.$$

It is well known that the polylogarithm can be extended analytically to a wider range of z and it satisfies several remarkable identities such as the two reflection properties,

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{\ln^2(-z)}{2} \quad \text{and} \quad \mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \ln(z)\ln(1-z).$$

These identities allow the explicit evaluation of the polylogarithm at some special values, such as

$$\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}, \quad \mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}, \quad \mathrm{Li}_2(\phi_-) = -\frac{\pi^2}{15} + \frac{\ln^2(\phi_+)}{2}.$$

where $\phi_\pm = (1 \pm \sqrt{5})/2$.

The *finite polylogarithm* function is the partial sum of the above series over the range $0 < k < p$ where p is a prime

$$£_d(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^d}.$$

It satisfies some nice properties that resemble the ones satisfied by the classical polylogarithm.

Here we restrict our attention to $£_2(x)$ (see [5] for more details): for all prime numbers $p > 3$,

$$\begin{aligned} £_2(x) &\equiv x^p £_2(1/x) \pmod{p}, \\ £_1(1-x) &\equiv -Q_p(x) - p£_2(x) \pmod{p^2}, \\ £_2(x) &\equiv £_2(1-x) + x^p £_2(1-1/x) \pmod{p}, \\ x^p £_2(x) + (1-x)^p £_2(1-x) &\equiv \frac{1}{2} Q_p^2(x) \pmod{p}. \end{aligned}$$

where

$$Q_p(x) = xq_p(x) + (1-x)q_p(1-x), \quad \text{with} \quad q_p(x) = \frac{x^{p-1}-1}{p}.$$

Several congruences for special values of $£_2(x)$ are known:

$$£_2(1) \equiv £_2(-1) \equiv 0, \quad £_2(2) \equiv 2£_2(1/2) \equiv -q_p^2(2) \pmod{p}.$$

Moreover

$$\begin{aligned} £_2\big((1 \pm i)/2\big) &\equiv -\frac{q_p^2(2)}{8} + \frac{1}{4}\left(\left(\frac{-1}{p}\right) \pm i\right) E_{p-3} \pmod{p}, \\ £_2(\omega_6^{\pm 1}) &\equiv \frac{1}{8}\left(\left(\frac{p}{3}\right) \pm i\frac{\sqrt{3}}{3}\right) B_{p-2}(1/3), \pmod{p} \end{aligned}$$

where $\omega_6 = (1 \pm i\sqrt{3})/2$ and E_n is n-th Euler number. Finally, for all prime numbers $p > 5$ we have

$$\pounds_2(\phi_\pm) \equiv \mp\frac{\sqrt{5}}{10}\left(\frac{p}{5}\right) q_L^2 \pmod{p},$$
$$\pounds_2(\phi_\pm^2) \equiv -\frac{1}{2}\left(1 \pm \frac{\sqrt{5}}{5}\left(\frac{p}{5}\right)\right) q_L^2 \pmod{p},$$
$$\pounds_2(-\phi_\pm) \equiv -\frac{1}{4}\left(1 \pm \frac{\sqrt{5}}{5}\left(\frac{p}{5}\right)\right) q_L^2 \pmod{p}.$$

Notice that the Lucas quotient satisfies (see [7]),

$$q_L = Q(\phi_\pm) \equiv \frac{5\, F_{p-\left(\frac{p}{5}\right)}}{2p} \pmod{p}.$$

3 Polynomial congruences for central binomial sums

In [5, 6], we studied various sum involving the central binomial coefficients. In particular, it has been shown that for all prime numbers $p > 3$,

$$\sum_{k=1}^{p-1}\binom{2k}{k}x^k \equiv \sum_{k=1}^{p-1}\binom{\frac{p-1}{2}}{k}(-4x)^k \equiv (1-4x)^{(p-1)/2} \pmod{p}, \tag{7}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k} \equiv \pounds_1(\alpha) + \pounds_1(\beta) \pmod{p}, \tag{8}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k^2} \equiv 2\pounds_2(\alpha) + 2\pounds_2(\beta) \pmod{p}, \tag{9}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}H_k^{(2)}x^k \equiv \frac{2(\pounds_2(\beta) - \pounds_2(\alpha))}{\sqrt{1-4x}} \pmod{p}. \tag{10}$$

where $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$.

In [1, Proposition 5], Boyadzhiev used the following Euler-type series transformation formula to handle series with central binomial coefficients: if $a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k b_k$ then in a neighborhood of $x = 0$,

$$\sum_{k=0}^{\infty}\binom{2k}{k}a_k x^k = \frac{1}{\sqrt{1-4x}}\sum_{j=0}^{\infty}\binom{2j}{j}b_j\left(\frac{-x}{1-4x}\right)^j.$$

It turns out that something similar holds for finite sum congruences:

$$\begin{aligned}
\sum_{k=0}^{p-1}\binom{2k}{k}a_kx^k &\equiv \sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}a_k(-4x)^k = \sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}(-4x)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^jb_j\\
&= \sum_{j=0}^{(p-1)/2}(-1)^jb_j\sum_{k=j}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}\binom{k}{j}(-4x)^k\\
&= \sum_{j=0}^{(p-1)/2}(-1)^jb_j\binom{\frac{p-1}{2}}{j}(-4x)^j(1-4x)^{\frac{p-1}{2}-j}\\
&\equiv (1-4x)^{\frac{p-1}{2}}\sum_{j=0}^{p-1}\binom{2j}{j}b_j\left(\frac{-x}{1-4x}\right)^j \pmod p. \qquad (11)
\end{aligned}$$

In the next theorem we apply the above transformation.

Theorem 1. *For all prime numbers $p>3$,*

$$\sum_{k=1}^{p-1}\binom{2k}{k}H_kx^k \equiv -2(1-4x)^{\frac{p-1}{2}}\pounds_1\left(-\frac{\beta}{\sqrt{1-4x}}\right) \pmod p, \qquad (12)$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{H_kx^k}{k} \equiv 2(1-4x)^{\frac{p}{2}}\left(\pounds_2\left(\frac{\alpha}{\sqrt{1-4x}}\right)-\pounds_2\left(-\frac{\beta}{\sqrt{1-4x}}\right)\right) \pmod p. \qquad (13)$$

Proof. It is easy to verify by induction that

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k(1) = -\frac{1}{n} \quad\text{and}\quad \sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k(2) = -\frac{H_n}{n}.$$

Moreover

$$\alpha\left(\frac{-x}{1-4x}\right) = \frac{\alpha}{\sqrt{1-4x}} \quad\text{and}\quad \beta\left(\frac{-x}{1-4x}\right) = -\frac{\beta}{\sqrt{1-4x}}.$$

Hence, by (11) and (8),

$$\begin{aligned}
\sum_{k=1}^{p-1}\binom{2k}{k}H_kx^k &\equiv -(1-4x)^{\frac{p-1}{2}}\sum_{j=1}^{p-1}\binom{2k}{k}\frac{1}{k}\left(\frac{-x}{1-4x}\right)^k\\
&\equiv -(1-4x)^{\frac{p-1}{2}}\left(\pounds_1\left(\frac{\alpha}{\sqrt{1-4x}}\right)+\pounds_1\left(-\frac{\beta}{\sqrt{1-4x}}\right)\right)\\
&\equiv -(1-4x)^{\frac{p-1}{2}}\left(\pounds_1\left(1-\frac{\alpha}{\sqrt{1-4x}}\right)+\pounds_1\left(-\frac{\beta}{\sqrt{1-4x}}\right)\right)\\
&\equiv -2(1-4x)^{\frac{p-1}{2}}\pounds_1\left(-\frac{\beta}{\sqrt{1-4x}}\right) \pmod p,
\end{aligned}$$

where we also used $\pounds_1(x) \equiv \pounds_1(1-x)$. Thus the proof of (12) is complete.

As regards (13), Eqns. (11) and (10) imply

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{H_k x^k}{k} \equiv -(1-4x)^{\frac{p-1}{2}}\sum_{j=1}^{p-1}\binom{2k}{k}H_k^{(2)}\left(\frac{-x}{1-4x}\right)^k$$
$$\equiv 2(1-4x)^{\frac{p}{2}}\left(\pounds_2\left(\frac{\alpha}{\sqrt{1-4x}}\right)-\pounds_2\left(-\frac{\beta}{\sqrt{1-4x}}\right)\right) \pmod p.$$

□

In the next theorem, we establish (2), the analogous congruence for the series (1).

Theorem 2. *For all prime numbers $p>3$,*

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}x^k\right)\cdot\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right) \equiv 2\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k-1}-H_k)x^k \pmod p, \tag{14}$$

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right)^2 \equiv 4\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k-1}-H_k)\frac{x^k}{k} \pmod p. \tag{15}$$

Proof. Since p divides $\binom{2k}{k}$ for $(p-1)/2<k<p$, it follows that

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}x^k\right)\cdot\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right) \equiv \sum_{n=1}^{p-1}x^n\sum_{k=1}^{n-1}\left(\frac{1}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\right) \pmod p.$$

In a similar way,

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right)^2 \equiv \sum_{n=1}^{p-1}x^n\sum_{k=1}^{n-1}\left(\frac{1}{k(n-k)}\binom{2k}{k}\binom{2(n-k)}{n-k}\right)$$
$$\equiv \sum_{n=1}^{p-1}\frac{x^n}{n}\sum_{k=1}^{n-1}\left(\left(\frac{1}{k}+\frac{1}{n-k}\right)\binom{2k}{k}\binom{2(n-k)}{n-k}\right)$$
$$\equiv 2\sum_{n=1}^{p-1}\frac{x^n}{n}\sum_{k=1}^{n-1}\left(\frac{1}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\right) \pmod p.$$

Therefore, it suffices to show by induction that

$$\sum_{k=1}^{n-1}F(n,k) = 2(H_{2n-1}-H_n) \quad \text{where} \quad F(n,k)=\frac{1}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\binom{2n}{n}^{-1}.$$

It holds for $n=1$, and it is straightforward to verify that

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k) \;\text{ with }\; G(n,k)=-\frac{k^2(2n-2k+1)F(n,k)}{(n+1)(2n+1)(n+1-k)}.$$

Hence, by the inductive assumption,

$$\begin{aligned}\sum_{k=1}^{n} F(n+1,k) &= \sum_{k=1}^{n} F(n,k) + \sum_{k=1}^{n}(G(n,k+1)-G(n,k)) \\ &= 2(H_{2n-1}-H_n) + F(n,n) + G(n,n+1) - G(n,1) \\ &= 2(H_{2n-1}-H_n) + \frac{1}{n} + 0 + \frac{(2n-1)F(n,1)}{(n+1)(2n+1)n} \\ &= 2(H_{2n-1}-H_n) + \frac{1}{n} + \frac{1}{(n+1)(2n+1)} = 2(H_{2n+1}-H_{n+1}).\end{aligned}$$

□

Now we are ready to show that our main result (3) and the congruence corresponding to the series [2, Theorem 6]: for $|x| < 1/4$,

$$\sum_{k=1}^{\infty} \binom{2k}{k} H_{2k} x^k = \frac{1}{\sqrt{1-4x}} \left(\ln\left(\frac{1+\sqrt{1-4x}}{2}\right) - 2\ln(\sqrt{1-4x}) \right). \tag{16}$$

Theorem 3. *For all prime numbers $p > 3$,*

$$\sum_{k=1}^{p-1} \binom{2k}{k} H_{2k} x^k \equiv (1-4x)^{(p-1)/2} \left(\pounds_1(\beta) - 2\pounds_1\left(-\frac{\beta}{\sqrt{1-4x}}\right)\right) \pmod p \tag{17}$$

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} \equiv (2x-\alpha)^p \pounds_2(-\beta/\alpha) + 2\alpha^p \pounds_2(\beta/\alpha) \pmod{p}. \tag{18}$$

Proof. As regards (17), since $H_{2k} = \frac{1}{2k} + (H_{2k-1} - H_k) + H_k$, it follows immediately that,

$$\sum_{k=1}^{p-1} \binom{2k}{k} H_{2k} x^k = \frac{1}{2}\sum_{k=1}^{p-1} \binom{2k}{k} \frac{x^k}{k} + \sum_{k=1}^{p-1} \binom{2k}{k} (H_{2k-1}-H_k) x^k + \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k$$

and we apply (7), (14), and (12). In a similar way, for (18),

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} = \frac{1}{2}\sum_{k=1}^{p-1} \binom{2k}{k} \frac{x^k}{k^2} + \sum_{k=1}^{p-1} \binom{2k}{k} (H_{2k-1}-H_k) \frac{x^k}{k} + \sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k x^k}{k}$$

and then we use (9), (15), and (13). □

As a remark, we point out that although the series (16) does not converge for $x = 1/4$, by letting $f(x)$ be the left-hand side of (16) then

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{H_{2k}}{4^k k} = \int_0^{\frac{1}{4}} \frac{f(x)}{x}\, dx = \frac{5\pi^2}{12}.$$

On the other hand, it can be verified that the congruence (18) holds even for $x = 1/4$, and for all prime numbers $p > 3$,

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{4^k k} \equiv \pounds_2(1) \equiv 0 \pmod{p}.$$

4 Congruences with Fibonacci and Lucas numbers

By looking at this table and by using the values of $\pounds_1$ and $\pounds_2$, we can easily obtain the explicit values of the congruences established in the previous section.

x	α	β
1	ω_6	ω_6^{-1}
-1	ϕ_+	ϕ_-
-2	2	-1
$1/2$	$(1+i)/2$	$(1-i)/2$
$1/3$	$(1+\omega_6)/3$	$(1+\omega_6^{-1})/3$
$1+i$	$1-i$	i
$1-i$	$1+i$	$-i$
$\pm i\sqrt{3}$	$1+\omega_6^{\mp 1}$	$-\omega_6^{\mp 1}$
$-\phi_-^3$	$-\phi_-$	ϕ_-^2
$-\phi_+^3$	ϕ_+^2	$-\phi_+$

For example, for all prime numbers $p > 3$, by taking $x = 1, 1/2, 1/3$ in (18), we get respectively (4), and the next two congruences,

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{2^k k} \equiv \frac{3}{16} \left(\frac{-1}{p}\right) B_{p-2}(1/4) \pmod{p},$$
$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{3^k k} \equiv \frac{2}{9} \left(\frac{p}{3}\right) B_{p-2}(1/3) \pmod{p}.$$

To order to get the congruences with F_n and L_n we need consider the cases $x = -\phi_\pm^3$. If $x = -\phi_-^3$ then $2x - \alpha = -\phi_-^4$ and

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k}\phi_-^{3k} &\equiv (-\phi_-^4)^p \pounds_2(\phi_-) + 2(-\phi_-)^p \pounds_2(-\phi_-) \\ &\equiv \frac{1}{2}\left(-7 + 3\left(\frac{p}{5}\right)\sqrt{5}\right)\pounds_2(\phi_-) + \left(-1 + \left(\frac{p}{5}\right)\sqrt{5}\right)\pounds_2(-\phi_-) \\ &\equiv \left(\frac{5}{4} - \frac{13}{20}\left(\frac{p}{5}\right)\sqrt{5}\right) q_L^2 \pmod{p}.\end{aligned}$$

where we used the fact that $2\phi_{\pm}^{p} \equiv 1 \pm \left(\frac{p}{5}\right)\sqrt{5} \pmod{p}$. In a similar way, we find that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k}\binom{2k}{k} H_{2k}\phi_{\pm}^{3k} \equiv \left(\frac{5}{4} \pm \frac{13}{20}\left(\frac{p}{5}\right)\sqrt{5}\right) q_L^2 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k}\binom{2k}{k} H_{k}\phi_{\pm}^{3k} \equiv \left(\frac{1}{2} \pm \frac{3}{10}\left(\frac{p}{5}\right)\sqrt{5}\right) q_L^2 \pmod{p}.$$

Since $\sqrt{5}F_{3k} = \phi_{+}^{3k} - \phi_{-}^{3k}$ and $L_{3k} = \phi_{+}^{3k} + \phi_{-}^{3k}$, it follows that for $p > 5$, (5) and (6) hold and also we find that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k}\binom{2k}{k} H_k F_{3k} \equiv \frac{3}{5}\left(\frac{p}{5}\right) q_L^2 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k}\binom{2k}{k} H_k L_{3k} \equiv q_L^2 \pmod{p}.$$

References

[1] K. N. Boyadzhiev, Series with central binomial coefficients, Catalan numbers, and harmonic numbers, *J. Integer Sequences* **15** (2012), Article 12.1.7.

[2] H. Chen, Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers, *J. Integer Sequences* **19** (2016), Article 16.1.5.

[3] D. E. Knuth, Problem 11832, *Amer. Math. Monthly* **122** (2015), 390.

[4] G.-S. Mao and Z.-W. Sun, Two congruences involving harmonic numbers with applications, *Int. J. Number Theory* **12** (2016), 527–539.

[5] S. Mattarei and R. Tauraso, Congruences for central binomial sums and finite polylogarithms, *J. Number Theory* **133** (2013), 131–157.

[6] S. Mattarei and R. Tauraso, From generating series to polynomial congruences, *in preparation*.

[7] H. C. Williams, A note on the Fibonacci quotient $F_{p-\varepsilon}/p$, *Can. Math. Bull.* **25** (1982), 366–370.

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