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On the Log-Concavity of the Hyperfibonacci Numbers and the Hyperlucas Numbers

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Abstract

In this paper, we discuss the properties of the hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$. We investigate the log-concavity (log-convexity) of hyperfibonacci numbers and hyperlucas numbers. For example, we prove that $\{F_n^{[r]}\}_{n\geq 1}$ is log-concave. In addition, we also study the log-concavity (log-convexity) of generalized hyperfibonacci numbers and hyperlucas numbers.

1 Introduction

Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ denote the Fibonacci and Lucas sequence, respectively. It is well known that the Binet forms of F_n and L_n are

$$F_n = \frac{\alpha^n - (-1)^n\alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n\alpha^{-n}, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$. It is evident that $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ satisfy

$$W_n = W_{n-1} + W_{n-2}, \quad n \geq 2. \tag{2}$$

For positive integers r, hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ are defined as follows [5]:

$$F_n^{[r]} = \sum_{k=0}^{n} F_k^{[r-1]}, \quad L_n^{[r]} = \sum_{k=0}^{n} L_k^{[r-1]},$$

where $F_n^{[0]} = F_n$ and $L_n^{[0]} = L_n$. Initial values of $\{F_n^{[1]}\}$ and $\{L_n^{[1]}\}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_n^{[1]}$	0	1	2	4	7	12	20	33	54	88	143	232	376	609	986
$L_n^{[1]}$	2	3	6	10	17	28	46	75	122	198	321	520	842	1363	2206

In [3, 5, 9], some properties of hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ are given. In this paper, we continue discussing the properties of $F_n^{[r]}$ and $L_n^{[r]}$. Now we recall some other definitions involved in this paper.

Definition 1. Let $\{a_n\}_{n\geq 0}$ be a sequence of positive numbers. If for all $j \geq 1$, $a_j^2 \geq a_{j-1}a_{j+1}$ (respectively $a_{j-1}a_{j+1} \geq a_j^2$), the sequence $\{a_n\}_{n\geq 0}$ is called log-concave (respectively log-convex).

Definition 2. [6] Let $\{a_n\}_{n\geq 0}$ be a sequence of positive real numbers. We say that $\{a_n\}_{n\geq 0}$ is *log-balanced* if $\{a_n\}_{n\geq 0}$ is log-convex and $\{\frac{a_n}{n!}\}_{n\geq 0}$ is log-concave.

The log-convexity and log-concavity are important properties of combinatorial sequences, and they play an important role in many fields such as quantum physics, white noise theory, probability, economics and mathematical biology. See for instance [1, 2, 6, 7, 8, 10, 11]. Clearly, log-balancedness implies log-convexity. It is well known that $\{a_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{a_{n+1}/a_n\}_{n\geq 0}$ is nondecreasing (nonincreasing). Naturally, the quotient sequence of a log-balanced sequence does not grow too fast. For the Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$, their log-concavity (log-convexity) are related to the parity of n. It is well known that $\{F_{2n+1}\}$ and $\{L_{2n}\}$ are log-convex and $\{F_{2n}\}$ and $\{L_{2n+1}\}$ are log-concave. In this paper, we discuss the log-concavity (log-convexity) of hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$. In Section 2, we show that $\{F_n^{[r]}\}_{n\geq 1}$ and $\{L_n^{[r]}\}_{n\geq 3}$ are log-concave for $r \geq 1$. In addition, we also consider the log-concavity (log-convexity) of generalized hyperfibonacci numbers and generalized hyperlucas numbers.

2 The log-concavity of hyperfibonacci numbers and hyperlucas numbers

In this section, we state and prove the main results of this paper.

Lemma 3. [4, 12] *If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}, n = 0, 1, 2, \ldots$.*

We note that above $\{F_n^{[r]}\}_{n\geq 1}$ is the convolution of $\{F_n^{[r]}\}_{n\geq 1}$ and $\{1\}_{n\geq 0}$.

Lemma 4. *For $n \geq 0$, the following equalities hold:*

$$L_{n+2}^2 - L_{n+1}L_{n+3} = 5(-1)^n, \quad F_{n+2}^2 - F_{n+1}F_{n+3} = (-1)^{n+1}$$

Proof. From (1), we can immediately prove that this lemma holds. □

Theorem 5. *For $r \geq 1$, the sequences $\{F_n^{[r]}\}_{n\geq 1}$, $\{L_n^{[1]}\}_{n\geq 3}$, and $\{L_n^{[r]}\}_{n\geq 0}(r \geq 2)$ are log-concave.*

Proof. A simple induction using the defining recurrence (2) shows that

$$F_n^{[1]} = F_{n+2} - 1, \quad L_n^{[1]} = L_{n+2} - 1. \tag{3}$$

By using (3) and (2), we can verify that $\{F_n^{[1]}\}$ and $\{L_n^{[1]}\}$ satisfy the following recurrence relation

$$W_{n+1} = 2W_n - W_{n-2}, \quad n \geq 2. \tag{4}$$

For $n \geq 1$, let $x_n = F_{n+1}^{[1]}/F_n^{[1]}$. We prove by induction that $\{x_n\}_{n\geq 1}$ is decreasing. Clearly, $x_1 = x_2 = 2 > x_3 = 7/4$. For $n \geq 3$, assume that $x_{k-1} \geq x_k$ for all 1≤ k ≤ n. It follows from (4) that

$$x_n = 2 - \frac{1}{x_{n-1}x_{n-2}}.$$

Then we have

$$x_n - x_{n+1} = \frac{x_{n-2} - x_n}{x_{n-2}x_{n-1}x_n}.$$

Since $x_{n-2} \geq x_{n-1} \geq x_n$, it follows that $x_n \geq x_{n+1}$. Then $\{x_n\}_{n\geq 1}$ is decreasing and $\{F_n^{[1]}\}_{n\geq 1}$ is log-concave. For $\{L_n^{[1]}\}_{n\geq 0}$, using a similar method, we can prove that $\{L_n^{[1]}\}_{n\geq 3}$ is log-concave. It is clear that the initial cases $(0 \leq n \leq 2)$ of $\{L_n^{[1]}\}_{n\geq 0}$ are not log-concave. According to Lemma 3, we know that $\{F_n^{[r]}\}_{n\geq 1}$ is log-concave. Now we show that $\{L_n^{[2]}\}_{n\geq 0}$ is log-concave. We can verify that

$$L_n^{[2]} = L_{n+4} - n - 5. \tag{5}$$

By using (5), (2) and Lemma 4, we get

$$\begin{aligned}
\left(L_n^{[2]}\right)^2 - L_{n-1}^{[2]}L_{n+1}^{[2]} &= (L_{n+4} - n - 5)^2 - (L_{n+3} - n - 4)(L_{n+5} - n - 6) \\
&= L_{n+4}^2 - L_{n+3}L_{n+5} + nL_{n+1} + 2L_{n-1} + 1 \\
&= 5(-1)^n + nL_{n+1} + 2L_{n-1} + 1.
\end{aligned}$$

When $n \geq 1$,

$$\left(L_n^{[2]}\right)^2 - L_{n-1}^{[2]}L_{n+1}^{[2]} \geq 3n - 2 > 0.$$

Then $\{L_n^{[2]}\}_{n\geq 0}$ is log-concave. According to Lemma 3, we know that $\{L_n^{[r]}\}_{n\geq 0}(r \geq 3)$ is log-concave. □

Theorem 6. *The sequences* $\{n!F_n^{[1]}\}_{n\geq 1}$ *and* $\{n!L_n^{[1]}\}_{n\geq 3}$ *are log-balanced.*

Proof. By Theorem 5, in order to prove the log-balancedness of $\{n!F_n^{[1]}\}_{n\geq 1}$ and $\{n!L_n^{[1]}\}_{n\geq 3}$, we only need to show that they are log-convex. It follows from (3), Lemma 4 and (2) that

$$\left(F_n^{[1]}\right)^2 - F_{n-1}^{[1]}F_{n+1}^{[1]} = (-1)^{n+1} + F_{n-1}, \tag{6}$$

$$\left(L_n^{[1]}\right)^2 - L_{n-1}^{[1]}L_{n+1}^{[1]} = 5(-1)^n + L_{n-1}. \tag{7}$$

From (3), (6) and (7), we have

$$\begin{aligned}
n\left(F_n^{[1]}\right)^2 - (n+1)F_{n-1}^{[1]}F_{n+1}^{[1]} &= (n+1)[(-1)^{n+1} + F_{n-1}] - (F_{n+2} - 1)^2, \\
n\left(L_n^{[1]}\right)^2 - (n+1)L_{n-1}^{[1]}L_{n+1}^{[1]} &= (n+1)[5(-1)^n + L_{n-1}] - (L_{n+2} - 1)^2.
\end{aligned}$$

Let $S_n = (n+1)[(-1)^{n+1} + F_{n-1}] - (F_{n+2} - 1)^2$, $\quad T_n = (n+1)[5(-1)^n + L_{n-1}] - (L_{n+2} - 1)^2$. Clearly, $S_k < 0$ for $2 \le k \le 5$ and $T_k < 0$ for $k = 4$ or $k = 5$. We can prove by induction that

$$F_n \ \ge \ n, \quad n \ge 5, \tag{8}$$

$$L_n \ \ge \ n, \quad n \ge 0. \tag{9}$$

For $n \ge 6$, by applying (8), (9) and (2), we obtain

$$\begin{aligned} S_n &\le (n+1)(1 + F_{n-1}) - (n+1)(F_{n+2} - 1) \\ &= (n+1)(2 - 2F_n) \\ &< 0, \\ T_n &\le (n+1)(6 - 2L_n) \\ &< 0. \end{aligned}$$

Noting that

$$\begin{aligned} \left(n!F_n^{[1]}\right)^2 - (n-1)!(n+1)!F_{n-1}^{[1]}F_{n+1}^{[1]} &= (n-1)!n!S_n, \\ \left(n!L_n^{[1]}\right)^2 - (n-1)!(n+1)!L_{n-1}^{[1]}L_{n+1}^{[1]} &= (n-1)!n!T_n, \end{aligned}$$

we have

$$\left(n!F_n^{[1]}\right)^2 - F_{n-1}^{[1]}F_{n+1}^{[1]} < 0 \quad for \ \ n \ge 2; \quad \left(n!L_n^{[1]}\right)^2 - L_{n-1}^{[1]}L_{n+1}^{[1]} < 0 \quad for \ \ n \ge 4.$$

Hence $\{n!F_n^{[1]}\}_{n\ge1}$ and $\{n!L_n^{[1]}\}_{n\ge3}$ are log-convex. As the sequences $\{F_n^{[1]}\}_{n\ge1}$ and $\{L_n^{[1]}\}_{n\ge3}$ are log-concave, the sequences $\{n!F_n^{[1]}\}_{n\ge1}$ and $\{n!L_n^{[1]}\}_{n\ge3}$ are log-balanced. □

In the final part of this section, we discuss the log-concavity (log-convexity) of generalized hyperfibonacci numbers and hyperlucas numbers. Let $\{U_n\}_{n\ge0}$ and $\{V_n\}_{n\ge0}$ denote the generalized Fibonacci and Lucas sequence, respectively. Their Binet forms of U_n and V_n are

$$U_n = \frac{\tau^n - (-1)^n\tau^{-n}}{\sqrt{\Delta}}, \quad V_n = \tau^n + (-1)^n\tau^{-n}, \tag{10}$$

where $\Delta = p^2 + 4, \tau = (p + \sqrt{\Delta})/2$, and $p \ge 1$. It is clear that (10) is a generalization of (1), and $\{U_n\}_{n\ge0}$ and $\{V_n\}_{n\ge0}$ satisfy the recurrence

$$W_{n+1} = pW_n + W_{n-1} \quad (n \ge 1), \quad with \quad U_0 = 0, U_1 = 1, V_0 = 2, V_1 = p. \tag{11}$$

For positive integers r, the generalized hyperfibonacci numbers $U_n^{[r]}$ and generalized hyperlucas numbers $V_n^{[r]}$ are defined as follows:

$$U_n^{[r]} = \sum_{k=0}^{n} U_k^{[r-1]}, \quad V_n^{[r]} = \sum_{k=0}^{n} V_k^{[r-1]},$$

where $U_n^{[0]} = U_n$ and $V_n^{[0]} = V_n$. For some properties of $\{U_n^{[r]}\}$ and $\{V_n^{[r]}\}$, see [3, 9]. Now we prove the log-concavity of $\{U_n^{[r]}\}$ and $\{V_n^{[r]}\}$.

Theorem 7. *For $r \geq 1$ and $p \geq 1$, the sequences $\{U_n^{[r]}\}_{n\geq 1}$ and $\{V_n^{[1]}\}_{n\geq 3}$ are log-concave.*

Proof. From the definitions of $\{U_n^{[1]}\}$ and $\{V_n^{[1]}\}$ and (11), we can verify that

$$W_{n+1} = (1+p)W_n + (1-p)W_{n-1} - W_{n-2}, \quad n \geq 2. \tag{12}$$

We first show that $\{U_n^{[1]}\}_{n\geq 1}$ is log-concave. Let $x_n = U_{n+1}^{[1]}/U_n^{[1]}$ for $n \geq 1$. It follows from (12) that

$$x_n = 1 + p + \frac{1-p}{x_{n-1}} - \frac{1}{x_{n-1}x_{n-2}}, \quad n \geq 3.$$

Then we have

$$x_{n+1} - x_n = \frac{(p-1)(x_n - x_{n-1})}{x_{n-1}x_n} + \frac{x_n - x_{n-2}}{x_{n-2}x_{n-1}x_n}, \quad n \geq 3. \tag{13}$$

Now we prove by induction that $\{x_n\}_{n\geq 1}$ is decreasing. It is clear that

$$\begin{aligned}
&U_1^{[1]} = 1, \quad U_2^{[1]} = p+1, \quad U_3^{[1]} = p^2+p+2, \quad U_4^{[1]} = p^3+p^2+3p+2,\\
&x_1 = p+1 \geq x_2 = p + \frac{2}{p+1} > x_3 = p + \frac{p+2}{p^2+p+2}.
\end{aligned}$$

Assume that $x_{k-1} \geq x_k$ for all $2 \leq k \leq n$. It follows from this assumption and (13) that $x_n \geq x_{n+1}$ for $n \geq 1$. Hence $\{x_n\}_{n\geq 1}$ is decreasing, and $\{U_n^{[1]}\}_{n\geq 1}$ is log-concave. Using this method, we can also prove that $\{V_n^{[1]}\}_{n\geq 3}$ is log-concave. It follows from Lemma 3 that $\{U_n^{[r]}\}_{n\geq 1}$ is log-concave for $r \geq 2$. $\square$

Theorem 8. *For $p \geq 1$, the sequences $\{n!U_n^{[1]}\}_{n\geq 5}$ and $\{n!V_n^{[1]}\}_{n\geq 5}$ are log-balanced.*

Proof. By (10), we derive

$$U_n^{[1]} = \frac{U_n + U_{n+1} - 1}{p}, \quad V_n^{[1]} = \frac{V_n + V_{n+1} + p - 2}{p}. \tag{14}$$

By (14) and (11), we can verify that

$$
\begin{aligned}
n\left(U_n^{[1]}\right)^2-(n+1)U_{n-1}^{[1]}U_{n+1}^{[1]} &= \frac{(n+1)[(-1)^{n+1}+U_{n+1}-U_n]}{p}-\frac{(U_n+U_{n+1}-1)U_n^{[1]}}{p},\\
n\left(V_n^{[1]}\right)^2-(n+1)V_{n-1}^{[1]}V_{n+1}^{[1]} &= \frac{(n+1)[(-1)^n\Delta-(p-2)(V_{n+1}-V_n)]}{p}\\
&\quad -\frac{(V_{n+1}+V_n+p-2)V_n^{[1]}}{p}.
\end{aligned}
$$

We can prove that

$$
\begin{aligned}
U_n^{[1]} &\geq n+1, \quad n\geq 4,\\
V_n^{[1]} &\geq n+1, \quad n\geq 4.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
U_n &\geq 1, \quad n\geq 1,\\
V_n &\geq p^2+2, \quad n\geq 2.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
n\left(U_n^{[1]}\right)^2-(n+1)U_{n-1}^{[1]}U_{n+1}^{[1]} &\leq \frac{2(n+1)(1-U_n)}{p}\\
&\leq 0,\\
n\left(V_n^{[1]}\right)^2-(n+1)V_{n-1}^{[1]}V_{n+1}^{[1]} &\leq \frac{(n+1)[\Delta-(p-2)(V_{n+1}-V_n)-(V_{n+1}+V_n+p-2)]}{p}\\
&= \frac{(n+1)[p^2-p+6-(p-1)(V_{n+1}-V_n)-2V_n]}{p}\\
&\leq \frac{(n+1)[-p^2-p+2-(p-1)(V_{n+1}-V_n)]}{p}\\
&\leq 0.
\end{aligned}
$$

Then the sequences $\{n!U_n^{[1]}\}_{n\geq 5}$ and $\{n!V_n^{[1]}\}_{n\geq 5}$ are log-convex for $p\geq 1$. Hence the sequences $\{n!U_n^{[1]}\}_{n\geq 5}$ and $\{n!V_n^{[1]}\}_{n\geq 5}$ are log-balanced. □

3 Conclusions

We have discussed the log-concavity of hyperfibonacci numbers and hyperlucas numbers. Our next work is to study the log-concavity (log-convexity) of various recurrence sequences appearing in combinatorics.

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