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Dedekind Sums with Arguments near Certain Transcendental Numbers

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Abstract

We study the asymptotic behavior of the classical Dedekind sums $s(s_k/t_k)$ for the sequence of convergents s_k/t_k $k \geq 0$, of the transcendental number

$$\sum_{j=0}^{\infty} \frac{1}{b^{2^j}}, \ b \geq 3.$$

In particular, we show that there are infinitely many open intervals of constant length such that the sequence $s(s_k/t_k)$ has infinitely many transcendental cluster points in each interval.

1 Introduction and result

Dedekind sums have quite a number of interesting applications in analytic number theory (modular forms), algebraic number theory (class numbers), lattice point problems and algebraic geometry (for instance [1, 6, 7, 10]).

Let n be a positive integer and $m \in \mathbb{Z}$, $(m, n) = 1$. The classical Dedekind sum $s(m/n)$ is defined by

$$s(m/n) = \sum_{k=1}^{n} ((k/n))((mk/n))$$

where $((\cdots))$ is the usual sawtooth function (for example, [7, p. 1]). In the present setting it is more natural to work with

$$S(m/n) = 12s(m/n)$$

instead.

In the previous paper [3] we used the Barkan-Hickerson-Knuth-formula to study the asymptotic behavior of $S(s_k/t_k)$ for the convergents s_k/t_k of transcendental numbers like e or e^2. In this situation the limiting behavior of $S(s_k/t_k)$ was fairly simple. It is much more complicated, however, for the transcendental number

$$x(b) = \sum_{j=0}^{\infty} \frac{1}{b^{2^j}}, \ b \geq 3. \tag{1}$$

In fact, we have no full description of what happens in this case. Its complexity is illustrated by the following theorem, which forms the main result of this paper.

Theorem 1. *Let s_k/t_k, $k \geq 0$, be the sequence of convergents of the number $x(b)$ of (1). Then the sequence $S(s_k/t_k)$, $k \geq 0$, has infinitely many transcendental cluster points in each of the intervals*

$$\left(b - 10 - 2i + \frac{1}{b}, b - 9 - 2i + \frac{1}{b-1}\right), \ i \geq 0.$$

Note that each of the intervals of Theorem 1 has the length $1 + 1/(b(b-1))$, whereas the distance between two neighboring intervals is $1 - 1/(b(b-1))$.

2 The integer part

We start with the continued fraction expansion $[a_0, a_1, a_2, \ldots]$ of an arbitrary irrational number x. The numerators and denominators of its convergents

$$s_k/t_k = [a_0, a_1, \ldots, a_k] \tag{2}$$

are defined by the recursion formulas

$$\begin{aligned} &s_{-2} = 0, \quad s_{-1} = 1, \quad s_k = a_k s_{k-1} + s_{k-2} \ \text{and} \\ &t_{-2} = 1, \quad t_{-1} = 0, \quad t_k = a_k t_{k-1} + t_{k-2}, \ \text{for } k \geq 0. \end{aligned} \tag{3}$$

Henceforth we will assume $0 < x < 1$, so $a_0 = 0$. Then the Barkan-Hickerson-Knuth formula says that for $k \geq 0$

$$S(s_k/t_k) = \sum_{j=1}^{k} (-1)^{j-1} a_j + \begin{cases} (s_k + t_{k-1})/t_k - 3, & \text{if } k \text{ is odd;} \\ (s_k - t_{k-1})/t_k, & \text{if } k \text{ is even;} \end{cases} \tag{4}$$

see [2, 4, 5].

In the case of the number $x = x(b)$, the continued fraction expansion has been given in [9]. It is defined recursively. To this end put

$$C(1) = C(1, b) = [0, b-1, b+2]$$

in the sense of (2) and (3). If $C(j) = C(j, b)$ has been defined for $j \geq 1$ and $C(j) = [0, a_1, \ldots, a_n]$ (where $n = 2^j$), then

$$C(j+1) = C(j+1, b) = [0, a_1, \ldots, a_n, a_n - 2, a_{n-1}, a_{n-2}, \ldots, a_2, a_1 + 1].$$

Then $x = \lim_{j\to\infty} C(j)$. In particular, $x = [0, a_1, a_2, \ldots]$, where a_k is the corresponding partial denominator of each $C(j)$ with $2^j \geq k$.

In view of formula (4) for $x = x(b)$, it is natural to investigate

$$L(k) = L(k, b) = \sum_{j=1}^{k-1} (-1)^{j-1} a_j, \;\; k \geq 0,$$

first. For the sake of simplicity we call $L(k)$ the *integer part* of the Dedekind sum $S(s_k/t_k)$.

The following lemma comprises three easy observations.

Lemma 2. *Let $[0, a_1, a_2, \ldots]$ be the continued fraction expansion of $x = x(b)$ and $n = 2^j$, $j \geq 0$.*

(a) *If $n \geq 4$, then*

$$a_{n+k} = a_{n-k+1} \text{ for } 2 \leq k \leq n-1.$$

(b) *If $n \geq 8$, then*

$$a_k = a_{n-k+1} \text{ for } 2 \leq k \leq n/2 - 1.$$

(c) *If $n \geq 8$, then*

$$a_k = a_{n+k} \text{ for } 2 \leq k \leq n/2 - 1.$$

Proof. Obviously, assertion (c) follows from (a) and (b). Assertion (a) is immediate from the definition of the continued fraction expansion of $x(b)$. In order to deduce (b) from (a), we assume $n \geq 4$ and put $l = n - k + 1$, $2 \leq k \leq n-1$. Then $a_l = a_{n-k+1} = a_{n+k}$, by (a). Since $k = n - l + 1$, this gives $a_l = a_{n+(n-l+1)} = a_{2n-l+1}$. So we have, for $n \geq 8$ and $2 \leq l \leq n/2 - 1$: $a_l = a_{n-l+1}$, which is (b). □

Lemma 3. *Let $n = 2^j$, $n \geq 4$. For $1 \leq k \leq n-1$ we have*

$$L(n+k) = -2 + L(n-k).$$

Proof. Since $L(n+1) = L(n-1) + (-1)^{n-1} a_n + (-1)^n (a_n - 2) = L(n-1) - 2$, the assertion holds for $k = 1$. Let $2 \leq k \leq n-1$. Then

$$L(n+k) = L(n-1) - 2 + \sum_{i=2}^{k} (-1)^{n+i-1} a_{n+i}.$$

By assertion (a) of Lemma 2, the sum on the right hand side equals

$$\sum_{i=2}^{k}(-1)^{n+i-1}a_{n-i+1} = \sum_{i=2}^{k}(-1)^{i-1}a_{n-i+1} = \sum_{i=1}^{k-1}(-1)^{i}a_{n-i}.$$

We observe

$$\sum_{i=1}^{k-1}(-1)^{i}a_{n-i} = \sum_{i=n-k+1}^{n-1}(-1)^{i}a_{i}.$$

This gives

$$L(n+k) = -2 + \sum_{i=1}^{n-1}(-1)^{i-1}a_i + \sum_{i=n-k+1}^{n-1}(-1)^{i}a_i = -2 + L(n-k).$$

□

Remark 4. By the construction of the sequence $C(j)$, we have $a_n = b$ for each $n = 2^j, j \geq 2$. From Lemma 3 we obtain $L(2n) = L(2n-1)+(-1)^{2n-1}a_{2n} = L(n+(n-1))-b = L(1)-2-b = b-1-2-b = -3$.

Lemma 5. *Let $n = 2^j$, $n \geq 8$. For $2 \leq k \leq n/2-1$,*

$$L(n+k) = -4 + L(k).$$

In particular, $L(n+k) = L(2n+k) = L(4n+k) = \cdots$

Proof. We have $L(n) = -3$, by the remark. Hence $L(n+1) = L(n)+(-1)^n a_{n+1} = -3+b-2 = b-5$. From Lemma 2, (c) we obtain

$$L(n+k) = b-5+(-1)^{n+1}a_{n+2}+\cdots+(-1)^{n+k-1}a_{n+k} =$$

$$b-5+(-1)^{1}a_2+\cdots+(-1)^{k-1}a_k = b-5+L(k)-a_1 = -4+L(k).$$

□

Let $n = 2^j$, $n \geq 8$. We define a sequence k_i, $i \geq 0$, in the following way:

$$k_0 = n-1. \tag{5}$$

If k_{i-1} has been defined, $i \geq 1$, then

$$k_i = 2^i n - k_{i-1}. \tag{6}$$

Induction based on (5) and (6) gives

$$2 \leq k_i \leq 2^i n - 1, \tag{7}$$

and

$$k_i = \frac{2^{i+1} + (-1)^i}{3} n + (-1)^{i-1} \tag{8}$$

for all $i \geq 0$. We have

$$L(k_0) = L(n-1) = L(n) + a_n = -3 + b$$

from the remark. Further, Lemma 3 gives, by induction,

$$L(k_i) = -3 - 2i + b.$$

Indeed, if $L(k_{i-1}) = -3 - 2(i-1) + b$, $L(k_i) = L(2^i n - k_{i-1}) = L(2^{i-1}n + (2^{i-1}n - k_{i-1})) = -2 + L(k_{i-1}) = -3 - 2i + b$. Altogether, we know the numbers k_i and the integer part of $S(s_{k_i}/t_{k_i})$ explicitly, namely

Lemma 6. *Let $n = 2^j$, $n \geq 8$. For $i \geq 0$ let k_i be defined by (8). Then*

$$L(k_i) = b - 3 - 2i.$$

Lemma 6 says that the integer part $L(k_i)$ of $S(s_{k_i}/t_{k_i})$ is independent of n if $n \geq 8$ is a power of 2. Suppose, therefore, that $n_l = 2^{2+l}$, $l = 1, \ldots, r$. Fix $i \geq 0$ for the time being and define

$$k_{i,l} = \frac{2^{i+1} + (-1)^i}{3} n_l + (-1)^{i-1}. \tag{9}$$

By (7),

$$k_{i,l} \leq 2^i n_l - 1 \leq 2^i n_r - 1 = 2^{i+r+2} - 1.$$

Suppose that $\widehat{n}$ is a power of 2, $\widehat{n} \geq 2^{i+r+3}$. Then we have

$$2 \leq k_{i,l} \leq \frac{\widehat{n}}{2} - 1$$

for all $l = 1, \ldots, r$. Therefore, Lemma 5 and Lemma 6 give

Proposition 7. *Let $i \geq 0$ and $r \geq 1$ be given and $n_l = 2^{2+l}$, $l = 1, \ldots r$. Suppose that the numbers $k_{i,l}$ are defined as in (9). If $\widehat{n}$ is a power of 2, $\widehat{n} \geq 2^{i+r+3}$, then*

$$L(\widehat{n} + k_{i,l}) = -4 + L(k_{i,l}) = b - 7 - 2i.$$

3 The fractional part

Note that the numbers $k_{i,l}$ of the foregoing section are all odd. Hence Lemma 9 and the Barkan-Hickerson-Knuth formula give

$$S(s_{\widehat{n}+k_{i,l}}/t_{\widehat{n}+k_{i,l}}) = b - 7 - 2i + \frac{s_{\widehat{n}+k_{i,l}}}{t_{\widehat{n}+k_{i,l}}} + \frac{t_{\widehat{n}+k_{i,l-1}}}{t_{\widehat{n}+k_{i,l}}} - 3. \tag{10}$$

If $\widehat{n}$ tends to infinity $s_{\widehat{n}+k_{i,l}}/t_{\widehat{n}+k_{i,l}}$ tends to $x = x(b)$. Accordingly, we have to investigate the limiting behavior of $t_{\widehat{n}+k_{i,l-1}}/t_{\widehat{n}+k_{i,l}}$ in order to understand the fractional part of formula (10).

To this end we suppose that n is a power of 2, $n \geq 8$, and k is an integer, $2 \leq k \leq n/2-1$. From (3) we have $t_{n+k} = a_{n+k}t_{n+k-1} + t_{n+k-2}$, hence

$$\frac{t_{n+k}}{t_{n+k-1}} = a_{n+k} + \frac{t_{n+k-2}}{t_{n+k-1}} = [a_{n+k}, \frac{t_{n+k-1}}{t_{n+k-2}}].$$

When we repeat this procedure, we obtain the well-known fact

$$\frac{t_{n+k}}{t_{n+k-1}} = [a_{n+k}, a_{n+k-1}, \frac{t_{n+k-2}}{t_{n+k-3}}] = [a_{n+k}, a_{n+k-1}, \ldots, a_1].$$

From Lemma 2, (c), we infer

$$a_{n+k} = a_k, a_{n+k-1} = a_{k-1}, \ldots, a_{n+2} = a_2.$$

Moreover, $a_{n+1} = a_n - 2 = b - 2$ and $a_n = b$. Finally, Lemma 2, (b) says

$$a_{n-1} = a_2, a_{n-2} = a_3, \ldots, a_{n/2+2} = a_{n/2-1}.$$

Altogether,

$$\frac{t_{n+k}}{t_{n+k-1}} = [a_k, a_{k-1}, \ldots, a_2, b-2, b, a_2, a_3, \ldots a_{n/2-1}, a_{n/2+1}, \ldots, a_1].$$

The final terms $a_{n/2+1}, a_{n/2}, \ldots, a_1$ are not of interest. It suffices to write

$$\frac{t_{n+k}}{t_{n+k-1}} = [a_k, a_{k-1}, \ldots, a_2, b-2, b, a_2, a_3, \ldots, a_{n/2-1}, c(n)] \tag{11}$$

for some $c(n) \in \mathbb{Q}$. From [9, Theorem 8] we know that all numbers $a_1, a_2, \ldots$ are ≥ 1 and $\leq b+2$, hence we have

$$1 \leq c(n) \leq b + 3.$$

Proposition 8. *Suppose that k remains fixed, $2 \leq k \leq n/2-1$, but $n = 2^j$ tends to infinity. Then t_{n+k}/t_{n+k-1} converges to*

$$t(k) = t(k, b) = [a_k, a_{k-1}, \ldots, a_2, b-2, (x+1)/x],$$

where $x = x(b)$ is defined by (1).

Proof. We have $x = \lim_{i\to\infty} C(i) = [0, b-1, y]$ with $y = [a_2, a_3, \ldots]$. A short calculation shows

$$[b, y] = [b, a_2, a_3, \ldots] = (x + 1)/x.$$

Let p_i/q_i, $i = 0, 1, 2, \ldots$ be the convergents of t_{n+k}/t_{n+k-1} (where the numbers p_i, q_i are defined in the same way as the numbers s_i, t_i in (3)). We have, by (11),

$$\frac{t_{n+k}}{t_{n+k-1}} = \frac{pc(n) + p'}{qc(n) + q'}$$

with $p = p_{k+n/2-1}$, $p' = p_{k+n/2-2}$, $q = q_{k+n/2-1}$, $q' = q_{k+n/2-2}$. We write

$$t(k) = [a_k, \ldots, a_2, b-2, b, a_2, \ldots, a_{n/2-1}, z(n)],$$

where $z(n)$ satisfies $1 \leq z(n) \leq b+3$ by the argument above. Accordingly,

$$t(k) = \frac{pz(n) + p'}{qz(n) + q'}.$$

This gives

$$t(k) - \frac{t_{n+k}}{t_{n+k-1}} = \frac{pz(n) + p'}{qz(n) + q'} - \frac{pc(n) + p'}{qc(n) + q'}. \tag{12}$$

The expression on the right hand side of (12) simplifies to

$$\frac{(pq' - p'q)z(n) + (p'q - pq')c(n)}{(qz(n) + q')(qc(n) + q')}.$$

However, it is well-known that $pq' - p'q = \pm 1$. Observing $1 \leq z(n), c(n) \leq b+3$, we obtain

$$\left| t(k) - \frac{t_{n+k}}{t_{n+k-1}} \right| \leq \frac{2b+6}{(q+q')^2}.$$

Since q and q' tend to infinity for $n \to \infty$, our proof is complete. □

We conclude this section with two observations.

Lemma 9. *In the above setting, let $2 \leq k < k'$ be integers. Then $t(k) \neq t(k')$.*

Proof. Suppose $t(k) = t(k')$, so

$$[a_{k'}, \ldots, a_{k+1}, t(k)] = t(k).$$

An identity of this kind can only hold if $t(k)$ is a quadratic irrationality. However, $t(k)$ is a transcendental number since x is transcendental (see [8, p. 35, Satz 8]). □

Lemma 10. *Let $k \geq 2$ be an integer. Then $x + 1/t(k)$ is a transcendental number.*

Proof. Suppose $\alpha = x + t(k)$ is algebraic. Since we may write

$$1/t(k) = [0, t(k)] = \frac{p(x+1)/x + p'}{q(x+1)/x + q'} = \frac{p(x+1) + p'x}{q(x+1) + q'x}$$

with integers p, p', q, q', $q > 0$, $q' \geq 0$, we obtain

$$x + \frac{p(x+1) + p'x}{q(x+1) + q'x} = \alpha.$$

This, however, means that x satisfies a quadratic equation over the field $\mathbb{Q}(\alpha)$. Accordingly, x is algebraic, a contradiction. □

4 Proof of Theorem 1

As in the setting of Proposition 7, let $i \geq 0$ and $r \geq 1$ be given and $n_l = 2^{2+l}$, $l = 1, \ldots, r$. Suppose that the numbers $k_{i,l}$ are defined as in (9). Let $\widehat{n}$ be a power of 2, $\widehat{n} \geq 2^{i+r+3}$. By Proposition 7,

$$L(\widehat{n} + k_{i,l}) = b - 7 - 2i.$$

If $\widehat{n}$ tends to infinity, Proposition 8 says that $t_{\widehat{n}+k_{i,l}}/t_{\widehat{n}+k_{i,l}-1}$ tends to

$$t(k_{i,l}) = [a_{k_{i,l}}, a_{k_{i,l}-1}, \ldots, a_2, b-2, (x+1)/x].$$

Therefore $t_{\widehat{n}+k_{i,l}-1}/t_{\widehat{n}+k_{i,l}}$ tends to $1/t(k_{i,l})$. Altogether, we have

$$S(s_{\widehat{n}+k_{i,l}}/t_{\widehat{n}+k_{i,l}}) \to b - 10 - 2i + x + \frac{1}{t(k_{i,l})}.$$

For $l < l' \leq r$ we obtain $k_{i,l} < k_{i,l'}$ from (9). By Lemma 9, $t(k_{i,l}) \neq t(k_{i,l'})$. Accordingly, the numbers $1/t(k_{i,l})$ are pairwise different for $1 \leq l \leq r$. Further, $x + 1/t(k_{i,l})$ is transcendental, by Lemma 10. The inequalities

$$1/b < x < 1/(b-1) \text{ and } 0 < 1/t(k_{i,l}) < 1$$

are obvious by (1) and $x = [0, b-1, \ldots]$, $1/t(k_{i,l}) = [0, a_{k_{i,l}}, \ldots]$. Therefore, the sequence $S(s_j/t_j)$, $j \geq 1$, has r distinct transcendental cluster points in the interval

$$\left(b - 10 - 2i + \frac{1}{b}, b - 9 - 2i + \frac{1}{b-1}\right).$$

Since r can be chosen arbitrarily large, this proves Theorem 1.

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