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Families of Sequences From a Class of Multinomial Sums

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Abstract

In this paper we obtain formulas for certain sums of products involving multinomial coefficients and Fibonacci numbers. The sums studied here may be regarded as generalizations of the binomial transform of the sequence comprising the even-numbered terms of the Fibonacci sequence. The general formulas, involving both Fibonacci and Lucas numbers, give rise to infinite sequences that are parameterized by two positive integers. Links to the exponential partial Bell polynomials are also established.

1 Introduction

The sequences being considered in much of what follows may in some sense be regarded as generalizations of the *binomial transform* $\{b_n\}$ of $\{F_{2n}\}$, where the nth *Fibonacci number* F_n may be defined recursively by setting $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ [2, 3, 5]. The nth term of $\{b_n\}$ is given by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} F_{2k}. \tag{1}$$

This sequence, which starts $1, 5, 20, 75, 275, \ldots$, appears in Sloane's *On-line Encyclopedia of Integer Sequences* [10] as A093131. It may be shown that

$$b_{2n-1} = 5^{n-1} L_{2n-1} \qquad \text{and} \qquad b_{2n} = 5^n F_{2n}, \tag{2}$$

where L_n, the nth *Lucas number*, is defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Our generalization leads us to study certain sums of products involving multinomial coefficients and Fibonacci numbers. These give rise to sequences parameterized by two positive integers, s and t. In other words, each pair (s, t) is associated with a particular infinite sequence. Note, by way of contrast, that $\{b_n\}$ does not possess any parameters. A further result concerning 1-parameter sequences is then derived. Finally, we go on to obtain some results associated with our sequences, including one in connection with the exponential partial Bell polynomials.

2 Some general results

Throughout this paper we use the notation

$$\binom{n}{a_0, a_2, \ldots, a_{t-1}}$$

to represent the *multinomial coefficient* [6]

$$\frac{(a_0 + a_2 + \cdots + a_{t-1})!}{a_0! a_1! \cdots a_{t-1}!},$$

where each a_i, $i = 0, 1, \ldots, t-1$, is a non-negative integer such that $n = a_0 + a_1 + \cdots + a_{t-1}$. The binomial coefficients correspond to the special case $t = 2$.

Furthermore, we shall find it useful to extend the definition of the Fibonacci numbers to negative subscripts as in [5]. This may be achieved by rearranging the Fibonacci recurrence relation to give $F_{n-2} = F_n - F_{n-1}$ and then using it in a recursive manner for $n = 1, 0, -1, -2 \ldots$. From this we obtain $F_{-1} = F_1 - F_0 = 1$, $F_{-2} = F_0 - F_{-1} = -1$, and so on. It is in fact straightforward to show that for any $n \in \mathbb{N}$,

$$F_{-n} = (-1)^{n+1} F_n. \tag{3}$$

In order to obtain some of the results that follow, we will, at appropriate points, introduce and utilize various mathematical properties of the *golden ratio* ϕ [4, 5, 6], where

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

We shall have cause to make frequent use of the result

$$\phi^j = F_j \phi + F_{j-1}, \tag{4}$$

which is given in [5, 6]. This may be proved by induction using the Fibonacci recurrence relation and the fact that $\phi^2 = \phi + 1$. Note, by virtue of (3) and the result $\frac{1}{\phi} = \phi - 1$, that (4) is valid for all $j \in \mathbb{Z}$. Similarly, we may obtain

$$L_{-n} = (-1)^n L_n \tag{5}$$

and

$$\sqrt{5}\ \phi^j = L_j\phi + L_{j-1}. \tag{6}$$

Result (6) appears in [5] and, as with (4), is valid for all $j \in \mathbb{Z}$.

In Theorem 2 below, use shall be made of the following simple Lemma concerning irrational numbers in general:

Lemma 1. *For any irrational number α and $p, q, r, s \in \mathbb{Q}$, $p\alpha + r = q\alpha + s$ if, and only if, $p = q$ and $r = s$.*

Proof. Suppose that $p\alpha + r = q\alpha + s$. If $p = q$ then it is clear that $r = s$. Let us assume, therefore, that $p \neq q$. We then can write

$$\alpha = \frac{s-r}{p-q} \in \mathbb{Q}.$$

This, however, is a contradiction since α is irrational. It must thus be the case that $p\alpha + r = q\alpha + s$ implies $p = q$ and $r = s$. The converse is obviously true, thereby proving the lemma. □

Theorem 2.

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} F_{u_1(k,m)} = F_{2kn(2m-1)} \left(\frac{F_{4km}}{F_{2k}}\right)^n,$$

where

$$u_1(k,m) = 4k\,(a_1 + 2a_2 + \cdots + (2m-1)a_{2m-1})$$

and the summation is to be taken over all possible $a_i \geq 0$, $i = 0, 1, 2, \ldots, 2m-1$, such that $a_0 + a_1 + \cdots + a_{2m-1} = n$.

Proof. First, let

$$\begin{aligned} g(s,t) &= 1 + \phi^{2s} + \phi^{4s} + \cdots + \phi^{2s(t-1)} \\ &= \sum_{j=0}^{t-1} \phi^{2js}, \end{aligned}$$

where $s, t \in \mathbb{N}$ such that $t \geq 2$. Then, on using (4), it follows that for any $n \in \mathbb{N}$ we have

$$\begin{aligned} [g(2k,2m)]^n &= \sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} \left(\phi^{4k}\right)^{a_1} \left(\phi^{8k}\right)^{a_2} \cdots \left(\phi^{4k(2m-1)}\right)^{a_{2m-1}} \\ &= \sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} \phi^{4k(a_1+2a_2+\cdots+(2m-1)a_{2m-1})} && (7) \\ &= \sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} \left(F_{u_1(k,m)}\phi + F_{u_1(k,m)-1}\right), && (8) \end{aligned}$$

where $u_1(k,m)$ is as in the statement of the theorem, and the summation is to be taken over all possible $a_i \geq 0$, $i = 0, 1, 2, \ldots, 2m-1$, such that $a_0 + a_1 + \cdots + a_{2m-1} = n$.

Next, on using the formula for the sum of a finite geometric progression [9], we have

$$
\begin{aligned}
g(s,t) &= \frac{\left(\phi^{2s}\right)^t - 1}{\phi^{2s} - 1} \\
&= \frac{\phi^{st}\left(\phi^{st} - \phi^{-st}\right)}{\phi^{s}\left(\phi^{s} - \phi^{-s}\right)} \\
&= \phi^{s(t-1)}\left(\frac{\phi^{st} - \phi^{-st}}{\phi^{s} - \phi^{-s}}\right). \qquad (9)
\end{aligned}
$$

From Binet's formula [2, 3, 4, 5, 6]

$$F_j = \frac{1}{\sqrt{5}}\left(\phi^j - \left(-\frac{1}{\phi}\right)^j\right)$$

and the corresponding formula for the Lucas numbers [2, 4]

$$L_j = \phi^j + \left(-\frac{1}{\phi}\right)^j$$

we may obtain the result

$$\phi^j - \phi^{-j} = \begin{cases} \sqrt{5}\,F_j, & \text{if } j \text{ is even;} \\ L_j, & \text{if } j \text{ is odd.} \end{cases} \qquad (10)$$

This implies, by way of (9), that

$$g(2k, 2m) = \phi^{2k(2m-1)}\frac{F_{4km}}{F_{2k}},$$

and hence, using (4), that

$$
\begin{aligned}
\left[g(2k,2m)\right]^n &= \phi^{2kn(2m-1)}\left(\frac{F_{4km}}{F_{2k}}\right)^n \qquad (11) \\
&= \left(\phi F_{2kn(2m-1)} + F_{2kn(2m-1)-1}\right)\left(\frac{F_{4km}}{F_{2k}}\right)^n. \qquad (12)
\end{aligned}
$$

The theorem then follows from (8), (12) and Lemma 1 with $\alpha = \phi$. □

In the following corollary we show how to generalize the result from Theorem 2 yet further:

Corollary 3. *For any $j \in \mathbb{Z}$,*

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} F_{u_1(k,m)+j} = F_{2kn(2m-1)+j}\left(\frac{F_{4km}}{F_{2k}}\right)^n,$$

where the notation is identical to that used in the statement of Theorem 2.

Proof. From (7) and (11), we obtain

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} \phi^{4k(a_1+2a_2+\cdots+(2m-1)a_{2m-1})} = \phi^{2kn(2m-1)} \left(\frac{F_{4km}}{F_{2k}}\right)^n.$$

Both sides of this equality are then multiplied by ϕ^j for some $j \in \mathbb{Z}$, and the result follows from (4) and Lemma 1. $\square$

Similarly, (9) and (10) give

$$g(2k-1, 2m) = \phi^{(2k-1)(2m-1)} \frac{\sqrt{5}\, F_{2m(2k-1)}}{L_{2k-1}}, \tag{13}$$

$$g(2k, 2m-1) = \phi^{4k(m-1)} \frac{F_{2k(2m-1)}}{F_{2k}}, \tag{14}$$

$$g(2k-1, 2m-1) = \phi^{2(2k-1)(m-1)} \frac{L_{(2k-1)(2m-1)}}{L_{2k-1}}. \tag{15}$$

On obtaining the multinomial expansions for $[g(2k, 2m-1)]^n$ and $[g(2k-1, 2m-1)]^n$, using (14) and (15), and then employing a similar idea to that utilized in Corollary 3, we have the following results:

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-2}} F_{u_3(k,m)+j} = F_{4kn(m-1)+j} \left(\frac{F_{2k(2m-1)}}{F_{2k}}\right)^n \tag{16}$$

and

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-2}} F_{u_4(k,m)+j} = F_{2n(2k-1)(m-1)+j} \left(\frac{L_{(2k-1)(2m-1)}}{L_{2k-1}}\right)^n, \tag{17}$$

where

$$u_3(k, m) = 4k\,(a_1 + 2a_2 + \cdots + (2m-2)a_{2m-2})$$

and

$$u_4(k, m) = 2(2k-1)\,(a_1 + 2a_2 + \cdots + (2m-2)a_{2m-2})\,.$$

The presence of the factor $\sqrt{5}$ in (13) means that, in order to use Lemma 1, a slightly different treatment is required. We use (4) and (6) to obtain

$$\begin{aligned}[g(2k-1, 2m)]^{2n-1} &= 5^{n-1}\sqrt{5}\,\phi^{(2n-1)(2k-1)(2m-1)} \left(\frac{F_{2m(2k-1)}}{L_{2k-1}}\right)^{2n-1} \\ &= 5^{n-1}\left(L_{(2n-1)(2k-1)(2m-1)}\phi + L_{(2n-1)(2k-1)(2m-1)}\right) \left(\frac{F_{2m(2k-1)}}{L_{2k-1}}\right)^{2n-1}\end{aligned}$$

and

$$\begin{aligned}[g(2k-1, 2m)]^{2n} &= 5^n \phi^{2n(2k-1)(2m-1)} \left(\frac{F_{2m(2k-1)}}{L_{2k-1}}\right)^{2n} \\ &= 5^n \left(F_{2n(2k-1)(2m-1)}\phi + F_{2n(2k-1)(2m-1)}\right) \left(\frac{F_{2m(2k-1)}}{L_{2k-1}}\right)^{2n}.\end{aligned}$$

These results lead to

$$\sum \binom{2n-1}{a_0, a_1, \ldots, a_{2m-1}} F_{u_2(k,m)+j} = 5^{n-1} L_{(2n-1)(2k-1)(2m-1)+j} \left(\frac{F_{2m(2k-1)}}{L_{2k-1}} \right)^{2n-1} \tag{18}$$

and

$$\sum \binom{2n}{a_0, a_1, \ldots, a_{2m-1}} F_{u_2(k,m)+j} = 5^{n} F_{2n(2k-1)(2m-1)+j} \left(\frac{F_{2m(2k-1)}}{L_{2k-1}} \right)^{2n}, \tag{19}$$

respectively, where

$$u_2(k,m) = 2(2k-1)\left(a_1 + 2a_2 + \cdots + (2m-1)a_{2m-1}\right).$$

Note that a formula for the right-hand side of (1) may be obtained by considering the expansion of $(1+\phi^2)^n = [g(1,2)]^n$. The results $b_{2n-1} = 5^{n-1}L_{2n-1}$ and $b_{2n} = 5^n F_{2n}$ given in (2) may thus be seen to be specializations of (18) and (19), respectively. It is also worth noting, for example, that both Identity 3 in [1] and Result (31) in [7] are specializations of Theorem 2. The sequence arising from Identity 3 in [1] appears in [10] as A087426.

Since

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-1}} F_{u_1(k,m)}$$

is an integer, it follows from Theorem 2 that

$$F_{2k}^n | F_{2kn(2m-1)} F_{4km}^n.$$

Although not necessarily following from this divisibility result, it is in fact well-known that $F_a | F_{ab}$ and hence $F_a^n | F_{ab}^n$ for any $a, b, n \in \mathbb{N}$ [2]. Indeed, a similar observation may be made regarding (16). On considering (17) we see that

$$L_{2k-1}^n | F_{2n(2k-1)(m-1)} L_{(2k-1)(2m-1)}^n.$$

Again, although not necessarily following from this, it is true, if rather less well-known, that $L_a | L_{ab}$ for any odd positive integers a and b. Similarly, (18) and (19) hint at the possibility that, for any odd positive integer a and even positive integer b, $L_a | F_{ab}$. This result also happens to be true (see [5] for a summary of such results).

Leading on from this, since

$$\frac{F_{2k(2m-1)}}{F_{2k}} = 1 + \sum_{j=1}^{m-1} L_{4jk} \qquad \text{and} \qquad \frac{L_{(2k-1)(2m-1)}}{L_{2k-1}} = 1 + \sum_{j=1}^{m-1} L_{2j(2k-1)},$$

we may in fact combine (16) and (17) to give

$$\sum \binom{n}{a_0, a_1, \ldots, a_{2m-2}} F_{u_5(s,m)+j} = F_{2sn(m-1)+j} \left(1 + \sum_{j=1}^{m-1} L_{2js} \right)^n,$$

where

$$u_5(s,m) = 2s\left(a_1 + 2a_2 + \cdots + (2m-2)a_{2m-2}\right)$$

We have thus combined two cases at the expense of a somewhat more unwieldy expression on the right-hand side.

3 An alternative family of sequences

We now consider a family of sequences parameterized by $t \in \mathbb{N}$ such that t is even, noting that these are not specializations of any of the sequences derived in Section 2. Let us start by defining the expression $h(t)$ given by

$$h(t) = 1 + \phi + \phi^2 + \cdots + \phi^{(t-1)}.$$

Then, since $\phi = \frac{1}{\phi - 1}$, we may obtain

$$\begin{aligned} h(4m) &= \frac{\phi^{4m} - 1}{\phi - 1} \\ &= \phi \cdot \phi^{2m} \left(\phi^{2m} - \phi^{-2m}\right) \\ &= \phi^{2m+1} \left(\phi^{2m} - \left(-\frac{1}{\phi}\right)^{2m}\right) \\ &= \phi^{2m+1}\sqrt{5}\, F_{2m}, \end{aligned} \tag{20}$$

and similarly

$$\begin{aligned} h(4m-2) &= \phi^{2m} \left(\phi^{2m-1} + \left(-\frac{1}{\phi}\right)^{2m-1}\right) \\ &= \phi^{2m} L_{2m-1}. \end{aligned} \tag{21}$$

Result (20) gives rise to

$$\sum \binom{2n-1}{a_0, a_1, \ldots, a_{4m-1}} F_{v_1(m)+j} = 5^{n-1} L_{(2n-1)(2m+1)+j} F_{2m}^{2n-1}$$

and

$$\sum \binom{2n}{a_0, a_1, \ldots, a_{4m-1}} F_{v_1(m)+j} = 5^n F_{2n(2m+1)+j} F_{2m}^{2n},$$

while (21) leads to

$$\sum \binom{n}{a_0, a_1, \ldots, a_{4m-3}} F_{v_2(m)+j} = F_{2mn+j} L_{2m-1}^n, \tag{22}$$

where

$$v_1(m) = a_1 + 2a_2 + \cdots + (4m-1)a_{4m-1}$$

and

$$v_2(m) = a_1 + 2a_2 + \cdots + (4m-3)a_{4m-3}.$$

4 Further results

As we now show, it is possible to express (22) in a form such that the subscripts of the Fibonacci numbers are given explicitly. On treating $(1+x+x^2+\cdots+x^{t-1})^n$ as a formal power series and ignoring issues of convergence, we have

$$\begin{aligned}\left(1+x+x^2+\cdots+x^{t-1}\right)^n &= \left(\frac{1-x^t}{1-x}\right)^n \\ &= \left(1-x^t\right)^n (1-x)^{-n} \\ &= \sum_{i=0}^{n}\binom{n}{i}\left(-x^t\right)^i \sum_{j=0}^{\infty}\binom{n-1+j}{n-1}x^j.\end{aligned}$$

Our aim is to find the coefficient of x^r in the above product of sums. To this end, consider the following typical term that arises when the product is expanded:

$$(-1)^i\binom{n}{i}\binom{n-1+j}{n-1}x^{ti+j}.$$

It is clear from this that

$$(-1)^i\binom{n}{i}\binom{n-1+j}{n-1}$$

makes a non-zero contribution to the coefficient of x^r if, and only if, $0 \le i \le \left\lfloor \frac{r}{t} \right\rfloor$. The coefficient of x^r is thus

$$\sum_{i=0}^{\lfloor \frac{r}{t} \rfloor}(-1)^i\binom{n}{i}\binom{n-1+r-ti}{n-1}.$$

In conjunction with (22) this then leads to the result

$$\sum_{r=0}^{n(4m-3)}\sum_{i=0}^{\lfloor \frac{r}{4m-2} \rfloor}(-1)^i\binom{n}{i}\binom{n-1+r-i(4m-2)}{n-1}F_{r+j} = F_{2mn+j}L_{2m-1}^n$$

for any $j \in \mathbb{Z}$.

Finally, we demonstrate a connection between the results obtained in Section 2 and the *exponential partial Bell polynomials* $B_{k,n}$ [8, 11]. The latter are defined by way of

$$\sum_{k=n}^{\infty}B_{k,n}(x_1,x_2,\ldots)\frac{t^k}{k!} = \frac{1}{n!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}x_j\right)^n.$$

There are a number of well-known results associated with these polynomials. For example,

$$\begin{aligned}\sum_{k=n}^{\infty}B_{k,n}(1,1,1,\ldots)\frac{t^k}{k!} &= \frac{1}{n!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\right)^n \\ &= \frac{1}{n!}\left(e^t-1\right)^n,\end{aligned}$$

which is the exponential generating function for the nth column of the Stirling number triangle of the second kind. Incidentally, it follows from this that

$$B_{k,n}(1,1,1,\ldots) = S(k,n),$$

the Stirling number of the second kind enumerating the ways of partitioning a set of k labeled objects into n non-empty disjoint parts. In addition, the kth Bell number B_k arises by way of

$$\sum_{n=1}^{k} B_{k,n}(1,1,1,\ldots) = B_k.$$

This number enumerates all possible partitions of k labeled objects.

We obtain here the result

$$\sum_{k=2n-1}^{\infty} B_{k,2n-1}\left(1!,2!,\ldots,(4m)!,0,0,\ldots\right)\frac{F_{k+j}}{k!} = \frac{5^{n-1}}{(2n-1)!}L_{2(m+1)(2n-1)+j}F_{2m}^{2n-1} \tag{23}$$

for any $j \in \mathbb{Z}$, noting that the left-hand side is in fact a finite sum. First, with the help of (10):

$$\begin{aligned}
\sum_{k=2n-1}^{\infty} B_{k,2n-1}\left(1!,2!,\ldots,(4m)!,0,0,\ldots\right)\frac{\phi^k}{k!} &= \frac{1}{(2n-1)!}\left(\sum_{j=1}^{4m}\frac{\phi^j}{j!}j!\right)^{2n-1} \\
&= \frac{\phi^{2n-1}}{(2n-1)!}\left(\sum_{j=0}^{4m-1}\phi^j\right)^{2n-1} \\
&= \frac{\phi^{2n-1}}{(2n-1)!}\left(\frac{\phi^{4m}-1}{\phi-1}\right)^{2n-1} \\
&= \frac{\phi^{4n-2}}{(2n-1)!}\left(\phi^{2m}\left(\phi^{2m}-\phi^{-2m}\right)\right)^{2n-1} \\
&= \frac{\sqrt{5}\,\phi^{2(m+1)(2n-1)}}{(2n-1)!}5^{n-1}F_{2m}^{2n-1}.
\end{aligned}$$

Then, on utilizing Lemma 1, (6) and the idea from Corollary 3, we obtain (23). Similarly,

$$\sum_{k=2n}^{\infty} B_{k,2n}\left(1!,2!,\ldots,(4m)!,0,0,\ldots\right)\frac{F_{k+j}}{k!} = \frac{5^n}{(2n)!}F_{4n(m+1)+j}F_{2m}^{2n}$$

and

$$\sum_{k=n}^{\infty} B_{k,n}\left(1!,2!,\ldots,(4m-2)!,0,0,\ldots\right)\frac{F_{k+j}}{k!} = \frac{1}{n!}F_{n(2m+1)+j}L_{2m-1}^{n}.$$

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