Journal of Integer Sequences, Vol. 13 (2010),
Article 10.7.5

Ramanujan Cubic Polynomials of the Second Kind

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Abstract

We present generalizations of some identities discussed earlier by Shevelev. Moreover, we introduce Ramanujan cubic polynomials of the second kind (RCP2). This new type of cubic polynomial is closely related to the Ramanujan cubic polynomials (RCP) defined by Shevelev. We also give many fundamental properties of RCP2's.

1 Shevelev type identities

V. Shevelev [2] gave a trigonometric equality of the form

$$\sqrt{\frac{\cos\frac{2\pi}{5}}{\cos\frac{\pi}{5}}}+\sqrt{\frac{\cos\frac{\pi}{5}}{\cos\frac{2\pi}{5}}}=\sqrt{5}. \tag{1}$$

The theorem, given below, shows that (1) is a special case of a large class of identities for Fibonacci numbers F_n:

Theorem 1. *We have*

$$\sqrt[r]{\frac{F_{n-1}\varphi^{r-1}}{\varphi^{n-1}-F_n}}+\sqrt[r]{\frac{\varphi^{n-1}-F_n}{F_{n-1}\varphi^{r-1}}}=\sqrt{5}, \qquad n,r\in\mathbb{N}, \tag{2}$$

where φ denotes the golden ratio $\left(\varphi=\frac{1+\sqrt{5}}{2}\right)$.

Proof. We note that (2) is a consequence of the following identities

$$\varphi + \varphi^{-1} = \sqrt{5}$$

and

$$\varphi^n = F_n \varphi + F_{n-1}$$

(which is proved by a simple induction) or, equivalently,

$$\varphi^r = \frac{F_{n-1}\varphi^{r-1}}{\varphi^{n-1} - F_n}.$$

□

In the next theorem we present identities (2) for the general Fibonacci sequences

$$\begin{cases} F_0^* = 0, \quad F_1^* = 1, \\ F_{n+1}^* = \lambda_1 F_n^* + \lambda_2 F_{n-1}^*, \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1^2 + 4\lambda_2 \neq 0$, $\lambda_2 \neq 0$.

Let x_1, x_2 be two roots of the characteristic equation

$$x^2 - \lambda_1 x - \lambda_2 = 0.$$

We note that $x_1 \neq x_2$. Then we have

Theorem 2. *The following identities hold:*

$$x_l^n = F_n^* x_l + \lambda_2 F_{n-1}^*, \tag{4}$$

$$\frac{\lambda_2 F_{n-1}^*}{x_l^{n-1} - F_n^*} - \frac{x_l^{n-1} - F_n^*}{F_{n-1}^*} = \lambda_1, \tag{5}$$

$$\frac{x_l^n - \lambda_2 F_{n-1}^*}{\sqrt{\lambda_2} F_n^*} - \frac{\sqrt{\lambda_2} F_n^*}{x_l^n - \lambda_2 F_{n-1}^*} = \frac{\lambda_1}{\sqrt{\lambda_2}}, \tag{6}$$

$$\left(\frac{\lambda_2 F_{n-1}^*}{x_1} + F_n^*\right)^{\frac{k}{n-1}} + \left(\frac{\lambda_2 F_{n-1}^*}{x_2} + F_n^*\right)^{\frac{k}{n-1}} = \lambda_1 F_k^* + 2\lambda_2 F_{k-1}^*, \tag{7}$$

for any $l = 1, 2$ *and* $k, n \in \mathbb{N}$.

Proof. (4). Equality (4) can be proven by induction with respect to $n \in \mathbb{N}$.

(5). From (4) we get

$$x_l = \frac{\lambda_2 F_{n-1}^*}{x_l^{n-1} - F_n^*}.$$

Next, we note that

$$x_l^2 - \lambda_1 x_l - \lambda_2 = 0 \quad \Leftrightarrow \quad x_l - \lambda_2 x_l^{-1} = \lambda_1. \tag{8}$$

(6). From (4) we have

$$x_l = \frac{x_l^n - \lambda_2 F^*_{n-1}}{F^*_n}.$$

Hence, by (8) we obtain (6).

(7). From (4) we receive ($l = 1, 2$):

$$x_l^{n-1} = \frac{\lambda_2 F^*_{n-1}}{x_l} + F^*_n \quad \Rightarrow \quad x_l = \sqrt[n-1]{\frac{\lambda_2 F^*_{n-1}}{x_l} + F^*_n}$$

(the last one holds for the respective value of $(n-1)$-th root of the number $\frac{\lambda_2 F^*_{n-1}}{x_l} + F^*_n$), which implies the identity

$$\left(\frac{\lambda_2 F^*_{n-1}}{x_1} + F^*_n\right)^{\frac{k}{n-1}} + \left(\frac{\lambda_2 F^*_{n-1}}{x_2} + F^*_n\right)^{\frac{k}{n-1}} = x_1^k + x_2^k \overset{(4)}{=} \lambda_1 F^*_k + 2\lambda_2 F^*_{k-1}.$$

□

Corollary 3. *For any $k, n \in \mathbb{N}$ the following identity holds:*

$$\left(F_{n+1} + \frac{F_n}{\varphi}\right)^{k/n} + \left(F_{n+1} - \frac{F_n}{\varphi - 1}\right)^{k/n} =$$
$$= \left(F_{n-1} + \varphi\, F_n\right)^{k/n} + \left(F_{n+2} - \varphi^2\, F_n\right)^{k/n} = F_k + 2\, F_{k-1} = L_k, \quad (9)$$

where L_k denotes the k-th Lucas number.

Remark 4. Identities, similar to those discussed in the previous theorem, can be generated for the elements of linear recurrence equations of any order. See in particular the relations defining the so-called quasi-Fibonacci numbers [3, 4, 5, 8].

Remark 5. Shevelev's intention in the paper [2] was, it seems, to investigate the sum

$$\sqrt{\left|\frac{x_1}{x_2}\right|} + \sqrt{\left|\frac{x_2}{x_1}\right|},$$

where $x_1, x_2 \in \mathbb{R}$ are roots of the polynomial

$$x^2 - \lambda_1 x - \lambda_2$$

and $x_1 x_2 < 0$. Then we have

$$\sqrt{\left|\frac{x_1}{x_2}\right|} + \sqrt{\left|\frac{x_2}{x_1}\right|} = \frac{|x_1| + |x_2|}{\sqrt{|x_1 x_2|}} = \sqrt{\frac{(|x_1| + |x_2|)^2}{|x_1 x_2|}} =$$
$$= \sqrt{\frac{(x_1 + x_2)^2 + 2|x_1 x_2| - 2x_1 x_2}{|x_1 x_2|}} = \sqrt{\frac{\lambda_1^2 + 4\lambda_2}{\lambda_2}}. \quad (10)$$

In the particular case of

$$x^2 + x - 1 = \left(x + 2\cos\frac{\pi}{5}\right)\left(x - 2\cos\frac{2\pi}{5}\right),$$

the identity (1) follows from (10).

The extension of sums (10) to sums for roots of a given cubic polynomial is described in the next section.

Remark 6. We note that (see formula (7)):

$$\lambda_1 F_k^* + 2\,\lambda_2 F_{k-1}^* = F_{k+1}^* + \lambda_2 F_{k-1}^* = L_k^*, \tag{11}$$

where L_k^* denotes the generalized Lucas sequence

$$\begin{cases} L_0^* = 2, \quad L_1^* = \lambda_1, \\ L_{n+1}^* = \lambda_1 L_n^* + \lambda_2 L_{n-1}^*, \qquad n \in \mathbb{N}. \end{cases} \tag{12}$$

2 Cubic Shevelev sums

Let us assume that ξ_1, ξ_2, ξ_3 are complex roots of the following polynomial with complex coefficients

$$f(z) := z^3 + p\,z^2 + q\,z + r.$$

The symbols $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$, $\sqrt[3]{\xi_3}$ will denote any of the third complex roots of the numbers ξ_1, ξ_2 and ξ_3, respectively (only in the case that ξ_1, ξ_2 and ξ_3 are real numbers we will assume that $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$ and $\sqrt[3]{\xi_3}$ also denote the respective real roots).

Let us set

$$A := \left(\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3}\right)^3$$

and

$$B := \left(\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}\right)^3.$$

Thus, the numbers

$$\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} \qquad \text{and} \qquad \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}$$

belong to the sets of the third complex roots of A and B, respectively, which, for the conciseness of notation, will be denoted by the symbols $\sqrt[3]{A}$ and $\sqrt[3]{B}$, respectively. In other words, we have

$$\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} \in \sqrt[3]{A}$$

and

$$\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3} \in \sqrt[3]{B}.$$

Then we can deduce the relation

$$27AB = (A + p - 3\sqrt[3]{r})^3. \tag{13}$$

We note that

$$\xi_1^{1/3}\xi_2^{1/3}\xi_3^{1/3}\left(\frac{\xi_1^{1/3}}{\xi_2^{1/3}}+\frac{\xi_2^{1/3}}{\xi_1^{1/3}}+\frac{\xi_2^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_2^{1/3}}+\frac{\xi_1^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_1^{1/3}}\right)=$$
$$=\xi_1^{1/3}\xi_2^{1/3}\big(\xi_1^{1/3}+\xi_2^{1/3}\big)+\xi_2^{1/3}\xi_3^{1/3}\big(\xi_2^{1/3}+\xi_3^{1/3}\big)+\xi_1^{1/3}\xi_3^{1/3}\big(\xi_1^{1/3}+\xi_3^{1/3}\big)=$$
$$=\big(\xi_1^{1/3}+\xi_2^{1/3}+\xi_3^{1/3}\big)\big(\xi_1^{1/3}\xi_2^{1/3}+\xi_2^{1/3}\xi_3^{1/3}+\xi_1^{1/3}\xi_3^{1/3}\big)-3\xi_1^{1/3}\xi_2^{1/3}\xi_3^{1/3}.$$

Hence and from (13), for the respective values of $\sqrt[3]{A}$, $\sqrt[3]{B}$ and $\sqrt[3]{r}$ we get

$$\frac{\xi_1^{1/3}}{\xi_2^{1/3}}+\frac{\xi_2^{1/3}}{\xi_1^{1/3}}+\frac{\xi_2^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_2^{1/3}}+\frac{\xi_1^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_1^{1/3}}=$$
$$=\frac{\sqrt[3]{A}\sqrt[3]{B}}{-\sqrt[3]{r}}-3=\frac{1}{-3\sqrt[3]{r}}(A+p)-2$$

(from the formula (3.5) in [6])

$$=\frac{1}{\sqrt[3]{2}\sqrt[3]{r}}\left(\sqrt[3]{S+\sqrt{\tau}}+\sqrt[3]{S-\sqrt{\tau}}\right)$$
$$=\frac{1}{\sqrt[3]{2}}\left(\sqrt[3]{S_1+\sqrt{\tau_1}}+\sqrt[3]{S_1-\sqrt{\tau_1}}\right),$$

where

$$S=rS_1,\quad \tau=r^2\tau_1,$$
$$S_1=\frac{pq}{r}+\frac{6}{r^{2/3}}\big(q+p\sqrt[3]{r}+3\sqrt[3]{r^2}\big)-9,$$
$$\tau_1=\left(\frac{pq}{r}\right)^2-4\frac{q^3}{r^2}-4\frac{p^3}{r}+18\frac{pq}{r}-27=$$
$$=\left(\frac{pq}{r}+9\right)^2-\frac{4}{r^2}(q^3+p^3r+27r^2).$$

In consequence, if $f(z)$ is the RCP polynomial (see [2, 7]), then

$$pr^{1/3}+3r^{2/3}+q=0, \tag{14}$$

which implies

$$S_1=\frac{pq}{r}-9\quad\text{and}\quad \tau_1=\left(\frac{pq}{r}+9\right)^2-36\frac{pq}{r}=\left(\frac{pq}{r}-9\right)^2,$$

since

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)+3abc, \tag{15}$$

for $a,b,c\in\mathbb{C}$. Hence, we get the Shevelev formula

$$\frac{\xi_1^{1/3}}{\xi_2^{1/3}}+\frac{\xi_2^{1/3}}{\xi_1^{1/3}}+\frac{\xi_2^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_2^{1/3}}+\frac{\xi_1^{1/3}}{\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{\xi_1^{1/3}}=\left(\frac{pq}{r}-9\right)^{1/3}.$$

However, if we assume that

$$q^3 + p^3 r + 27r^2 = 0, \tag{16}$$

then we obtain

$$\sqrt{\tau_1} = \left|\frac{pq}{r} + 9\right|, \quad S_1 - \frac{pq}{r} - 9 = \frac{6}{r^{2/3}}\big(q + p\sqrt[3]{r}\big),$$
$$S_1 + \frac{pq}{r} + 9 = \frac{2pq}{r} + \frac{6}{r^{2/3}}\big(q + p\sqrt[3]{r} + 3\sqrt[3]{r^2}\big).$$

Hence, we get the formula

$$\frac{\xi_1^{1/3}}{\xi_2^{1/3}} + \frac{\xi_2^{1/3}}{\xi_1^{1/3}} + \frac{\xi_2^{1/3}}{\xi_3^{1/3}} + \frac{\xi_3^{1/3}}{\xi_2^{1/3}} + \frac{\xi_1^{1/3}}{\xi_3^{1/3}} + \frac{\xi_3^{1/3}}{\xi_1^{1/3}} =$$
$$= \sqrt[3]{\frac{3}{r^{2/3}}(q + p\sqrt[3]{r})} + \sqrt[3]{\frac{pq}{r} + \frac{3}{r^{2/3}}(q + p\sqrt[3]{r} + 3\sqrt[3]{r^2})}.$$

For example, let us set

$$f(z) = z^3 + 3z^2 - 3\sqrt[3]{2}z + 1. \tag{17}$$

Then the condition (16) is satisfied and the roots ξ_1, ξ_2 and ξ_3 of $f(z)$ are real: $\xi_1 = 0.56048$, $\xi_2 = 0.445392$ and $\xi_3 = -4.00587$. Furthermore, the following equality holds

$$\frac{\xi_1^{1/3}}{\xi_2^{1/3}} + \frac{\xi_2^{1/3}}{\xi_1^{1/3}} + \frac{\xi_2^{1/3}}{\xi_3^{1/3}} + \frac{\xi_3^{1/3}}{\xi_2^{1/3}} + \frac{\xi_1^{1/3}}{\xi_3^{1/3}} + \frac{\xi_3^{1/3}}{\xi_1^{1/3}} = \sqrt[3]{9(1 - \sqrt[3]{2})} + \sqrt[3]{18(1 - \sqrt[3]{2})} =$$
$$= \sqrt[3]{9(1 - \sqrt[3]{2})(1 + \sqrt[3]{2})} = -3. \tag{18}$$

We note that condition (16), by (15), is a condition of a type different from the condition (14).

3 RCP of the second kind

Shevelev in paper [1] (see also [7]) distinguished polynomials $f \in \mathbb{R}[z]$ of the form

$$f(z) = z^3 + p\,z^2 + q\,z + r, \tag{19}$$

having real roots and satisfying the condition (14), and called them Ramanujan cubic polynomials (shortly RCP).

Now we introduce a new family of cubic polynomials of the form (19), having real roots and satisfying the condition (16). We will call them Ramanujan cubic polynomials of the second kind (shortly RCP2). The polynomial (17) is an example of RCP2 which is not RCP. On the other hand, the polynomials (see [7]):

$$z^3 - \frac{3}{2}z^2 - \frac{3}{2}z + 1 = \Big(z - \frac{1}{2}\Big)(z+1)(z-2),$$
$$z^3 + z^2 - 2z - 1 = \Big(z - 2\cos\frac{2\pi}{7}\Big)\Big(z - 2\cos\frac{4\pi}{7}\Big)\Big(z - 2\cos\frac{8\pi}{7}\Big),$$

belong to the set RCP\RCP2. The polynomial

$$z^3 - 3\,z + 1 = \Big(z - 2\,\cos\frac{2\,\pi}{9}\Big)\,\Big(z - 2\,\cos\frac{4\,\pi}{9}\Big)\,\Big(z - 2\,\cos\frac{8\,\pi}{9}\Big)$$

belongs to the common part of families of RCP's and RCP2's (see [7] and Theorem 7 a) written below). The polynomial

$$z^3 - 3\,z + \sqrt{3} = \Big(z - 2\,\sin\frac{2\,\pi}{9}\Big)\,\Big(z + 2\,\sin\frac{4\,\pi}{9}\Big)\,\Big(z - 2\,\sin\frac{8\,\pi}{9}\Big)$$

is neither RCP nor RCP2.

In the next theorem we present the basic properties of RCP2's.

Theorem 7. *Let $f(z) \in \mathbb{R}[z]$ and be of the form (19). Then the following facts hold.*

a) If $f(z)$ is either RCP or RCP2 and $p\,q\,r = 0$, then $f(z)$ must be RCP and RCP2 simultaneously. Conversely, if $f(z)$ belongs to the intersection of the sets RCP and RCP2 then $p\,q\,r \neq 0$.

b) If $f(z)$ satisfies (16), then $f(z)$ is RCP2. In other words, the condition (16) implies that all the roots of $f(z)$ are real. Only in the case of $p\,q = -9\,r$ polynomial $f(z)$ possesses double root. In this case we have

$$g(z) := \frac{1}{p^3}\,f(p\,z) = z^3 + z^2 + \frac{\sqrt{5}-1}{6}\,z + \frac{1-\sqrt{5}}{54}. \tag{20}$$

Moreover, if ξ_1 ξ_2 and $\xi_3 = \xi_2$ are roots of $g(z)$, then we obtain (see formula (25) below):

$$\sqrt[3]{\xi_1} + 2\,\sqrt[3]{\xi_2} = \sqrt[3]{-1 + 2\,\sqrt[3]{\frac{\sqrt{5}-1}{2}} - 6\,\sqrt[3]{\frac{1}{3}\,\sqrt[3]{\Big(\frac{\sqrt{5}-1}{2}\Big)^2} - \frac{1}{3}\,\sqrt[3]{\Big(\frac{\sqrt{5}-1}{2}\Big)^4}}} \tag{21}$$

and

$$\sqrt[9]{\frac{\sqrt{5}-1}{2}}\,\Big(1 + \sqrt[3]{\frac{\xi_1}{\xi_2}} + \sqrt[3]{\frac{\xi_2}{\xi_1}}\Big) = \sqrt[3]{\sqrt[3]{\Big(\frac{\sqrt{5}-1}{2}\Big)^2} - 1}. \tag{22}$$

Next, whenever $f(z)$ satisfies the condition (14), then $f(z)$ is RCP if and only if $r > 0$.

c) If $f(z)$ is RCP2, then we have

$$r \neq 0 \quad\Longrightarrow\quad \frac{p\,q}{r} \leqslant \frac{9}{\sqrt[3]{4}}. \tag{23}$$

If $f(z)$ is RCP, then we have (see [1]):

$$r \neq 0 \quad\Longrightarrow\quad \frac{p\,q}{r} \leqslant \frac{9}{4}. \tag{24}$$

d) If $f(z)$ is RCP, then

$$p^2 \geqslant 12\,q,$$

whereas, if $f(z)$ is RCP2, then

$$p^2 \geqslant 3\,\sqrt[3]{4}\,q.$$

e) Let $f(z)$ belong to family of RCP2's and let ξ_1, ξ_2, ξ_3 be roots of $f(z)$. Then we have

$$\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}= \\ =\sqrt[3]{-p-6\,\sqrt[3]{r}-3\,\sqrt[3]{3\,\sqrt[3]{r}\,\big(q+p\,\sqrt[3]{r}\big)}-3\,\sqrt[3]{\big(p+3\,\sqrt[3]{r}\big)\,\big(q+3\,\sqrt[3]{r^2}\big)}}. \quad (25)$$

For example, for the polynomial (17) we obtain

$$\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}=0.$$

f) Let $f(z)$ belong to the family of RCP2's and $a, b \in \mathbb{R}$. Suppose that ξ_1, ξ_2, ξ_3 are roots of $f(z)$. If $a\,\xi_1+b$, $a\,\xi_2+b$, $a\,\xi_3+b$ are also roots of some RCP2, then

$$b\,\big(9\,b^2-9\,a\,b\,p+a^2\,(p^2+6\,q)\big)\big(9\,b^3-9\,a\,b^2\,p+a^2\,b\,(p^2+6\,q)-a^3\,(p\,q+9\,r)\big)=0. \quad (26)$$

g) If $a, a\,\varrho, a\,\varrho^2 \in \mathbb{R}$ are roots of some RCP2, then

$$2\,\sqrt[3]{2}\,\varrho=-\sqrt[3]{2}-3\pm\sqrt{3\,\big(3+2\,\sqrt[3]{2}-\sqrt[3]{4}\big)}.$$

Moreover, for $a=2\,\sqrt[3]{2}$ we have

$$a\,\varrho^2=6-\sqrt[3]{2}+\frac{9}{\sqrt[3]{2}}\mp\Big(1+\frac{3}{\sqrt[3]{2}}\Big)\,\sqrt{3\,\Big(3+2\,\sqrt[3]{2}-\sqrt[3]{4}\Big)}.$$

h) If $f(z)$ is RCP2, then

$$f(z)=z^3+\sqrt[3]{\Big(\alpha-\frac{27}{2}\Big)\,r}\,z^2-\sqrt[3]{\Big(\alpha+\frac{27}{2}\Big)\,r^2}\,z+r, \quad (27)$$

for any $\alpha, r \in \mathbb{R}$. We note that if $g(z)$ is RCP then from (18) in [7] we have

$$g(z)=z^3+\Big(\beta-\frac{3}{2}\Big)\,\varrho^{1/3}\,z^2-\Big(\beta+\frac{3}{2}\Big)\,\varrho^{2/3}\,z+\varrho$$

for some $\beta, \varrho \in \mathbb{R}$.

Proof. a) Both conclusions follow from (14), (15) and (16).

b) Suppose that $f(z)$ satisfies (16). Then

$$f'(z)=\Big(z+\frac{p-\sqrt{p^2-3\,q}}{3}\Big)\,\Big(z+\frac{p+\sqrt{p^2-3\,q}}{3}\Big)$$

and

$$f\left(\frac{-p+\sqrt{p^2-3\,q}}{3}\right)\,f\left(\frac{-p-\sqrt{p^2-3\,q}}{3}\right)\overset{(16)}{=}-\frac{1}{27}\left(p\,q+9\,r\right)^2,$$

which means that all the roots of $f(z)$ are real.

Now let $p\,q=-9\,r$. Then from (16) we get

$$q^2+\frac{3}{9}\,p^2\,q-\frac{1}{9}\,p^4=0,$$

which implies

$$q=\frac{\sqrt{5}-1}{6}\,p^2 \qquad \text{and} \qquad r=\frac{1-\sqrt{5}}{54}\,p^3,$$

and the relation (20) follows.

The equality (21) can be deduced from formula (25).

c) From (16) we get

$$\frac{p^3\,q^3}{r^3}=-\frac{p^6}{r^2}-27\,\frac{p^3}{r},$$

which implies

$$\frac{9^3}{4}-\left(\frac{p\,q}{r}\right)^3=\left(\frac{27}{2}+\frac{p^3}{r}\right)^2\geqslant 0, \tag{28}$$

i.e.,

$$\frac{p\,q}{r}\leqslant\frac{9}{\sqrt[3]{4}}.$$

d) From (16) we obtain

$$27\,r^2+p^3\,r+q^3=0,$$

$$\Delta_r=p^6-4\cdot 27\cdot q^3\geqslant 0,$$

i.e,

$$p^2\geqslant 3\,\sqrt[3]{4}\,q.$$

Similarly, if we have

$$3\,\sqrt[3]{r^2}+p\,\sqrt[3]{r}+q=0$$

and $p,q,r\in\mathbb{R}$, then

$$\Delta_{\sqrt[3]{r}}=p^2-12\,q\geqslant 0 \quad\Leftrightarrow\quad p^2\geqslant 12\,q.$$

e) We have the formula (see formula (3.5) in [6]):

$$\sqrt[3]{\xi_1}+\sqrt[3]{\xi_2}+\sqrt[3]{\xi_3}=\sqrt[3]{-p-6\,\sqrt[3]{r}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}+\sqrt{\mathcal{T}}}+\sqrt[3]{\mathcal{S}-\sqrt{\mathcal{T}}}\right)}, \tag{29}$$

where

$$\mathcal{S} := p\,q + 6\,q\,\sqrt[3]{r} + 6\,p\,\sqrt[3]{r^2} + 9\,r,$$
$$\mathcal{T} := p^2\,q^2 - 4\,q^3 - 4\,p^3\,r + 18\,p\,q\,r - 27\,r^2.$$

Hence, by (16) we get

$$\mathcal{T} = p^2\,q^2 + 18\,p\,q\,r + 81\,r^2 = r^2\Big(\frac{p\,q}{r} + 9\Big)^2, \tag{30}$$

which implies

$$\begin{aligned}
\big\{\mathcal{S} \pm \sqrt{\mathcal{T}}\big\} &= \big\{\mathcal{S} \pm \big(p\,q + 9\,r\big)\big\},\\
\mathcal{S} - p\,q - 9\,r &= 6\,\sqrt[3]{r}\,\big(q + p\,\sqrt[3]{r}\big),\\
\mathcal{S} + p\,q + 9\,r &= 2\,p\,q + 6\,q\,\sqrt[3]{r} + 6\,p\,\sqrt[3]{r^2} + 18\,r =\\
&= 2\,q\,\big(p + 3\,\sqrt[3]{r}\big) + 6\,\sqrt[3]{r^2}\,\big(p + 3\,\sqrt[3]{r}\big) = 2\,\big(p + 3\,\sqrt[3]{r}\big)\,\big(q + 3\,\sqrt[3]{r^2}\big)
\end{aligned}$$

and, at last, the formula (25) follows.

In consequence, if $f(z) = z^3 + 3\,z^2 - 3\,\sqrt[3]{2}\,z + 1$, then $p = 3$, $q = -3\,\sqrt[3]{2}$, $r = 1$ and from (25) we get

$$\begin{aligned}
\Big(\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3}\Big)^3 &= -9 - 3\,\sqrt[3]{9\,(1 - \sqrt[3]{2})} - 3\,\sqrt[3]{18\,(1 - \sqrt[3]{2})} =\\
&= -9 - 3\,\sqrt[3]{9\,(1 - \sqrt[3]{2})}\,\big(\sqrt[3]{2} + 1\big) \overset{(18)}{=} -9 + 9 = 0.
\end{aligned}$$

f) We have

$$\big(x - a\,\xi_1 - b\big)\,\big(x - a\,\xi_2 - b\big)\,\big(x - a\,\xi_3 - b\big) = x^3 + p_1\,x^2 + q_1\,x + r_1,$$

where

$$\begin{aligned}
p_1 &= a\,p - 3\,b,\\
q_1 &= a^2\,q + 3\,b^2 - 2\,a\,b\,p,\\
r_1 &= a^3\,r - a^2\,b\,q - b^3 + a\,b^2\,p.
\end{aligned}$$

If this polynomial is also RCP2, then $q_1^3 + p_1^3\,r_1 + 27\,r_1^2 = 0$, which (with assistance of Mathematica) implies the equation (26).

g) Suppose that $a \neq 0$ and

$$z^3 + p\,z^2 + q\,z + r = (z - a)\,(z - a\,\varrho)\,(z - a\,\varrho^2).$$

Then we have the relations

$$\begin{aligned}
r &= -(a\,\varrho)^3,\\
p &= -a\,(1 + \varrho + \varrho^2),\\
q &= a^2\,(\varrho + \varrho^2 + \varrho^3),
\end{aligned}$$

and the condition (16) has now the form

$$\left(\varrho+\varrho^2+\varrho^3\right)^3+27\,\varrho^6+\varrho^3\,(1+\varrho+\varrho^2)^3=0$$

or

$$2\,(1+\varrho+\varrho^2)^3+27\,\varrho^3=0.$$

Hence

$$\sqrt[3]{2}\,(1+\varrho+\varrho^2)=-3\,\varrho,$$

$$\varrho^2+\left(1+\frac{3}{\sqrt[3]{2}}\right)\varrho+1=0,$$

which implies

$$2\,\sqrt[3]{2}\,\varrho=-\sqrt[3]{2}-3\pm\sqrt{3\left(3+2\,\sqrt[3]{2}-\sqrt[3]{4}\right)}.$$

h) Let us set

$$\alpha:=\frac{27}{2}+\frac{p^3}{r}. \tag{31}$$

Then from (28) we generate the relation

$$\frac{9^3}{4}-\left(\alpha-\frac{27}{2}\right)\frac{q^3}{r^2}=\alpha^2,$$

i.e.,

$$\frac{9^3}{4}-\alpha^2=\left(\alpha-\frac{27}{2}\right)\frac{q^3}{r^2},$$

$$q^3=-\left(\alpha+\frac{27}{2}\right)r^2.$$

From (31) we obtain

$$p^3=\left(\alpha-\frac{27}{2}\right)r.$$

□

The following theorem, proved by Shevelev for RCP's [1], holds also for RCP2's.

Theorem 8. *If for two RCP2's of the form*

$$y^3+p_1\,y^2+q_1\,y+r_1,\qquad z^3+p_2\,z^2+q_2\,z+r_2$$

the following condition holds ($r_1\,r_2\neq 0$)*:*

$$\frac{p_1\,q_1}{r_1}=\frac{p_2\,q_2}{r_2},$$

then for their roots y_1*,* y_2 y_3 *and* z_1*,* z_2*,* z_3*, respectively, the sequence of numbers*

$$\frac{y_1}{y_2},\frac{y_2}{y_1},\frac{y_1}{y_3},\frac{y_3}{y_1},\frac{y_2}{y_3},\frac{y_3}{y_2},$$

is a permutation of the sequence

$$\frac{z_1}{z_2},\frac{z_2}{z_1},\frac{z_1}{z_3},\frac{z_3}{z_1},\frac{z_2}{z_3},\frac{z_3}{z_2}.$$

Proof. The proof runs like Shevelev's proof of Theorem 5 in [1]. Only one change is needed, for the case of RCP2 in formula (38) we have

$$\frac{p^3\, r + q^3}{r^2} \overset{(16)}{=} -27.$$

□

4 Acknowledgments

I wish to express my gratitude to the referee for several helpful comments and suggestions concerning the first version of the paper.

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2000 *Mathematics Subject Classification*: Primary 11C08; Secondary 11B83, 33B10.
Keywords: Ramanujan cubic polynomial.

Received June 17 2010; revised version received July 1 2010. Published in *Journal of Integer Sequences*, July 9 2010.

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