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On Recurrences of Fahr and Ringel Arising in Graph Theory

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Abstract

We solve certain recurrences given by Fahr and Ringel, and confirm their conjecture that two sequences are identical.

1 Introduction

Fahr and Ringel introduce two tables of numbers, the $b_t[r]$ and $c_t[r]$, given by

$b_t[r]$

$t \backslash r$	0	1	2	3	4	$\cdots$
0	1					
1	2	1				
2	7	4	1			
3	29	18	6	1		
4	130	85	33	8	1	

$c_t[r]$

$t \backslash r$	0	1	2	3	4	$\cdots$
0	1					
1	3	1				
2	12	5	1			
3	53	25	7	1		
4	247	126	42	9	1	

and the recurrences

$$b_{t+1}[r] = c_t[r-1] + 2c_t[r] - b_t[r],$$
$$c_{t+1}[r] = b_{t+1}[r] + 2b_{t+1}[r+1] - c_t[r],$$

which hold for $t, r \geq 0$, with the understanding that $c_t[-1] = c_t[0]$.

The object of this note is to determine the generating functions of the $b_t[r]$ and $c_t[r]$, and to prove that

$$F_{4t+2} = b_t[0] + 3\sum_{r\geq 1} 2^{2r-1} b_t[r], \qquad F_{4t+4} = 3\sum_{r\geq 0} 2^{2r} c_t[r],$$

where the F_t are the Fibonacci numbers, given by $F_0 = 0,\ F_1 = 1$ and $F_t = F_{t-1} + F_{t-2}$ for $t \geq 2$.

2 The solution

Let us define

$$B_r = B_r(q) = \sum_{t\geq 0} b_t[r]q^t, \quad C_r = C_r(q) = \sum_{t\geq 0} c_t[r]q^t.$$

The first few B_r, C_r are

$$\begin{aligned}
B_0 &= 1 + 2q + 7q^2 + 29q^3 + 130q^4 + \cdots , \\
B_1 &= q + 4q^2 + 18q^3 + 85q^4 + \cdots , \\
B_2 &= q^2 + 6q^3 + 33q^4 + \cdots , \\
C_0 &= 1 + 3q + 12q^2 + 53q^3 + 247q^4 + \cdots \\
C_1 &= q + 5q^2 + 25q^3 + 126q^4 + \cdots , \\
C_2 &= q^2 + 7q^3 + 42q^4 + \cdots .
\end{aligned}$$

Note that for $r \geq 0$,

$$B_r \equiv 0 \pmod{q^r}, \qquad C_r \equiv 0 \pmod{q^r}.$$

From the recurrences given above, we have

$$B_0 = 1 + 3qC_0 - qB_0,$$

or,

$$B_0 = \frac{1 + 3qC_0}{1+q}.$$

Also, for $r \geq 1$,

$$B_r = qC_{r-1} + 2qC_r - qB_r$$

and for $r \geq 0$,

$$C_r = B_r + 2B_{r+1} - qC_r.$$

That is, for $r \geq 0$,

$$B_{r+1} = \frac{1}{2}\left((1+q)C_r - B_r\right)$$

and

$$C_{r+1} = \frac{1}{2q}\left((1+q)B_{r+1} - qC_r\right).$$

It follows that

$$
\begin{aligned}
B_1 &= \frac{1}{2}\left((1+q)C_0 - \frac{1+3qC_0}{1+q}\right) = \frac{(1-q+q^2)C_0 - 1}{2(1+q)},\\
C_1 &= \frac{1}{2q}\left((1+q)\frac{(1-q+q^2)C_0-1}{2(1+q)} - qC_0\right) = \frac{(1-3q+q^2)C_0-1}{4q},\\
B_2 &= \frac{(1-3q-2q^2-3q^3+q^4)C_o - (1+q^2)}{8q(1+q)},\\
C_2 &= \frac{(1-5q+4q^2-5q^3+q^4)C_0 - (1-2q+q^2)}{16q^2},\\
B_3 &= \frac{(1-5q+q^2+2q^3+q^4-5q^5+q^6)C_0 - (1-2q-2q^2-2q^3+q^4)}{32q^2(1+q)},\\
C_3 &= \frac{(1-7q+11q^2-6q^3+11q^4-7q^5+q^6)C_0 - (1-4q+2q^2-4q^3+q^4)}{64q^3},
\end{aligned}
$$

and so on.

It is clear that we can write

$$
\begin{aligned}
B_r &= \frac{P_rC_0 - Q_r}{2^{2r-1}q^{r-1}(1+q)},\\
C_r &= \frac{S_rC_0 - T_r}{2^{2r}q^r},
\end{aligned}
$$

where $P_r,\ Q_r,\ S_r,\ T_r$ are polynomials in q. The first few are

$$
\begin{aligned}
P_1 &= 1-q+q^2,\\
P_2 &= 1-3q-2q^2-3q^3+q^4,\\
P_3 &= 1-5q+q^2+2q^3+q^4-5q^5+q^6,\\
Q_1 &= 1,\\
Q_2 &= 1+q^2,\\
Q_3 &= 1-2q-2q^2-2q^3+q^4,\\
S_1 &= 1-3q+q^2,\\
S_2 &= 1-5q+4q^2-5q^3+q^4,\\
S_3 &= 1-7q+11q^2-6q^3+11q^4-7q^5+q^6,\\
T_1 &= 1,\\
T_2 &= 1-2q+q^2,\\
T_3 &= 1-4q+2q^2-4q^3+q^4.
\end{aligned}
$$

It follows from the recurrences for the B_r and C_r that $P_r,\ Q_r,\ S_r$ and T_r all satisfy the recurrence

$$X_{r+2} - (1-q)^2X_{r+1} + 4q^2X_r = 0.$$

Indeed, in terms of α and β, the roots of $z^2-(1-q)^2z+4q^2=0$,

$$\alpha = \frac{(1-q)^2+(1+q)\sqrt{1-6q+q^2}}{2} = 1-2q-3q^2-8q^3-28q^4-112q^5- \cdots ,$$
$$\beta = \frac{(1-q)^2-(1+q)\sqrt{1-6q+q^2}}{2} = 4q^2+8q^3+28q^4+112q^5+ \cdots ,$$

we have

$$\begin{aligned}
P_r &= \left(\frac{3}{4}+\frac{1+q}{4\sqrt{1-6q+q^2}}\right)\alpha^r + \left(\frac{3}{4}-\frac{1+q}{4\sqrt{1-6q+q^2}}\right)\beta^r,\\
Q_r &= \left(\frac{1+q}{4q\sqrt{1-6q+q^2}}-\frac{1}{4q}\right)\alpha^r - \left(\frac{1+q}{4q\sqrt{1-6q+q^2}}+\frac{1}{4q}\right)\beta^r,\\
S_r &= \left(\frac{1}{2}+\frac{1-4q+q^2}{2(1+q)\sqrt{1-6q+q^2}}\right)\alpha^r + \left(\frac{1}{2}-\frac{1-4q+q^2}{2(1+q)\sqrt{1-6q+q^2}}\right)\beta^r,\\
T_r &= \frac{1}{(1+q)\sqrt{1-6q+q^2}}\,\alpha^r - \frac{1}{(1+q)\sqrt{1-6q+q^2}}\,\beta^r.
\end{aligned}$$

It follows that

$$\begin{aligned}
&2^{2r-1}q^{r-1}(1+q)B_r = P_rC_0-Q_r\\
&= \left(\left(\frac{3}{4}+\frac{1+q}{4\sqrt{1-6q+q^2}}\right)C_0-\left(\frac{1+q}{4q\sqrt{1-6q+q^2}}-\frac{1}{4q}\right)\right)\alpha^r\\
&+\left(\left(\frac{3}{4}-\frac{1+q}{4\sqrt{1-6q+q^2}}\right)C_0+\left(\frac{1+q}{4q\sqrt{1-6q+q^2}}+\frac{1}{4q}\right)\right)\beta^r
\end{aligned}$$

and

$$\begin{aligned}
2^{2r}q^rC_r = &\left(\left(\frac{1}{2}+\frac{1-4q+q^2}{2(1+q)\sqrt{1-6q+q^2}}\right)C_0-\frac{1}{(1+q)\sqrt{1-6q+q^2}}\right)\alpha^r\\
+&\left(\left(\frac{1}{2}-\frac{1-4q+q^2}{2(1+q)\sqrt{1-6q+q^2}}\right)C_0+\frac{1}{(1+q)\sqrt{1-6q+q^2}}\right)\beta^r.
\end{aligned}$$

Since the left–hand–sides of the above two equations are congruent to 0 modulo q^{2r-1}, we deduce that

$$\left(\frac{3}{4}+\frac{1+q}{4\sqrt{1-6q+q^2}}\right)C_0-\left(\frac{1+q}{4q\sqrt{1-6q+q^2}}-\frac{1}{4q}\right)=0,$$

alternatively that

$$\left(\frac{1}{2}+\frac{1-4q+q^2}{2(1+q)\sqrt{1-6q+q^2}}\right)C_0-\frac{1}{(1+q)\sqrt{1-6q+q^2}}=0.$$

It follows that

$$C_0=\frac{(1+q)\sqrt{1-6q+q^2}-(1-4q+q^2)}{2q(1-7q+q^2)}$$

(this confirms the conjecture of Fahr and Ringel [1] that $\{c_t[0]\},\ t=0,1,2,\ \dots$ is A110122 in the On-Line Encyclopedia of Integer Sequences [2]),

that

$$B_0=\frac{3\sqrt{1-6q+q^2}-(1+q)}{2(1-7q+q^2)},$$

(this is the generating function for A132262 in the On-Line Encyclopedia of Integer Sequences [2]),

and with some work we find that for $r\geq 1$,

$$B_r=B_0\left(\frac{\beta}{4q}\right)^r=B_0\left(\frac{(1-q)^2-(1+q)\sqrt{1-6q+q^2}}{8q}\right)^r,$$
$$C_r=C_0\left(\frac{\beta}{4q}\right)^r=C_0\left(\frac{(1-q)^2-(1+q)\sqrt{1-6q+q^2}}{8q}\right)^r.$$

Also, we can confirm the following result of Fahr and Ringel.

Theorem 1.

$$b_t[0]+3\sum_{r=1}^{t}2^{2r-1}b_t[r]=F_{4t+2},\quad 3\sum_{r=0}^{t}2^{2r}c_r[r]=F_{4t+4},$$

where the F_t are the Fibonacci numbers, given by $F_0=0,\ F_1=1$ and $F_t=F_{t-1}+F_{t-2}$ for $t\geq 2$.

Proof. We have

$$\begin{aligned}B_0+3\sum_{r\geq 1}2^{2r-1}B_r &= B_0\left(1+6\left(\frac{\beta}{4q}\right)+24\left(\frac{\beta}{4q}\right)^2+\ \cdots\right)\\ &= B_0\left(1+\frac{6\beta}{4q(1-\frac{\beta}{q})}\right)\\ &= \frac{1+q}{1-7q+q^2}\\ &= \sum_{t\geq 0}F_{4t+2}\,q^t\end{aligned}$$

and

$$\begin{aligned}
3\sum_{r\geq 0} 2^{2r} C_r &= 3C_0 \left(1 + 4\left(\frac{\beta}{4q}\right) + 16\left(\frac{\beta}{4q}\right)^2 + \cdots\right) \\
&= 3C_0 \left(1 + \frac{\beta}{q(1-\frac{\beta}{q})}\right) \\
&= \frac{3}{1-7q+q^2} \\
&= \sum_{t\geq 0} F_{4t+4}\, q^t.
\end{aligned}$$

□

References

[1] P. Fahr and C. M. Ringel, A partition formula for Fibonacci numbers, *J. Integer Sequences*, **11** (2008), Paper 08.1.4.

[2] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available electronically at `http://www.research.att.com/~njas/sequences/`.

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(Concerned with sequences A110122 and A132262.)

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