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A Note on the Average Order of the gcd-sum Function

Olivier Bordellès
2, Allée de la Combe
La Boriette
43000 Aiguilhe
France
borde43@wanadoo.fr

Abstract

We prove an asymptotic formula for the average order of the gcd-sum function by using a new convolution identity.

1 Introduction and main result

In 2001, Broughan [1] studied the gcd-sum function g defined for any positive integer n by

$$g(n) = \sum_{k=1}^{n} (k, n),$$

where (a, b) denotes the greatest common divisor of a and b. The author showed that g is multiplicative, and satisfies the convolution identity

$$g = \varphi * \ \mathrm{Id}, \tag{1}$$

where φ is the Euler totient function, Id is the completely multiplicative function defined by $\mathrm{Id}(n) = n$ and $*$ is the usual Dirichlet convolution product.

The function g appears in a specific lattice point problem [1, 6], where it can be used to estimate the number of integer coordinate points under the square-root curve. As a multiplicative function, the question of its average order naturally arises. By using the

Dirichlet hyperbola principle, Broughan [1, Theorem 4.7] proved the following result: for any real number $x \geqslant 1$, the following estimate

$$\sum_{n \leqslant x} g(n) = \frac{x^2 \log x}{2\zeta(2)} + \frac{\zeta(2)^2}{2\zeta(3)} x^2 + O\left(x^{3/2} \log x\right) \tag{2}$$

holds.

The aim of this paper is to prove another convolution identity for g, and then get a fairly more precise estimate than (2).

In what follows, τ is the well-known divisor function, μ is the Möbius function, $\mathbf{1}$ is the completely multiplicative function defined by $\mathbf{1}(n) = 1$, $F * G$ is the Dirichlet convolution product of the arithmetical functions F and G, and we denote by θ the smallest positive real number such that

$$\sum_{n \leqslant x} \tau(n) = x \log x + x(2\gamma - 1) + O_\varepsilon\left(x^{\theta+\varepsilon}\right) \tag{3}$$

holds for any real numbers $x \geqslant 1$ and $\varepsilon > 0$. The following inequality

$$\theta \geqslant \frac{1}{4}$$

is well-known [3]. On the other hand, Huxley [4] showed that

$$\theta \leqslant \frac{131}{416} \approx 0.3149\ldots$$

holds. Now we are able to prove the following result

Theorem 1.1. *For any real numbers $x \geqslant 1$ and $\varepsilon > 0$, we have*

$$\sum_{n \leqslant x} g(n) = \frac{x^2 \log x}{2\zeta(2)} + \frac{x^2}{2\zeta(2)} \left(\gamma - \frac{1}{2} + \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right) + O_\varepsilon\left(x^{1+\theta+\varepsilon}\right)$$

where $\mathcal{A} \approx 1.282\,427\,129\ldots$ is the Glaisher-Kinkelin constant.

For further details about the Glaisher-Kinkelin constant, see [2, 5]. The reader interested in gcd-sum integer sequences should refer to Sloane's sequence A018804.

2 A convolution identity

The proof uses the following lemmas.

Lemma 2.1. *For any real number $z \geqslant 1$ and any $\varepsilon > 0$, we have*

$$\sum_{n \leqslant z} n\tau(n) = \frac{z^2}{2} \log z + z^2 \left(\gamma - \frac{1}{4}\right) + O_\varepsilon\left(z^{1+\theta+\varepsilon}\right).$$

Proof. The result follows easily from (3) and Abel's summation. □

Lemma 2.2. *We have*

$$g = \mu * (Id \cdot \tau).$$

Proof. Since $\varphi = \mu * \mathrm{Id}$, we have, using (1),

$$g = \varphi * \mathrm{Id} = \mu * (\mathrm{Id} * \mathrm{Id}) = \mu * (\mathrm{Id} \cdot \tau)$$

which is the desired result. □

3 Proof of Theorem 1.1

By using Lemma 2.2, we get

$$\sum_{n \leqslant x} g(n) = \sum_{d \leqslant x} \mu(d) \sum_{k \leqslant x/d} k\tau(k)$$

and Lemma 2.1 applied to the inner sum gives

$$\begin{aligned}
\sum_{n \leqslant x} g(n) &= \sum_{d \leqslant x} \mu(d) \left\{ \frac{x^2}{d^2} \left(\frac{1}{2} \log\left(\frac{x}{d}\right) + \gamma - \frac{1}{4} \right) + O_\varepsilon\left(\left(\frac{x}{d}\right)^{1+\theta+\varepsilon} \right) \right\} \\
&= x^2 \left\{ \left(\frac{1}{2}\log x + \gamma - \frac{1}{4} \right) \sum_{d \leqslant x} \frac{\mu(d)}{d^2} - \sum_{d \leqslant x} \frac{\mu(d)\log d}{2d^2} \right\} + O_\varepsilon\left(x^{1+\theta+\varepsilon} \sum_{d \leqslant x} \frac{1}{d^{1+\theta+\varepsilon}} \right) \\
&= x^2 \left\{ \left(\frac{1}{2}\log x + \gamma - \frac{1}{4} \right) \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=1}^{\infty} \frac{\mu(d)\log d}{2d^2} + O\left(\frac{\log x}{x}\right) \right\} + O_\varepsilon\left(x^{1+\theta+\varepsilon}\right).
\end{aligned}$$

Now it is well-known that, for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s}$$

which gives by differentiation

$$\frac{\zeta'(s)}{(\zeta(s))^2} = \sum_{d=1}^{\infty} \frac{\mu(d)\log d}{d^s}$$

for $\operatorname{Re} s > 1$, and hence

$$\sum_{n \leqslant x} g(n) = \frac{x^2}{2\zeta(2)} \left(\log x - \frac{\zeta'(2)}{\zeta(2)} + 2\gamma - \frac{1}{2} \right) + O_\varepsilon\left(x^{1+\theta+\varepsilon}\right),$$

and we use

$$\frac{\zeta'(2)}{\zeta(2)} = \gamma - \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right).$$

The proof of the theorem is now complete.

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(Concerned with sequence A018804.)

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