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SPECTRAL DOMINANCE AND YOUNG'S INEQUALITY IN TYPE III FACTORS

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ABSTRACT. Let $p,\ q > 0$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. We prove that for any positive invertible operators a and b in σ-finite type III factors acting on Hilbert spaces, there is a unitary u, depending on a and b such that

$$u^*|ab|u \le \frac{1}{p}a^p + \frac{1}{q}b^q\,.$$

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1. INTRODUCTION

Young's inequality asserts that if p and q are positive real numbers for which $p^{-1}+q^{-1}=1$, then $|\lambda\mu| \le p^{-1}|\lambda|^p + q^{-1}|\mu|^q$, for all complex numbers λ and μ, and the equality holds if and only if $|\mu|^q = |\lambda|^p$.

R. Bhatia and F. Kittaneh [3] established a matrix version of the Young inequality for the special case $p=q=2$. T. Ando [2] proved that for any pair A and B of $n \times n$ complex matrices there is a unitary matrix U, depending on A and B such that

$$U^*|AB|U \le \frac{1}{p}|A|^P + \frac{1}{q}|B|^q. \tag{1.1}$$

Ando's methods were adapted recently to the case of compact operators acting on infinite-dimensional separable Hilbert spaces by Erlijman, Farenick, and Zeng [4]. In this paper by using the concept of `spectral dominance` in type III factors, we prove a version of Young's inequality for positive operators in a type III factor N.

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If $\mathfrak{H}$ is an n-dimensional Hilbert space and if a and b are positive operators acting on $\mathfrak{H}$, then a is said to be spectrally dominated by b if

$$\alpha_j \leq \beta_j, \quad \text{for every } 1 \leq j \leq n, \tag{1.2}$$

where $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$ are the eigenvalues of a and b, respectively, in nonincreasing order and with repeats according to geometric multiplicities. It is a simple consequence of the Spectral Theorem and the Min-Max Variational Principle that inequalities (1.2) are equivalent to a single operator inequality:

$$a \leq u^*bu, \quad \text{for some unitary operator } u : \mathfrak{H} \to \mathfrak{H}, \tag{1.3}$$

where $h \leq k$, for Hermitian operators h and k, denotes $\langle h\xi, \xi\rangle \leq \langle k\xi, \xi\rangle$ for all $\xi \in \mathfrak{H}$. One would like to investigate inequalities (1.2) and (1.3) for operators acting on infinite-dimensional Hilbert spaces. Of course, as many operators on infinite-dimensional space fail to have eigenvalues, inequality (1.2) requires a somewhat more general formulation. This can be achieved through the use of spectral projections.

Let $\mathcal{B}(\mathfrak{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathfrak{H}$, and suppose that $N \subseteq \mathcal{B}(\mathfrak{H})$ is a von Neumann algebra. The cone of positive operators in N and the projection lattice in N are denoted by N^+ and $\mathcal{P}(N)$ respectively. The notation $e \sim f$, for $e, f \in \mathcal{P}(N)$, shall indicate the Murray–von Neumann equivalence of e and f : $e = v^*v$ and $f = vv^*$ for some $v \in N$. The notation $f \precsim e$ denotes that there is a projection $e_1 \in N$ with $e_1 \leq e$ and $f \sim e_1$; that is, f is subequivalent to e.

Recall that a nonzero projection $e \in N$ is infinite if there exists a nonzero projection $f \in N$ such that $e \sim f \leq e$ and $f \neq e$. In a factor of type III, all nonzero projections are infinite; in a σ-finite factor, all infinite projections are equivalent. Thus, in a σ-finite type III factor N, any two nonzero projections in N are equivalent. (Examples, constructions, and properties of factors [von Neumann algebras with 1-dimensional center] are described in detail in [5], as are the assertions above concerning the equivalence of nonzero projections in σ-finite type III factors.)

The spectral resolution of the identity of a Hermitian operator $h \in N$ is denoted here by p^h. Thus, the spectral representation of h is

$$h = \int_{\mathbb{R}} s \, dp^h(s).$$

In [1], Akemann, Anderson, and Pedersen studied operator inequalities in various von Neumann algebras. In so doing they introduced the following notion of spectral preorder called "spectral dominance." If $h, k \in N$ are Hermitian, then we say that k `spectrally dominates` h, which is denoted by the notation

$$h \precsim_{sp} k,$$

if, for every $t \in \mathbb{R}$,

$$p^h[t, \infty) \precsim p^k[t, \infty) \quad \text{and} \quad p^k(-\infty, t] \precsim p^h(-\infty, t].$$

h and k are said to be equivalent in the spectral dominance sense if, $h \precsim_{sp} k$ and $k \precsim_{sp} h$.

If N is a type I_n factor—say, $N = \mathcal{B}(\mathfrak{H})$, where $\mathfrak{H}$ is n-dimensional—then, for any positive operators $a, b \in N$,

$$a \precsim_{sp} b \quad \text{if and only if} \quad \alpha_j \leq \beta_j, \text{ for every } 1 \leq j \leq n, \tag{1.4}$$

where $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$ are the eigenvalues (with multiplicities) of a and b in nonincreasing order. The first main result of the present paper is Theorem 1.1 below, which shows that in type III factors the condition $a \precsim_{sp} b$ is equivalent to an operator inequality in the form of (1.3), thereby giving a direct analogue of (1.4).

Theorem 1.1. *If N is a σ-finite type III factor and if $a, b \in N^+$, then $a \precsim_{sp} b$ if and only if there is a unitary $u \in N$ such that $a \le u\, b\, u^*$.*

The second main result established herein is the following version of Young's inequality, which extends Ando's result (Equation (1.1)) to positive operators in type III factors.

Theorem 1.2. *If a and b are positive operators in type III factor N such that b is invertible, then there is a unitary u, depending on a and b such that*

$$u|ab|u^* \le \frac{1}{p}a^p + \frac{1}{q}b^q,$$

for any $p, q \in (1, \infty)$ that satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

2. SPECTRAL DOMINANCE

The pupose of this section is to record some basic properties of spectral dominance in arbitrary von Neumann algebras and to then prove Theorem 1.1 for σ-finite type III factors. Some of the results in this section have been already proved or outlined in [1]. However, the presentation here simplifies or provides additional details to several of the original arguments.

Unless it is stated otherwise, N is assumed to be an arbitrary von Neumann algebra acting on a Hilbert space $\mathfrak{H}$.

Lemma 2.1. *If $0 \ne h \in N$ is Hermitian, $\eta \in \mathfrak{H}$ is a unit vector, and $t \in \mathbb{R}$, then:*

(1) $p^h\,[t\,,\,\infty)\,\eta = 0$ *implies that* $\langle h\,\eta\,,\,\eta\rangle < t$;
(2) $p^h\,(-\infty\,,\,t]\,\eta = 0$ *implies that* $\langle h\,\eta\,,\,\eta\rangle > t$;
(3) $p^h\,[t\,,\,\infty)\,\eta = \eta$ *implies that* $\langle h\,\eta\,,\,\eta\rangle \ge t$;
(4) $p^h\,(-\infty\,,\,t]\,\eta = \eta$ *implies that* $\langle h\,\eta\,,\,\eta\rangle \le t$.

Proof. This is a standard application of the spectral theorem. □

Lemma 2.2. *If $h, k \in N$ are hermitian and $h \le k$, then $h \precsim_{sp} k$.*

Proof. Fix $t \in \mathbb{R}$. We first prove that $p^k(-\infty, t] \precsim p^h(-\infty, t]$. Note that the condition $h \le k$ implies that $p^k(-\infty, t] \wedge p^h(t, \infty) = 0$, for if ξ is a unit vector in $p^k(-\infty, t](\mathfrak{H}) \cap p^h(t, \infty)(\mathfrak{H})$, then we would have that $\langle k\xi, \xi\rangle \le t < \langle h\xi, \xi\rangle$, which contradicts $h \le k$. Kaplansky's formula [5, Theorem 6.1.7] and $p^k(-\infty, t] \wedge p^h(t, \infty) = 0$ combine to yield

$$\begin{aligned} p^k(-\infty, t] &= p^k(-\infty, t] - \left(p^k(-\infty, t] \wedge p^h(t, \infty)\right) \\ &\sim \left(p^k(-\infty, t] \vee p^h(t, \infty)\right) - p^h(t, \infty) \\ &\le 1 - p^h(t, \infty) \\ &= p^h(-\infty, t]\,. \end{aligned}$$

Using $p^h[t, \infty) \wedge p^k(-\infty, t) = 0$, one concludes that $p^h[t, \infty) \precsim p^k[t, \infty)$ by a proof similar to the one above. □

Theorem 2.3. *Assume that $a, b, u \in N$, with a and b positive and u unitary. If $a \le ubu^*$, then $a \precsim_{sp} b$.*

Proof. By Lemma 2.2, $a \le ubu^*$ implies that $a \precsim ubu^*$. However, because $u \in N$ is unitary, we have $p^b(\Omega) \sim p^{ubu^*}(\Omega)$, for every Borel set Ω. Hence, $a \precsim_{sp} b$. □

The converse of Theorem 2.3 will be shown to hold in Theorem 2.7 under the assumption that N is a σ-finite factor of type III. To arrive at the proof, we follow [1] and define, for Hermitians h and k, the following real numbers:

$$\begin{array}{rclcrcl} \alpha^+ & = & \max\{\lambda : \lambda \in \sigma(h)\}, & & \alpha^- & = & \min\{\lambda : \lambda \in \sigma(h)\}, \\ \beta^+ & = & \max\{\nu : \nu \in \sigma(k)\}, & & \beta^- & = & \min\{\nu : \nu \in \sigma(k)\}. \end{array}$$

Lemma 2.4. *If $h, k \in N$ are Hermitian and $h \precsim_{sp} k$, then*

(1) $\alpha^+ \le \beta^+$ *and* $p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\})$, *and*
(2) $\beta^- \le \alpha^-$ *and* $p^k(\{\alpha^-\}) \precsim p^h(\{\alpha^-\})$.

Proof. To prove statement (1), we prove first that $\alpha^+ \le \beta^+$. Assume, contrary to what we wish to prove, that $\beta^+ < \alpha^+$. Because $h \precsim_{sp} k$,

$$p^h[t, \infty) \precsim p^k[t, \infty), \quad \forall t \in \mathbb{R}.$$

In particular, $p^h[\alpha^+, \infty) \precsim p^k[\alpha^+, \infty)$. The assumption $\beta^+ < \alpha^+$ implies that $p^k[\alpha^+, \infty) = 0$, and so, also,

$$p^h[\alpha^+, \infty) = 0.$$

By a similar argument, $p^h[r, \infty) = 0$, for each $r \in (\beta^+, \alpha^+)$. Hence, α^+ is an isolated point of the spectrum of h and, therefore, α^+ is an eigenvalue of h. Thus,

$$p^h[\alpha^+, \infty) \neq 0,$$

which is a contradiction. Therefore, it must be true that $\alpha^+ \le \beta^+$.

To prove that $p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\})$, we consider two cases. In the first case, suppose that $\alpha^+ < \beta^+$. Then

$$p^h(\{\beta^+\}) = 0,$$

which leads, trivially, to $p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\})$. In the second case, assume that $\alpha^+ = \beta^+$. Then

$$p^h(\{\beta^+\}) = p^h[\alpha^+, \infty) \precsim p^k[\alpha^+, \infty) = p^k(\{\beta^+\}),$$

which completes the proof of statement (1).

The proof of statement (2) follows the arguments in the proof of (1), except that we use $p^k(-\infty, t] \precsim p^h(-\infty, t]$ in place of $p^h[t, \infty) \precsim p^k[t, \infty)$. The details are, therefore, omitted. □

If N is a σ-finite type III factor, then Lemma 2.4 has the following converse.

Lemma 2.5. *Let N be a σ-finite factor of type III. If Hermitian operators $h, k \in N$ satisfy*

(1) $\alpha^+ \le \beta^+$ *and* $p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\})$, *and*
(2) $\beta^- \le \alpha^-$ *and* $p^k(\{\alpha^-\}) \precsim p^h(\{\alpha^-\})$,

then $h \precsim_{sp} k$.

Proof. We need to show that, for each $t \in \mathbb{R}$,

$$p^h[t, \infty) \precsim p^k[t, \infty) \quad \text{and} \quad p^k(-\infty, t] \precsim p^h(-\infty, t].$$

Fix $t \in \mathbb{R}$. Because N is a σ-finite type III factor, the projections $p^h[t, \infty)$ and $p^k[t, \infty)$ will be equivalent if they are both zero or if they are both nonzero. Thus, we shall show that if $p^k[t_0, \infty) = 0$, then $p^h[t_0, \infty) = 0$. To this end, if $p^k[t, \infty) = 0$, then $t \ge \beta^+ \ge \alpha^+$. If, on the one hand, it is the case that $t > \alpha^+$, then $p^h\,[t\,,\infty) \;=\; 0$ and we have the result. If, on the other hand, $t \;=\; \alpha^+$, then $t \;=\; \alpha^+ \;=\; \beta^+$ and

$$\begin{aligned} p^h\,[t\,,\infty) &= p^h\,[\alpha^+\,,\infty) \;=\; p^h\,(\{\alpha^+\}) \\ &= \; p^h\,(\{\beta^+\}) \;\precsim\; p^k\,(\{\beta^+\}) \\ &= p^k\,[\beta^+\,,\infty) \;=\; p^k\,[t\,,\infty)\,. \end{aligned}$$

A similar argument proves that $p^k\,(-\infty\,,t] \;\precsim\; p^h\,(-\infty\,,t]$. □

A Hermitian operator h in a von Neumann algebra N is said to be a *diagonal operator* if

$$h = \sum_n \alpha_n\, e_n \quad \text{and} \quad 1 = \sum_n e_n,$$

where $\{\alpha_n\}$ is a sequence of real numbers (not necessarily distinct) and $\{e_n\} \subset \mathcal{P}(N)$ is a sequence of mutually orthogonal nonzero projections in N.

The following interesting and useful theorem is due to Akemann, Anderson, and Pedersen.

Theorem 2.6 ([1])**.** *Let N be a σ-finite type III factor, and suppose that Hermitian operators $h, k \in N$ are diagonal operators. If $h \precsim_{sp} k$, then there is a unitary $u \in N$ such that $h \leq uku^*$.*

The proof of the characterisation of spectral dominance by an operator inequality (Theorem 1.1) is completed by the following result. The method of proof again borrows ideas from [1].

Theorem 2.7. *If N is a σ-finite type III factor, and $a, b \in N^+$ satisfy $a \;\precsim_{sp}\; b$, then there is a unitary $u \in N$ such that $a \;\leq\; u\,b\,u^*$.*

Proof. It is enough to prove that there are diagonal operators $h, k \in N$ such that $a \leq h$, $k \leq b$, and $h \precsim_{sp} k$—because, by Theorem 2.6, there is a unitary $u \in N$ such that $h \leq uku^*$, which yields $a \leq ubu^*$.

Because N is σ-finite, the point spectra $\sigma_p(a)$ and $\sigma_p(b)$ of a and b are countable. Let $\sigma_p(b) \;=\; \{\beta_n \;:\; n \in \Lambda\}$, where Λ is a countable set. Let f_n be a projection with kernel $(b - \beta_n 1)$ and

$$q \;=\; \sum_{n\in\Lambda} f_n\,.$$

Then

$$qb \;=\; bq \;=\; \sum_{n\in\Lambda} \beta_n f_n\,.$$

Let $b_1 \;=\; (1-q)b \;(= b(1-q))$. Thus, we may write

$$b \;=\; \sum_n \beta_n\, f_n + b_1\,.$$

By a similar argument for a, we may write

$$a \;=\; \sum_n \alpha_n\, e_n + a_1\,,$$

where a_1 and b_1 have continuous spectrum.

For any Borel set Ω, we define

$$p^{b_1}(\Omega) \;=\; (1-q)p^b(\Omega)(1-q)\,.$$

Thus p^{b_1} is a spectral measure on the Borel sets of $\sigma(b_1)$. For each $n \in \Lambda$ and Borel set Ω we have

$$f_n p^{b_1}(\Omega) \;=\; p^{b_1}(\Omega) f_n \;=\; 0. \tag{2.1}$$

Let β^+ and β^- denote the spectral endpoints of b and choose infinite sequences $\{\beta_n^+\}$ and $\{\beta_n^-\}$ such that $\beta_n^+, \beta_n^- \in (\beta^-, \beta^+)$ and

$$\beta_0^+ = \frac{1}{2}(\beta^+ + \beta^-) < \beta_1^+ < \beta_2^+ < \cdots < \beta_n^+ \to \beta^+,$$

$$\beta_0^- = \frac{1}{2}(\beta^+ + \beta^-) > \beta_1^- > \beta_2^- > \cdots > \beta_n^- \to \beta^-.$$

Let f_n^+ denote the spectral projection of b_1 associated with the interval $[\beta_n^+, \beta_{n+1}^+)$, $n = 0, 1, 2, \dots$, and f_n^- denote the spectral projection associated with $[\beta_{n+1}^-, \beta_n^-)$. Write

$$k = \sum_n \beta_n f_n + \sum_n \beta_n^+ f_n^+ + \sum_n \beta_{n+1}^- f_n^- ,$$

and observe that k is a diagonal operator. Moreover, by the choice of β_n^+ and β_n^-,

$$\sum_n \beta_n^+ f_n^+ + \sum_n \beta_{n+1}^- f_n^- \le b_1 .$$

The construction of k yields

$$\begin{aligned} \sigma_p(b) \subseteq \sigma_p(k) &= \{\beta_n : n \in \Lambda\} \cup \{\beta_m^+ : m \in \Lambda_1\} \cup \{\beta_{m+1}^+ : m \in \Lambda_2\} \\ &\subseteq \operatorname{conv} \sigma(b), \end{aligned}$$

where Λ, Λ_1 and Λ_2 are countable sets and $\operatorname{conv}\sigma(b)$ denotes the convex hull of the spectrum of b. Thus, $0 \le k \le b$ and k has the same spectral endpoints as b. Furthermore, k has an eigenvalue at a spectral endpoint if and only if b has an eigenvalue at that same point.

Arguing similarly for a, let α^+ and α^- denote the spectral endpoints of a, and select sequences $\{\alpha_n^+\}$ and $\{\alpha_n^-\}$ such that $\alpha_n^+, \alpha_n^- \in (\alpha^-, \alpha^+)$ and

$$\alpha_0^+ = \frac{1}{2}(\alpha^+ + \alpha^-) < \alpha_1^+ < \alpha_2^+ < \cdots < \alpha_n^+ \to \alpha^+$$

$$\alpha_0^- = \frac{1}{2}(\alpha^+ + \alpha^-) > \alpha_1^- > \alpha_2^- > \cdots > \alpha_n^- \to \alpha^-.$$

Denote the spectral projection of a_1 associated with $[\alpha_n^+, \alpha_{n+1}^+)$ by e_n^+ and, similarly, e_n^- for $p^{a_1}[\alpha_{n+1}^-, \alpha_n^-)$. Let

$$h = \sum_n \alpha_n e_n + \sum_n \alpha_{n+1}^+ e_n^+ + \sum_n \alpha_n^- e_n^- .$$

Note that

$$a_1 \le \sum_n \alpha_{n+1}^+ e_n^+ + \sum_n \alpha_n^- e_n^- .$$

Thus, $a \le h$ and h has the same spectral endpoints as a; moreover, h has an eigenvalue at an endpoint if and only if a has an eigenvalue at that point.

By the hypothesis, $a \precsim_{sp} b$; thus, by Lemma 2.4,

$$\beta^+ \ge \alpha^+ \text{ and } \beta^- \le \alpha^- , \tag{2.2}$$

and

$$p^a(\{\beta^+\}) \precsim p^b(\{\beta^+\}) \text{ and } p^b(\{\alpha^-\}) \precsim p^a(\{\alpha^-\}). \tag{2.3}$$

Now, we use Lemma 2.5 to prove that $h \precsim_{sp} k$. Because the spectral endpoints of h are α^- and α^+, and the spectral endpoints of k are β^- and β^+, we need only to show that

$$p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\}) \quad \text{and} \quad p^k(\{\alpha^-\}) \precsim p^h(\{\alpha^-\}).$$

(We already know from (2.2) that $\alpha^+ \le \beta^+$ and $\alpha^- \ge \beta^-$.)

As we have pointed out in previous proofs, because N is a σ-finite type III factor, to prove that $p^h(\{\beta^+\}) \precsim p^k(\{\beta^+\})$ it is enough to show that if $p^k(\{\beta^+\}) = 0$, then $p^h(\{\beta^+\}) = 0$. Thus, assume that $p^k(\{\beta^+\}) = 0$; then, β^+ is not an eigenvalue of k and, therefore, it is not eigenvalue of b. Thus, $p^b(\{\beta^+\}) = 0$. But $p^a(\{\beta^+\}) \precsim p^b(\{\beta^+\})$, by (2.3), and so $p^a(\{\beta^+\}) = 0$. Hence, $p^h(\{\beta^+\}) = 0$.

By a similar argument, we can prove $p^k(\{\alpha^-\}) \precsim p^h(\{\alpha^-\})$. □

Corollary 2.8 (Theorem 1.1)**.** *Let N be a σ-finite type III factor and $a, b \in N^+$. Then $a \precsim_{sp} b$ if and only if there is a unitary $u \in N$ such that $a \le u\,b\,u^*$.*

Proof. The sufficiency is Theorem 2.3 and the necessity is Theorem 2.7. □

3. YOUNG'S INEQUALITY

In this section we use properties of spectral dominance to prove the second main result. We begin with two lemmas that are needed in the proof of Theorem 3.3. A compressed form of Young's inequality was established in [4], based on an idea originating with Ando [2], and was used to prove Young's inequality—relative to the Löwner partial order of $\mathcal{B}(\mathfrak{H})$—for compact operators. Although the focus of [4] was upon compact operators, the following important lemma from [4] in fact holds in arbitrary von Neumann algebras.

Lemma 3.1. *Assume that $p \in (1, 2]$. If N is any von Neumann algebra and $a, b \in N^+$, with b invertible, then for any $s \in \mathbb{R}_0^+$,*

$$sf_s \le f_s \left(p^{-1}a^p + q^{-1}b^q\right) f_s \quad \text{and} \quad f_s \sim p^{|ab|}(\,[s,\infty)\,)\ ,$$

where $f_s = R[b^{-1}p^{|ab|}(\,[s,\infty)\,)]$.

Lemma 3.2. *If a and b are positive operators in a von-Neumann algebra N, then $|ab|$ and $|ba|$ are equivalent in the spectral dominance sense.*

Proof. It is well known that the spectral measures for $|x|$ and $|x^*|$ are equivalent in the Murry-von Neumann sense, the equivalence being given by the phase part of the polar decomposition of x. (If $x = w|x|$ is the polar decomposition of x, then $xx^* = w|x|^2w^*$, so $|x^*|^2 = (w|x|w^*)^2$, and therefore $|x^*| = (w|x|w^*)$.)

In particular, for $a, b \ge 0$ the two absolute value parts $|ab|$, $|ba|$ are equivalent in the spectral dominance sense. □

Theorem 3.3. *If a and b are positive invertible operators in type III factor N, then there is a unitary u, depending on a and b such that*

$$u|ab|u^* \le \frac{1}{p}a^p + \frac{1}{q}b^q,$$

for any $p, q \in (1, \infty)$ that satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Theorem 2.7, it is enough to prove that

$$|ab| \precsim_{sp} p^{-1}a^p + q^{-1}b^q. \tag{3.1}$$

We assume, that $p \in (1, 2]$ and that $b \in N^+$ is invertible. The assumption on p entails no loss of generality because if inequality (3.1) holds for $1 < p \le 2$, then in cases, where $p > 2$ the conjugate q satisfies $q < 2$, and so by Lemma 3.2

$$|ab| \precsim_{sp} |ba| \precsim_{sp} p^{-1}a^p + q^{-1}b^q\ . \tag{3.2}$$

To prove the inequality (3.1) we need to prove that for each real number t,

$$p^{|ab|}[t\,,\,\infty) \precsim p^{p^{-1}a^p + q^{-1}b^q}[t\,,\,\infty)$$

and

$$p^{p^{-1}a^p + q^{-1}b^q}(-\infty\,,\,t] \precsim p^{|ab|}(-\infty\,,\,t]\,.$$

Since M is a type III factor, it is sufficient to prove that if $p^{p^{-1}a^p + q^{-1}b^q}[t\,,\,\infty) = 0\,(p^{|ab|}(-\infty\,,\,t] = 0)$, then $p^{|ab|}[t\,,\,\infty) = 0\,(p^{p^{-1}a^p + q^{-1}b^q}(-\infty\,,\,t] = 0)\,.$

Suppose there is a $t_0 \in \mathbb{R}$ such that $p^{p^{-1}a^p + q^{-1}b^q}[t_0\,,\,\infty) = 0$ and $p^{|ab|}[t_0\,,\,\infty) \neq 0$. Then by the Compression Lemma, $f_{t_0} \neq 0$, so there is a unit vector $\eta \in \mathfrak{H}$ such that $f_{t_0}\eta = \eta$ and $p^{p^{-1}a^p + q^{-1}b^q}[t_0\,,\,\infty)\eta = 0$. Thus, by Lemma 2.1 and the Compression Lemma we have that

$$t_0 = \langle t_0 f_{t_0}\eta\,,\,\eta\rangle \leq \langle f_{t_0}(p^{-1}a^p + q^{-1}b^q)f_{t_0}\eta\,,\,\eta\rangle = \langle (p^{-1}a^p + q^{-1}b^q)\eta\,,\,\eta\rangle < t_0\,,$$

which is a contradiction.

Similarly, if $p^{|ab|}(-\infty\,,\,t_0] = 0$ and $p^{p^{-1}a^p + q^{-1}b^q}(-\infty\,,\,t_0] \neq 0$ for some $t_0 \in \mathbb{R}$, then $p^{|ab|}(t_0\,,\,\infty) = 1$ and $p^{p^{-1}a^p + q^{-1}b^q}(t_0\,,\,\infty) \neq 1$.

Let η be a unit vector in $\mathfrak{H}$ such that $p^{p^{-1}a^p + q^{-1}b^q}(t_0\,,\,\infty)\eta = 0$ and $p^{|ab|}(t_0\,,\,\infty)\eta = \eta$. Again we have contradiction by Lemma 2.1 and the Compression Lemma (3.1). Thus,

$$|ab| \precsim_{sp} p^{-1}a^p + q^{-1}b^q.$$

By Theorem 2.7, there is a unitary u in M such that

$$u\,|ab|\,u^* \leq p^{-1}a^p + q^{-1}b^q.$$

□

REFERENCES

[1] C.A. AKEMANN, J. ANDERSON AND G.K. PEDERSEN, Triangle inequalities in operators algebras, *Linear Multilinear Algebra*, **11** (1982), 167–178.

[2] T. ANDO, Matrix Young inequalities, *Oper. Theory Adv. Appl.*, **75** (1995), 33–38.

[3] R. BHATIA AND F. KITTANEH, On the singular values of a product of operators, *SIAM J. Matrix Anal. Appl.*, **11** (1990), 272–277.

[4] J. ERLIJMAN, D.R. FARENICK AND R. ZENG, Young's inequality in compact operators, *Oper. Theory Adv. Appl.*, **130** (2001), 171–184.

[5] R.V. KADISON AND J.R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, Volume II, Academic Press, New York, 1986.