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A MULTIPLICATIVE EMBEDDING INEQUALITY IN ORLICZ-SOBOLEV SPACES

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ABSTRACT. We prove an Orlicz type version of the multiplicative embedding inequality for Sobolev spaces.

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1. INTRODUCTION AND PRELIMINARY RESULTS

Let Ω be a non-empty bounded open set in $\mathbb{R}$, $n > 1$ and let $1 \leq p < n$. The most important result of Sobolev space theory is the well-known *Sobolev imbedding theorem* (see e.g. [1]), which - in the case of functions vanishing on the boundary - gives an estimate of the norm in the Lebesgue space $L^q(\Omega)$, $q = np/(n-p)$ of a function u in the Sobolev space $W_0^{1,p}(\Omega)$, in terms of its $W_0^{1,p}(\Omega)$-norm. Such an estimate, due to Gagliardo and Nirenberg ([6], [12]) can be stated in the following multiplicative form (see e.g. [4], [10]).

Theorem 1.1. *Let Ω be a non-empty bounded open set in $\mathbb{R}$, $n > 1$ and let $1 \leq p < n$. Let $u \in W_0^{1,p}(\Omega) \bigcap L^r(\Omega)$ for some $r \geq 1$. If q lies in the closed interval bounded by the numbers r and $np/(n-p)$, then the following inequality holds*

$$\|u\|_q \leq c\||Du|\|_p^\theta \|u\|_r^{1-\theta}, \tag{1.1}$$

where

$$\theta = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{n} - \frac{1}{p} + \frac{1}{r}} \in [0, 1]$$

and

$$c = c(n, p, \theta) = \left[\frac{p(n-1)}{n-p}\right]^\theta .$$

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The constant $c = c(n, p, \theta)$ is not optimal (see [16], [7] for details).

The goal of this paper is to provide an Orlicz version of inequality (1.1), in which the role of the parameter θ is played by a certain concave function. Our approach uses a generalized Hölder inequality proved in [8] (see Lemma 1.2 below).

We summarize some basic facts of Orlicz space theory; we refer the reader to Krasnosel'skiĭ and Rutickiĭ [9], Maligranda [11], or Rao and Ren [14] for further details.

A function $A : [0, \infty) \to [0, \infty)$ is an N-function if it is continuous, convex and strictly increasing, and if $A(0) = 0$, $A(t)/t \to 0$ as $t \to 0$, $A(t)/t \to +\infty$ as $t \to +\infty$.

If A, B are N-functions (in the following we will adopt the next symbol for the inverse function of N-functions, too), we write $A(t) \approx B(t)$ if there are constants $c_1, c_2 > 0$ such that $c_1 A(t) \le B(t) \le c_2 A(t)$ for all $t > 0$. Also, we say that B dominates A, and denote this by $A \preceq B$, if there exists $c > 0$ such that for all $t > 0$, $A(t) \le B(ct)$. If this is true for all $t \ge t_0 > 0$, we say that $A \preceq B$ near infinity.

An N-function A is said to be doubling if there exists a positive constant c such that $A(2t) \le cA(t)$ for all $t > 0$; A is called submultiplicative if $A(st) \le cA(s)A(t)$ for all $s, t > 0$. Clearly $A(t) = t^r$, $r \ge 1$, is submultiplicative. A straightforward computation shows that $A(t) = t^a[\log(e + t)]^b$, $a \ge 1$, $b > 0$, is also submultiplicative.

Given an $N-$function A, the Orlicz space $L_A(\Omega)$ is the Banach space of Lebesgue measurable functions f such that $A(|f|/\lambda)$ is (Lebesgue) integrable on A for some $\lambda > 0$. It is equipped with the Luxemburg norm $\|f\|_A = \inf \left\{ \lambda > 0 : \int_\Omega A\left(\frac{|f|}{\lambda}\right) dx \le 1 \right\}$.

If $A \preceq B$ near infinity then there exists a constant c, depending on A and B, such that for all functions f,

$$\|f\|_A \le c\|f\|_B. \tag{1.2}$$

This follows from the standard embedding theorem which shows that $L_B(\Omega) \subset L_A(\Omega)$.

Given an N-function A, the complementary N-function $\widetilde{A}$ is defined by

$$\widetilde{A}(t) = \sup_{s>0}\{st - A(s)\}, \qquad t \ge 0.$$

The N-functions A and $\widetilde{A}$ satisfy the following inequality (see e.g. [1, (7) p. 230]):

$$t \le A^{-1}(t)\widetilde{A}^{-1}(t) \le 2t. \tag{1.3}$$

The Hölder's inequality in Orlicz spaces reads as

$$\int_\Omega |fg|\, dx \le 2\|f\|_A \|g\|_{\widetilde{A}}.$$

We will need the following generalization of Hölder's inequality to Orlicz spaces due to Hogan, Li, McIntosh, Zhang [8] (see also [3] and references therein).

Lemma 1.2. *If A, B and C are $N-$functions such that for all $t > 0$,*

$$B^{-1}(t)C^{-1}(t) \le A^{-1}(t),$$

then

$$\|fg\|_A \le 2\|f\|_B \|g\|_C.$$

If A is an $N-$function, let us denote by $W^{1,A}(\Omega)$ the space of all functions in $L^A(\Omega)$ such that the distributional partial derivatives belong to $L^A(\Omega)$, and by $W_0^{1,A}(\Omega)$ the closure of the $\mathcal{C}_0^\infty(\Omega)$ functions in this space. Such spaces are well-known in the literature as *Orlicz-Sobolev* spaces (see e.g. [1]) and share various properties of the classical Sobolev spaces. References for main properties and applications are for instance [5] and [15].

If $u \in W_0^{1,A}(\Omega)$ and

$$\int_1^\infty \frac{\tilde{A}(s)}{s^{n'+1}} ds = +\infty, \qquad n' = n/(n-1)$$

then the continuous embedding inequality

$$\|u\|_{A^*} \leq c \, \||Du|\|_A \tag{1.4}$$

holds, where A^* is the so-called *Sobolev conjugate* of A, defined in [1], and c is a positive constant depending only on A and n. In the following it will be not essential, for our purposes, to know the exact expression of A^*. However, we stress here that one could consider the *best* function A^* such that inequality (1.4) holds (see [2], [13] for details).

In the sequel we will need the following definition.

Definition 1.1. Given an $N-$function A, define the function h_A by

$$h_A(s) = \sup_{t>0} \frac{A(st)}{A(t)}, \qquad 0 \leq s < \infty.$$

Remark 1.3. The function h_A could be infinite if $s > 1$, but if A is doubling then it is finite for all $0 < s < \infty$ (see Maligranda [11, Theorem 11.7]). If A is submultiplicative then $h_A \approx A$. More generally, given any A, for all $s, t \geq 0$, $A(st) \leq h_A(s)A(t)$.

The property of the function h_A which will play a role in the following is that it can be inverted, in fact the following lemma holds.

Lemma 1.4. *If A is a doubling $N-$function then h_A is nonnegative, submultiplicative, strictly increasing in $[0, \infty)$ and $h_A(1) = 1$.*

For the (easy) proof see [3, Lemma 3.1] or [11, p. 84].

2. THE MAIN RESULT

We will begin by proving two auxiliary results. The first one concerns two functions that we call $K = K(t)$ and $H = H(t)$: they are a way to "measure", in the final multiplicative inequality, how far the right hand side is with respect to the norms of u and of $|Du|$. In the standard case it is $K(t) = t^\theta$, $0 \leq \theta \leq 1$ and $H(t) = t^{1-\theta}$.

Lemma 2.1. *Let $K \in \mathcal{C}([0, +\infty[) \bigcap \mathcal{C}^2(]0, +\infty[)$ be:*
- a positive, constant function,
or
- $K(t) = \alpha t$ for some $\alpha > 0$,
or
- the inverse function of an $N-$function which is doubling together with its complementary $N-$function.
Then the function $H : [0, +\infty[\to [0, +\infty[$ defined by

$$H(t) = \begin{cases} \frac{t}{K(t)} & \text{if } t > 0 \\ \lim\limits_{t \to 0} \frac{t}{K(t)} & \text{if } t = 0 \end{cases}$$

belongs to $\mathcal{C}([0, +\infty[) \bigcap \mathcal{C}^2(]0, +\infty[)$, and is:
- a positive, constant function,
or
- $H(t) = \beta t$ for some $\beta > 0$,
or

- is equivalent to the inverse function of an $N-$function which is doubling together with its complementary $N-$function.

Proof. In the first two possibilities for K the statement is easy to prove. If K is the inverse of a doubling N-function A, it is sufficient to observe that from inequality (1.3) it is $H \approx \widetilde{A}^{-1}$. $\square$

Lemma 2.2. *Let Φ be an $N-$function, and let F be a doubling $N-$function such that $\Phi \circ F^{-1}$ is an $N-$function. The following inequality holds for every $u \in L^{\Phi}(\Omega)$:*

$$\|u\|_{\Phi} \leq \xi_{F^{-1}}(\|F \circ |u|\|_{\Phi\circ F^{-1}}), \tag{2.1}$$

where $\xi_{F^{-1}}$ is the increasing function defined by

$$\xi_{F^{-1}}(\mu) = \frac{1}{h_F^{-1}\left(\frac{1}{\mu}\right)} \qquad \forall \mu > 0. \tag{2.2}$$

Proof. By definition of h_F (see Definition 1.1; note that by the assumption that F is doubling, h_F is everywhere finite, see Remark 1.3) we have

$$F(s)h_F(t) \geq F(st) \qquad \forall\, s,t > 0$$

and therefore

$$sh_F(t) \geq F(F^{-1}(s)t) \qquad \forall\, s,t > 0,$$

$$F^{-1}(sh_F(t)) \geq F^{-1}(s)t \qquad \forall\, s,t > 0. \tag{2.3}$$

Setting

$$\mu = \mu(\lambda) = \frac{1}{h_F\left(\frac{1}{\lambda}\right)}$$

it is

$$\lambda = \frac{1}{h_F^{-1}\left(\frac{1}{\mu}\right)} := \xi_{F^{-1}}(\mu),$$

therefore from inequality (2.3), for $t = \frac{1}{\lambda}$ and $s = F(|u|)$, taking into account that $\xi_{F^{-1}}$ is increasing, we have

$$\begin{aligned}
\|u\|_{\Phi} &= \inf\left\{\lambda > 0 \,:\, \int_{\Omega} \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\} \\
&= \inf\left\{\lambda > 0 \,:\, \int_{\Omega} \Phi\left(\frac{F^{-1}(F(|u|))}{\lambda}\right) dx \leq 1\right\} \\
&\leq \inf\left\{\lambda > 0 \,:\, \int_{\Omega} \Phi\left(F^{-1}\left(F(|u|)h_F\left(\frac{1}{\lambda}\right)\right)\right) dx \leq 1\right\} \\
&= \inf\left\{\xi_{F^{-1}}(\mu) > 0 \,:\, \int_{\Omega} \Phi\left(F^{-1}\left(\frac{F(|u|)}{\mu}\right)\right) dx \leq 1\right\} \\
&= \xi_{F^{-1}}\left(\inf\left\{\mu > 0 \,:\, \int_{\Omega} \Phi\left(F^{-1}\left(\frac{F(|u|)}{\mu}\right)\right) dx \leq 1\right\}\right) \\
&= \xi_{F^{-1}}(\|F \circ |u|\|_{\Phi\circ F^{-1}})
\end{aligned}$$

$\square$

We can prove now the main theorem of the paper. The symbol ξ_K which appears in the statement is the function considered in Lemma 2.2, defined in equation (2.2). However, since this symbol is used for any function K considered in Lemma 2.1, we agree to denote

$$\xi_K(\mu) := 1 \quad \forall \mu \geq 0 \qquad \text{if} \qquad K \text{ is constant}$$

and

$$\xi_K(\mu) := \mu \quad \forall \mu \geq 0 \qquad \text{if} \qquad K(t) = \alpha t \text{ for some } \alpha > 0.$$

The same conventions will be adopted for the symbol ξ_H. Note that from Lemma 2.1 we know that H is *equivalent* to the inverse of a doubling $N-$function, let us call it B^{-1}. We will agree to denote still by ξ_H the function that we should denote by $\xi_{B^{-1}}$. This convention does not create ambiguities because if $B \approx C$ then $h_B \approx h_C$ and $\xi_{B^{-1}} \approx \xi_{C^{-1}}$, therefore ξ_H is well defined up to a multiplicative positive constant.

Theorem 2.3. *Let Ω be a non-empty bounded open set in $\mathbb{R}$, $n > 1$ and let P be an $N-$function satisfying*

$$\int_1^\infty \frac{\tilde{P}(s)}{s^{n'+1}}\, ds = +\infty, \qquad n' = n/(n-1).$$

Let $u \in W_0^{1,P}(\Omega) \bigcap L^R(\Omega)$ for some $N-$function R. If Q is an $N-$function such that

$$K((P^*)^{-1}(s)) \cdot H(R^{-1}(s)) \leq Q^{-1}(s) \qquad \forall s > 0 \tag{2.4}$$

then the following inequality holds

$$\|u\|_Q \leq \xi_K(c\||Du|\|_P)\xi_H(\|u\|_R), \tag{2.5}$$

where K and H are functions as in Lemma 2.1 and c is a constant depending only on n, P, K.

Proof. Let K and H be functions as in Lemma 2.1. If K is a positive, constant function or $K(t) = \alpha t$ for some $\alpha > 0$, then the statement reduces respectively to a direct consequence of inequality (1.2) (with A and B replaced respectively by Q and R) or to inequality (1.4) (with A replaced by P). We may therefore assume in the following that K is the inverse function of an $N-$function which is doubling together with its complementary $N-$function. Let

$$\Phi_1 = P^* \circ K^{-1} \qquad \Phi_2 = R \circ H^{-1}.$$

It is easy to verify that Φ_1 and Φ_2 are $N-$functions. By assumption (2.4) and Lemma 1.2 we have

$$\|u\|_Q = \|K(u)H(u)\|_Q \leq \|K(u)\|_{\Phi_1} \|H(u)\|_{\Phi_2}. \tag{2.6}$$

By inequality (2.1),

$$\|K(u)\|_{\Phi_1} \leq \xi_K(\|u\|_{\Phi_1 \circ K}) = \xi_K(\|u\|_{P^*}) \leq \xi_K(c\||Du|\|_P), \tag{2.7}$$

where c is a positive constant depending on n and P only. On the other hand,

$$\|H(u)\|_{\Phi_2} \leq \xi_H(\|u\|_{\Phi_2 \circ H}) = \xi_H(\|u\|_R). \tag{2.8}$$

From inequalities (2.6), (2.7), (2.8), we get the inequality (2.5) and the theorem is therefore proved. □

We remark that the natural choice of powers for P, Q, R, K, H reduce Theorem 2.3 to Theorem 1.1 (in Theorem 2.3 also the case $p = n$ is allowed); on the other hand, if inequality (2.5) allows growths of ξ_K different power types, in general it is not true that $\xi_K(t)\xi_H(t) = t$, and this is the "price" to pay for the major "freedom" given to the growth K.

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