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ON A HYBRID FAMILY OF SUMMATION INTEGRAL TYPE OPERATORS

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ABSTRACT. The present paper deals with the study of the mixed summation integral type operators having Szász and Baskakov basis functions in summation and integration respectively. Here we obtain the rate of point wise convergence, a Voronovskaja type asymptotic formula, an error estimate in simultaneous approximation. We also study some local direct results in terms of modulus of smoothness and modulus of continuity in ordinary and simultaneous approximation.

Key words and phrases: Linear positive operators, Summation-integral type operators, Rate of convergence, Asymptotic formula, Error estimate, Local direct results, K-functional, Modulus of smoothness, Simultaneous approximation.

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1. INTRODUCTION

The mixed summation-integral type operators discussed in this paper are defined as

$$\begin{aligned}(1.1)\qquad S_n(f,x) &= \int_0^\infty W_n(x,t)f(t)dt \\ &= (n-1)\sum_{\nu=1}^{\infty} s_{n,\nu}(x)\int_0^\infty b_{n,\nu-1}(t)f(t)dt + e^{-nx}f(0), \qquad x\in[0,\infty),\end{aligned}$$

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where

$$W_n(x,t) = (n-1)\sum_{\nu=1}^{\infty} s_{n,\nu}(x) b_{n,\nu-1}(t) + e^{-nx}\delta(t),$$

$\delta(t)$ being Dirac delta function,

$$s_{n,\nu}(x) = e^{-nx}\frac{(nx)^\nu}{\nu!}$$

and

$$b_{n,\nu}(t) = \binom{n+\nu-1}{\nu} t^\nu (1+t)^{-n-\nu}$$

are respectively Szász and Baskakov basis functions. It is easily verified that the operators (1.1) are linear positive operators, these operators were recently proposed by Gupta and Gupta in [3]. The behavior of these operators is very similar to the operators studied by Gupta and Srivastava [5], but the approximation properties of the operators S_n are different in comparison to the operators studied in [5]. The main difference is that the operators (1.1) are discretely defined at the point zero. Recently Srivastava and Gupta [8] proposed a general family of summation-integral type operators $G_{n,c}(f,x)$ which include some well known operators (see e.g. [4], [7]) as special cases. The rate of convergence for bounded variation functions was estimated in [8], Ispir and Yuksel [6] considered the Bézier variant of the operators $G_{n,c}(f,x)$ and studied the rate of convergence for bounded variation functions. We also note here that the results analogous to [6] and [8] cannot be obtained for the mixed operators $S_n(f,x)$ because it is not easier to write the integration of Baskakov basis functions in the summation form of Szász basis functions, which is necessary in the analysis for obtaining the rate of convergence at the point of discontinuity. We propose this as an open problem for the readers.

In the present paper we study some direct results, for the class of unbounded functions with growth of order $t^\gamma,\ \gamma > 0,$ for the operators S_n we obtain a point wise rate of convergence, asymptotic formula of Voronovskaja type, and an error estimate in simultaneous approximation. We also estimate local direct results in terms of modulus of smoothness and modulus of continuity in ordinary and simultaneous approximation.

2. AUXILIARY RESULTS

We will subsequently need the following lemmas:

Lemma 2.1. *For $m \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, if the m-th order moment is defined as*

$$U_{n,m}(x) = \sum_{\nu=0}^{\infty} s_{n,\nu}(x)\left(\frac{\nu}{n} - x\right)^m,$$

then $U_{n,0}(x) = 1,\ U_{n,1}(x) = 0$ and

$$nU_{n,m+1}(x) = x\left[U^{(1)}_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently

$$U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Lemma 2.2. *Let the function $\mu_{n,m}(x)$, $m \in \mathbb{N}^0$, be defined as*

$$\mu_{n,m}(x) = (n-1)\sum_{\nu=1}^{\infty} s_{n,\nu}(x)\int_0^\infty b_{n,\nu-1}(t)(t-x)^m dt + (-x)^m e^{-nx}.$$

Then

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{2x}{n-2}, \quad \mu_{n,2}(x) = \frac{nx(x+2)+6x^2}{(n-2)(n-3)},$$

also we have the recurrence relation:

$$(n-m-2)\mu_{n,m+1}(x) = x\left[\mu_{n,m}^{(1)}(x) + m(x+2)\mu_{n,m-1}(x)\right] \\ + \left[m + 2x(m+1)\right]\mu_{n,m}(x); \quad n > m+2.$$

Consequently for each $x \in [0,\infty)$ we have from this recurrence relation that

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Remark 2.3. It is easily verified from Lemma 2.2 that for each $x \in (0,\infty)$

$$S_n(t^i,x) = \frac{(n-i-2)!}{(n-2)!}(nx)^i + i(i-1)\frac{(n-i-2)!}{(n-2)!}(nx)^{i-1} + O(n^{-2}). \tag{2.1}$$

Lemma 2.4. [5]. *There exist the polynomials $Q_{i,j,r}(x)$ independent of n and ν such that*

$$x^r D^r\left[s_{n,\nu}(x)\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(\nu - nx)^j Q_{i,j,r}(x) s_{n,\nu}(x),$$

where $D \equiv \frac{d}{dx}$.

Lemma 2.5. *Let $n > r \ge 1$ and $f^{(i)} \in C_B[0,\infty)$ for $i \in \{0,1,2,\dots,r\}$ (cf. Section 3). Then*

$$S_n^{(r)}(f,x) = \frac{n^r}{(n-2)\cdots(n-r-1)}\sum_{\nu=0}^{\infty} s_{n,\nu}(x)\int_0^\infty b_{n-r,\nu+r-1}(t) f^{(r)}(t)dt.$$

3. DIRECT RESULTS

In this section we consider the class $C_\gamma[0,\infty)$ of continuous unbounded functions, defined as

$$f \in C_\gamma[0,\infty) \equiv \left\{f \in C[0,\infty) : |f(t)| \le Mt^\gamma, \text{ for some } M>0,\ \gamma>0\right\}.$$

We prove the following direct estimates:

Theorem 3.1. *Let $f \in C_\gamma[0,\infty),\ \gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} S_n^{(r)}(f(t),x) = f^{(r)}(x). \tag{3.1}$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Thus, using the above, we have

$$\begin{aligned} S_n^{(r)}(f,x) &= \int_0^\infty W_n^{(r)}(t,x)f(t)dt \\ &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt + \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt \\ &= R_1 + R_2, \quad \text{say.} \end{aligned}$$

First to estimate R_1, using a binomial expansion of $(t-x)^m$ and applying (2.1), we have

$$\begin{aligned}
R_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{\nu=0}^{i} \binom{i}{\nu} (-x)^{i-\nu} \frac{\partial^r}{\partial x^r} \int_0^\infty W_n(t,x) t^\nu dt \\
&= \frac{f^{(r)}(x)}{r!} \frac{d^r}{dx^r} \left[\frac{(n-r-2)!n^r}{(n-2)!} x^r + \text{terms containing lower powers of } x \right] \\
&= f^{(r)}(x) \left[\frac{(n-r-2)!n^r}{(n-2)!} \right] \to f^{(r)}(x) \text{ as } n \to \infty.
\end{aligned}$$

Next using Lemma 2.4, we obtain

$$\begin{aligned}
|R_2| \le (n-1) &\sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \\
&\times \sum_{\nu=1}^{\infty} |\nu - nx|^j s_{n,\nu}(x) \int_0^\infty b_{n,\nu-1}(t) \left|\varepsilon(t,x)\right| (t-x)^r dt \\
&+ (-n)^r e^{-nx} \left|\varepsilon(0,x)\right| (-x)^r \\
= R_3 + R_4&, \quad \text{say.}
\end{aligned}$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. Further if $s \ge \max\{\gamma, r\}$, where s is any integer, then we can find a constant $M_1 > 0$ such that $|\varepsilon(t,x)(t-x)^r| \le M_1 |t-x|^s$, for $|t-x| \ge \delta$. Thus

$$\begin{aligned}
R_3 &\le M_2(n-1) \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \sum_{\nu=1}^{\infty} s_{n,\nu}(x) \\
&\qquad \times |\nu - nx|^j \left\{ \varepsilon \int_{|t-x|<\delta} b_{n,\nu-1}(t) |t-x|^r dt + \int_{|t-x|\ge\delta} b_{n,\nu-1}(t) M_1 |t-x|^s dt \right\} \\
&= R_5 + R_6,
\end{aligned}$$

say.

Applying the Schwarz inequality for integration and summation respectively, and using Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
R_5 &\le \varepsilon M_2 (n-1) \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \sum_{\nu=1}^{\infty} s_{n,\nu}(x) \\
&\qquad \times |\nu - nx|^j \left(\int_0^\infty b_{n,\nu-1}(t) dt \right)^{\frac{1}{2}} \left(\int_0^\infty b_{n,\nu-1}(t)(t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\le \varepsilon M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \left(\sum_{\nu=1}^{\infty} s_{n,\nu}(x)(\nu - nx)^{2j} \right)^{\frac{1}{2}} \\
&\qquad \times \left((n-1) \sum_{\nu=1}^{\infty} s_{n,\nu}(x) \int_0^\infty b_{n,\nu-1}(t)(t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\le \varepsilon M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i O\left(n^{j/2}\right) O\left(n^{-r/2}\right) = \varepsilon O(1).
\end{aligned}$$

Again using the Schwarz inequality, Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
R_6 &\leq M_3(n-1)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{\nu=1}^{\infty} s_{n,\nu}(x)\,|\nu-nx|^j \int_{|t-x|\geq\delta} b_{n,\nu-1}(t)\,|t-x|^s\,dt\\
&\leq M_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{\nu=1}^{\infty} s_{n,\nu}(x)(\nu-nx)^{2j}\right)^{\frac{1}{2}}\\
&\qquad\times\left((n-1)\sum_{\nu=1}^{\infty} s_{n,\nu}(x)\int_0^{\infty} b_{n,\nu-1}(t)(t-x)^{2s}dt\right)^{\frac{1}{2}}\\
&= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O\left(n^{j/2}\right) O\left(n^{-s/2}\right) = O\left(n^{(r-s)/2}\right) = o(1).
\end{aligned}$$

Thus due to the arbitrariness of $\varepsilon > 0$ it follows that $R_3 = o(1)$. Also $R_4 \to 0$ as $n \to \infty$ and therefore $R_2 = o(1)$. Collecting the estimates of R_1 and R_2, we get (3.1). □

Theorem 3.2. *Let* $f \in C_\gamma[0,\infty),\ \gamma > 0$. *If* $f^{(r+2)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\begin{aligned}
&\lim_{n\to\infty} n\left[S_n^{(r)}(f,x) - f^{(r)}(x)\right]\\
&\qquad= \frac{r(r+3)}{2} f^{(r)}(x) + \left[x(2+r)+r\right] f^{(r+1)}(x) + \frac{x}{2}(2+x) f^{(r+2)}(x).
\end{aligned}$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Applying Lemma 2.2 and the above Taylor's expansion, we have

$$\begin{aligned}
n\left[S_n^{(r)}(f(t),x) - f^{(r)}(x)\right] &= n\left[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)\right]\\
&\qquad + \left[n\int_0^{\infty} W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}dt\right]\\
&= E_1 + E_2, \quad \text{say.}
\end{aligned}$$

$$\begin{aligned}
E_1 &= n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^{\infty} W_n^{(r)}(t,x) t^j dt - n f^{(r)}(x)\\
&= \frac{f^{(r)}(x)}{r!} n\left[S_n^{(r)}(t^r,x) - r!\right] + \frac{f^{(r+1)}(x)}{(r+1)!} n\left[(r+1)(-x)S_n^{(r)}(t^r,x) + S_n^{(r)}(t^{r+1},x)\right]\\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!} n\left[\frac{(r+2)(r+1)}{2} x^2 S_n^{(r)}(t^r,x) + (r+2)(-x)S_n^{(r)}(t^{r+1},x) + S_n^{(r)}(t^{r+2},x)\right].
\end{aligned}$$

Therefore, using (2.1) we have

$$\begin{aligned}
E_1 &= nf^{(r)}(x)\left[\frac{n^r(n-r-2)!}{(n-2)!}-1\right] \\
&\quad + n\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)r!\left\{\frac{n^r(n-r-2)!}{(n-2)!}\right\}\right. \\
&\quad \left.+\left\{\frac{n^{r+1}(n-r-3)!}{(n-2)!}(r+1)!x+r(r+1)\frac{n^r(n-r-3)!}{(n-2)!}r!\right\}\right] \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)x^2}{2}(r!)\frac{n^r(n-r-2)!}{(n-2)!}\right. \\
&\quad + (r+2)(-x)\left\{\frac{n^{r+1}(n-r-3)!}{(n-2)!}(r+1)!x+r(r+1)\frac{n^r(n-r-3)!}{(n-2)!}r!\right\} \\
&\quad + \left\{\frac{n^{r+2}(n-r-4)!}{(n-2)!}\right\}\frac{(r+2)!}{2}x^2 \\
&\quad \left.+(r+1)(r+2)\frac{n^{r+1}(n-r-4)!}{(n-2)!}(r+1)!x+O\left(n^{-2}\right)\right].
\end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $E_2 \to 0$ as $n \to \infty$, which can easily be proved along the lines of the proof of Theorem 3.1 and by using Lemma 2.1, Lemma 2.2 and Lemma 2.4. □

Theorem 3.3. *Let $f \in C_\gamma[0,\infty),\ \gamma > 0$ and $r \le m \le r+2$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for n sufficiently large*

$$\left\|S_n^{(r)}(f,x) - f^{(r)}\right\| \le M_4 n^{-1}\sum_{i=1}^{m}\left\|f^{(i)}\right\| + M_5 n^{-1/2} w\left(f^{(r+1)}, n^{-1/2}\right) + O\left(n^{-2}\right),$$

where the constants M_4 and M_5 are independent of f and n, $w(f,\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$ and $\|\cdot\|$ denotes the sup-norm on the interval $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!} + (t-x)^m\zeta(t)\frac{f^m(\xi)-f^m(x)}{m!} + h(t,x)\left(1-\zeta(t)\right),$$

where ζ lies between t and x and $\zeta(t)$ is the characteristic function on the interval $(a-\eta,b+\eta)$. For $t \in (a-\eta,b+\eta)$, $x \in [a,b]$, we have

$$f(t) = \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!} + (t-x)^m\frac{f^m(\xi)-f^m(x)}{m!}.$$

For $t \in [0,\infty)\setminus(a-\eta,b+\eta)$, $x \in [a,b]$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}.$$

Thus

$$\begin{aligned} S_n^{(r)}(f,x) - f^{(r)}(x) &= \left\{ \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \right\} \\ &\quad + \left\{ \int_0^\infty W_n^{(r)}(t,x) \frac{f^m(\xi) - f^m(x)}{m!}(t-x)^m \zeta(t) dt \right\} \\ &\quad + \left\{ \int_0^\infty W_n^{(r)}(t,x) h(t,x)(1-\zeta(t)) dt \right\} \\ &= \Delta_1 + \Delta_2 + \Delta_3, \qquad \text{say.} \end{aligned}$$

Using (3.1), we obtain

$$\begin{aligned} \Delta_1 &= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{\partial^r}{\partial x^r} \int_0^\infty W_n(t,x) t^j dt - f^{(r)}(x) \\ &= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{\partial^r}{\partial x^r} \left[\frac{(n-j-2)!}{(n-2)!}(nx)^j \right. \\ &\qquad \left. + j(j-1) \frac{(n-j-2)!}{(n-2)!}(nx)^{j-1} + O\left(n^{-2}\right) \right] - f^{(r)}(x). \end{aligned}$$

Hence

$$\|\Delta_1\| \le M_4 n^{-1} \sum_{i=r}^{m} \left\| f^{(i)} \right\| + O\left(n^{-2}\right),$$

uniformly in $x \in [a,b]$. Next

$$\begin{aligned} |\Delta_2| &\le \int_0^\infty \left| W_n^{(r)}(t,x) \right| \frac{|f^m(\xi) - f^m(x)|}{m!} |t-x|^m \zeta(t) dt \\ &\le \frac{w\left(f^{(m)}, \delta\right)}{m!} \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left(1 + \frac{|t-x|}{\delta} \right) |t-x|^m dt. \end{aligned}$$

Next, we shall show that for $q = 0, 1, 2, \ldots$

$$(n-1) \sum_{\nu=1}^{\infty} s_{n,\nu}(x) |\nu - nx|^j \int_0^\infty b_{n,\nu-1}(t) |t-x|^q dt = O\left(n^{(j-q)/2}\right).$$

Now by using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} &(n-1) \sum_{\nu=1}^{\infty} s_{n,\nu}(x) |\nu - nx|^j \int_0^\infty b_{n,\nu-1}(t) |t-x|^q dt \\ &\le \left(\sum_{\nu=1}^{\infty} s_{n,\nu}(x)(\nu - nx)^{2j} \right)^{\frac{1}{2}} \left((n-1) \sum_{\nu=1}^{\infty} s_{n,\nu}(x) \int_0^\infty b_{n,\nu-1}(t) (t-x)^{2q} dt \right)^{\frac{1}{2}} \\ &= O\left(n^{j/2}\right) O\left(n^{-q/2}\right) = O\left(n^{(j-q)/2}\right), \end{aligned}$$

uniformly in x. Thus by Lemma 2.4, we obtain

$$\begin{aligned}
&(n-1)\sum_{\nu=1}^{\infty}\left|s_{n,\nu}^{(r)}(x)\right|\int_0^{\infty}b_{n,\nu-1}(t)\,|t-x|^q\,dt\\
&\le M_6\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\left[(n-1)\sum_{\nu=1}^{\infty}s_{n,\nu}(x)\,|\nu-nx|^j\int_0^{\infty}b_{n,\nu-1}(t)\,|t-x|^q\,dt\right]\\
&=O\left(n^{(r-q)/2}\right),
\end{aligned}$$

uniformly in x, where $M_6=\sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}}\sup\limits_{x\in[a,b]}|Q_{i,j,r}(x)|\,x^{-r}$. Choosing $\delta=n^{-1/2}$, we get for any $s>0$,

$$\begin{aligned}
\|\Delta_2\| &\le \frac{w\left(f^{(m)},n^{-1/2}\right)}{m!}\left[O(n^{(r-m)/2})+n^{1/2}O\left(n^{(r-m-1)/2}\right)+O\left(n^{-s}\right)\right]\\
&\le M_5 w\left(f^{(m)},n^{-1/2}\right)n^{-(m-r)/2}.
\end{aligned}$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose a $\delta>0$ in such a way that $|t-x|\ge\delta$ for all $x\in[a,b]$. Applying Lemma 2.4, we obtain

$$\begin{aligned}
\|\Delta_3\| \le (n-1)\sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\,|\nu-nx|^j\,\frac{|Q_{i,j,r}(x)|}{x^r}s_{n,\nu}(x)&\\
\times\int_{|t-x|\ge\delta}b_{n,\nu-1}(t)\,|h(t,x)|\,dt+n^re^{-nx}\,|h(0,x)|\,.&
\end{aligned}$$

If β is any integer greater than or equal to $\{\gamma,m\}$, then we can find a constant M_7 such that $|h(t,x)|\le M_7\,|t-x|^{\beta}$ for $|t-x|\ge\delta$. Now applying Lemma 2.1 and Lemma 2.2, it is easily verified that $\Delta_3=O\left(n^{-q}\right)$ for any $q>0$ uniformly on $[a,b]$. Combining the estimates $\Delta_1-\Delta_3$, we get the required result. □

4. LOCAL APPROXIMATION

In this section we establish direct local approximation theorems for the operators (1.1). Let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions f on $[0,\infty)$ endowed with the norm $\|f\|=\sup\limits_{x\ge 0}|f(x)|$. The K-functionals are defined as

$$K(f,\delta)=\inf\left\{\|f-g\|+\delta\,\|g''\|:g\in W_{\infty}^2\right\},$$

where $W_{\infty}^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [1, pp 177, Th. 2.4], there exists a constant $\widetilde{M}$ such that $K(f,\delta)\le\widetilde{M}w_2\left(f,\sqrt{\delta}\right)$, where $\delta>0$ and the second order modulus of smoothness is defined as

$$w_2\left(f,\sqrt{\delta}\right)=\sup_{0<h\le\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|\,,$$

where $f\in C_B[0,\infty)$. Furthermore, let

$$w\left(f,\delta\right)=\sup_{0<h\le\delta}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|$$

be the usual modulus of continuity of $f\in C_B[0,\infty)$.

Our first theorem in this section is in ordinary approximation which involves second order and ordinary moduli of smoothness:

Theorem 4.1. *Let $f \in C_B[0,\infty)$. Then there exists an absolute constant $M_8 > 0$ such that*

$$|S_n(f,x) - f(x)| \leq M_8 w_2\left(f, \sqrt{\frac{x(1+x)}{n-2}}\right) + w\left(f, \frac{x}{n-2}\right),$$

for every $x \in [0,\infty)$ and $n = 3, 4, \dots$.

Proof. We define a new operator $\hat{S}_n : C_B[0,\infty) \to C_B[0,\infty)$ as follows

$$\hat{S}_n(f,x) = S_n(f,x) - f(x) + f\left(\frac{nx}{n-2}\right). \tag{4.1}$$

Then by Lemma 2.2, we obtain $\hat{S}_n(t-x,x) = 0$. Now, let $x \in [0,\infty)$ and $g \in W_\infty^2$. From Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty)$$

we get

$$\begin{aligned} \hat{S}_n(g,x) - g(x) &= \hat{S}_n\left(\int_x^t (t-u)g''(u)du, x\right) \\ &= S_n\left(\int_x^t (t-u)g''(u)du, x\right) + \int_x^{nx/(n-2)}\left(\frac{n}{n-2}x - u\right)g''(u)du. \end{aligned} \tag{4.2}$$

On the other hand,

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \|g''\| \tag{4.3}$$

and

$$\begin{aligned} \left|\int_x^{nx/(n-2)}\left(\frac{n}{n-2}x - u\right)g''(u)du\right| &\leq \left(\frac{nx}{n-2} - x\right)^2 \|g''\| \\ &\leq \frac{4x^2}{(n-2)^2}\|g''\| \leq \frac{4x(1+x)}{(n-2)^2}\|g''\|. \end{aligned} \tag{4.4}$$

Thus by (4.2), (4.3), (4.4) and by the positivity of S_n, we obtain

$$\left|\hat{S}_n(g,x) - g(x)\right| \leq S_n\left((t-x)^2, x\right)\|g''\| + \frac{4x(1+x)}{(n-2)^2}\|g''\|.$$

Hence in view of Lemma 2.2, we have

$$\begin{aligned} \left|\hat{S}_n(g,x) - g(x)\right| &\leq \left(\frac{2nx + (n+6)x^2}{(n-2)(n-3)} + \frac{4x(1+x)}{(n-2)^2}\right)\|g''\| \\ &\leq \left(\frac{n}{n-3} + \frac{1}{n-2}\right)\frac{4x(1+x)}{n-2}\|g''\| \\ &\leq \frac{18}{n-2}x(1+x)\|g''\|. \end{aligned} \tag{4.5}$$

Again applying Lemma 2.2

$$|S_n(f,x)| \leq (n-1)\sum_{\nu=1}^\infty s_{n,\nu}(x)\int_0^\infty b_{n,\nu-1}(t)|f(t)|dt + e^{-nx}|f(0)| \leq \|f\|.$$

This means that S_n is a contraction, i.e. $\|S_n f\| \leq \|f\|$, $f \in C_B[0,\infty)$. Thus by (4.2)

$$\left\| \hat{S}_n f \right\| \leq \|S_n f\| + 2\|f\| \leq 3\|f\|, \qquad f \in C_B[0,\infty). \tag{4.6}$$

Using (4.1), (4.5) and (4.6), we obtain

$$\begin{aligned}
|S_n(f,x) - f(x)| &\leq \left| \hat{S}_n(f-g,x) - (f-g)(x) \right| + \left| \hat{S}_n(g,x) - g(x) \right| + \left| f(x) - f\left(\frac{nx}{n-2}\right) \right| \\
&\leq 4\|f-g\| + \frac{18}{n-2} x(1+x)\|g''\| + \left| f(x) - f\left(\frac{nx}{n-2}\right) \right| \\
&\leq 18 \left\{ \|f-g\| + \frac{x(1+x)}{n-2}\|g''\| \right\} + w\left(f, \frac{x}{n-2}\right).
\end{aligned}$$

Now taking the infimum on the right hand side over all $g \in W_\infty^2$ and using (4.1) we arrive at the assertion of the theorem. □

The following error estimation is in terms of ordinary modulus of continuity in simultaneous approximation:

Theorem 4.2. *Let* $n > r+3 \geq 4$ *and* $f^{(i)} \in C_B[0,\infty)$ *for* $i \in \{0,1,2,...,r\}$. *Then*

$$\begin{aligned}
\left| S_n^{(r)}(f,x) - f^{(r)}(x) \right| &\leq \left(\frac{n^r(n-r-2)!}{(n-2)!} - 1 \right) \left\| f^{(r)} \right\| + \frac{n^r(n-r-2)!}{(n-2)!} \\
&\quad \times \left(1 + \sqrt{\frac{[n+(r+1)(r+2)]\,x^2 + 2\,[n+r(r+2)]\,x + r(r+1)}{n-r-3}} \right) \\
&\quad \times w\left(f^{(r)}, (n-r-2)^{-1/2}\right)
\end{aligned}$$

where $x \in [0,\infty)$.

Proof. Using Lemma 2.5 and taking into account the well known property $w\left(f^{(r)}, \lambda\delta\right) \leq (1+\lambda) w\left(f^{(r)}, \delta\right)$, $\lambda \geq 0$, we obtain

$$\begin{aligned}
&\left| S_n^{(r)}(f,x) - f^{(r)}(x) \right| \\
&\leq \frac{n^r(n-r-1)!}{(n-2)!} \sum_{\nu=0}^{\infty} s_{n,\nu}(x) \int_0^\infty b_{n-r,\nu+r-1}(t) \left[f^{(r)}(t) - f^{(r)}(x) \right] dt \\
&\qquad + \left[\frac{n^r(n-r-2)!}{(n-2)!} - 1 \right] f^{(r)}(x) \\
&\leq \frac{n^r(n-r-1)!}{(n-2)!} \sum_{\nu=0}^{\infty} s_{n,\nu}(x) \times \int_0^\infty b_{n-r,\nu+r-1}(t) \left(1 + \delta^{-1}|t-x|\right) w\left(f^{(r)}, \delta\right) dt \\
&\qquad + \left[\frac{n^r(n-r-1)!}{(n-2)!} - 1 \right] \left\| f^{(r)} \right\|.
\end{aligned} \tag{4.7}$$

Further, using Cauchy's inequality, we have

$$\begin{aligned}
&(n-r-1) \sum_{\nu=0}^{\infty} s_{n,\nu}(x) \int_0^\infty b_{n-r,\nu+r-1}(t)\, |t-x|\, dt \\
&\qquad \leq \left\{ (n-r-1) \sum_{\nu=0}^{\infty} s_{n,\nu}(x) \int_0^\infty b_{n-r,\nu+r-1}(t)\, (t-x)^2\, dt \right\}^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

By direct computation

$$(4.9)\quad (n-r-1)\sum_{\nu=0}^{\infty} s_{n,\nu}(x)\int_0^{\infty} b_{n-r,\nu+r-1}(t)\,(t-x)^2\,dt = \frac{n+(r+1)(r+2)}{(n-r-3)(n-r-2)}x^2 + \frac{2n+2r(r+2)}{(n-r-3)(n-r-2)}x + \frac{r(r+1)}{(n-r-3)(n-r-2)}.$$

Thus by combining (4.7), (4.8) and (4.9) and choosing $\delta^{-1} = \sqrt{n-r-2}$, we obtain the desired result. □

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