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## ON ONE OF H. ALZER'S PROBLEMS

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## Abstract

In this short note, the author solves an inequality problem which was posed by H. Alzer with difference substitution.

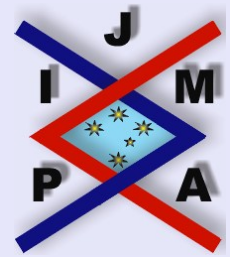
*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Problem, Inequality, Difference substitution.

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# 1. The Problem

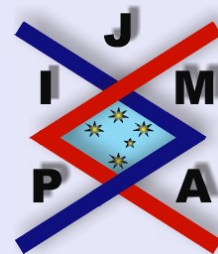
In 1993, H. Alzer posed the following inequality problem in [1]. In 2004, Ji-Chang Kuang reposed the problem in his monograph [2].

**Problem 1.** Let  $a_1, \dots, a_n$  ( $n \in \mathbb{N}^*$ ) be real numbers with  $a_i \in (0, \frac{1}{2}]$ , then prove or disprove

$$(1.1) \quad \prod_{k=1}^n \left( \frac{a_k}{1 - a_k} \right) \leq \frac{\sum_{k=1}^n a_k^n}{\sum_{k=1}^n (1 - a_k)^n}$$

where  $n = 4$  or  $n = 5$ .

In 1995, Michael Vowe pointed out that the inequality (1.1) holds when  $n \leq 3$  ( $n \in \mathbb{N}^*$ ) and does not hold when  $n \geq 6$  ( $n \in \mathbb{N}^*$ ) in [3].



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## 2. Solution of The Problem

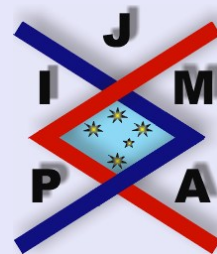
In this section, we show the reader the proof of the inequality (1.1) while  $n = 4$ .

*Proof.* We set  $a_k = \frac{b_k}{u}$  ( $k = 1, 2, 3, 4$ ), where  $u > 0$  and  $0 < a_k \leq \frac{1}{2}$  ( $k = 1, 2, 3, 4$ ), then  $0 < b_k \leq \frac{1}{2}u$  ( $k = 1, 2, 3, 4$ ). So the inequality (1.1) is equivalent to the inequality as follows.

$$(2.1) \quad \prod_{k=1}^4 \left( \frac{b_k}{u - b_k} \right) \leq \frac{\sum_{k=1}^4 b_k^4}{\sum_{k=1}^4 (u - b_k)^4}.$$

Inequality (2.1) is equivalent to the following inequality.

$$(2.2) \quad (b_3^4 + b_4^4 + b_2^4 + b_1^4 - 4b_1b_2b_3b_4)u^3 + (-b_1b_3^4 - b_1^5 + 4b_1b_2b_3b_4^2 - b_2b_3^4 - b_4b_3^4 - b_4b_1^4 - b_3b_4^4 - b_4^5 - b_4b_2^4 - b_1b_4^4 - b_3^5 - b_2b_4^4 - b_1b_2^4 - b_2b_1^4 - b_3b_1^4 + 4b_1b_2b_3^2b_4 - b_3b_2^4 - b_2^5 + 4b_1^2b_2b_3b_4 + 4b_1b_2^2b_3b_4)u^2 + (b_2b_3^5 + b_1b_3^5 + b_1^5b_3 + b_1b_4^5 + b_1^5b_4 + b_2^5b_3 - 6b_1b_2b_3b_4^3 - 6b_1b_2b_3^3b_4 - 6b_1b_2^3b_3b_4 - 6b_1^3b_2b_3b_4 + b_2b_4b_1^4 + b_3b_4b_2^4 + b_3b_4b_1^4 + b_1b_2b_4^4 + b_1b_2b_3^4 + b_1b_3b_4^4 + b_1b_3b_2^4 + b_1b_4b_3^4 + b_1b_4b_2^4 + b_2b_3b_4^4 + b_2b_3b_1^4 + b_2b_4b_3^4 + b_2b_4^5 + b_2^5b_4 + b_3b_4^5 + b_3^5b_4 + b_1^5b_2 + b_1b_2^5)u - b_2b_3b_4^5 + 3b_1b_2b_4b_3^4 - b_1b_3b_4^5 - b_1b_3^5b_4 + 3b_1b_3b_4b_2^4 + 3b_1b_2b_3b_4^4 - b_1^5b_3b_4 - b_2b_3^5b_4 - b_2^5b_3b_4 - b_1b_2^5b_4 + 3b_2b_3b_4b_1^4 - b_1^5b_2b_3 - b_1b_2b_4^5 - b_1b_2b_3^5 - b_1^5b_2b_4 - b_1b_2^5b_3 \geq 0.$$



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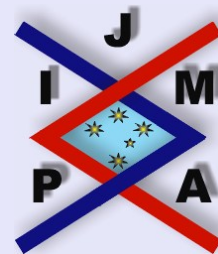
Inequality (2.2) is symmetrical for  $b_k$  ( $1 \leq k \leq 4, k \in N^*$ ), so there is no harm in supposing that  $b_1 \leq b_2 \leq b_3 \leq b_4$ . Then we can set

$$(2.3) \quad \begin{cases} b_2 = b_1 + c_1; \\ b_3 = b_1 + c_1 + c_2; \\ b_4 = b_1 + c_1 + c_2 + c_3; \\ u = 2(b_1 + c_1 + c_2 + c_3) + c_4, \end{cases}$$

where  $c_i \geq 0$  ( $1 \leq i \leq 4, i \in N^*$ ).

The substitution (2.3) was called a difference substitution in [4] (see also [5]). Substituting (2.3) in (2.2), we obtain the result (3.1) (see Appendix). Since every monomial on the left of (3.1) is nonnegative, the last inequality obviously holds, then the inequality (1.1) holds when  $n = 4$ .

Thus, the proof of the inequality (1.1) ( $n = 4$ ) is completed.  $\square$



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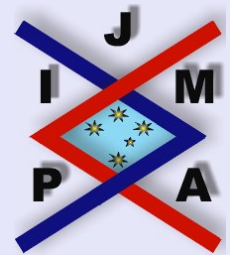
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### 3. Remarks

**Remark 1.** *In the same manner, we can also prove the inequality (1.1) holds when  $n = 5$ .*

**Remark 2.** *The operations in this paper were implemented using mathematics software Maple 9.0.*



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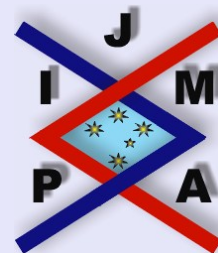
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# Appendix

$$(3.1) \quad 30b_1c_4^2c_1^4 + 10b_1c_1^6 + 120c_4b_1^3c_1c_2c_3 + 720c_4b_1c_1^2c_2c_3^2 \\ + 9c_4c_1^6 + 824c_4b_1c_1^2c_2^2c_3 + 10c_4c_2^6 + 498c_4b_1c_1^3c_2c_3 \\ + 13b_1^2c_1^5 + 51b_1^2c_3^5 + 44b_1^2c_2^5 + 8b_1^3c_1^4 + 40b_1^3c_3^4 \\ + 32b_1^3c_2^4 + 120c_4^2b_1^2c_1c_2c_3 + 10b_1^4c_3^3 + 20b_1^4c_1c_2c_3 \\ + 152b_1^3c_1^2c_2c_3 + 208b_1^3c_1c_2^2c_3 + 200b_1^3c_1c_2c_3^2 + 346b_1^2c_1^3c_2c_3 \\ + 624b_1^2c_1^2c_2^2c_3 + 600b_1^2c_1^2c_2c_3^2 + 550b_1^2c_1c_2c_3^3 \\ + 556b_1^2c_1c_2^3c_3 + 780b_1^2c_1c_2^2c_3^2 + 904b_1c_1c_2^2c_3^3 + 318b_1c_1^4c_2c_3 \\ + 696b_1c_1^3c_2^2c_3 + 652b_1c_1^3c_2c_3^2 + 756b_1c_1^2c_2c_3^3 + 824b_1c_1^2c_2^3c_3 \\ + 1128b_1c_1^2c_2^2c_3^2 + 490b_1c_1c_2c_3^4 + 504b_1c_1c_2^4c_3 + 912b_1c_1c_2^3c_3^2 \\ + 2b_1^4c_1^3 + 8b_1^4c_2^3 + 74b_1^2c_1^4c_2 + 40b_1^3c_1^3c_2 + 8b_1^4c_1^2c_2 \\ + 10b_1^4c_1^2c_3 + 12b_1^4c_1c_2^2 + 10b_1^4c_1c_3^2 + 20b_1^4c_2^2c_3 + 20b_1^4c_2c_3^2 \\ + 48b_1^3c_1^3c_3 + 80b_1^3c_1^2c_2^2 + 80b_1^3c_1^2c_3^2 + 80b_1^3c_1c_2^3 \\ + 80b_1^3c_1c_3^3 + 120b_1^3c_2c_3^3 + 112b_1^3c_2^3c_3 + 160b_1^3c_2^2c_3^2 \\ + 158b_1^2c_1c_2^4 + 165b_1^2c_1c_3^4 + 83b_1^2c_1^4c_3 + 178b_1^2c_1^3c_2^2 \\ + 180b_1^2c_1^3c_3^2 + 232b_1^2c_1^2c_2^3 + 220b_1^2c_1^2c_3^3 + 210b_1^2c_2c_3^4 \\ + 370b_1^2c_2^2c_3^3 + 194b_1^2c_2^4c_3 + 360b_1^2c_2^3c_3^2 + 112b_1c_1c_3^5 \\ + 116b_1c_1c_2^5 + 236b_1c_1^2c_2^4 + 210b_1c_1^2c_3^4 + 62b_1c_1^5c_2 + 64b_1c_1^5c_3 \\ + 168b_1c_1^4c_2^2 + 158b_1c_1^4c_3^2 + 260b_1c_1^3c_2^3 + 224b_1c_1^3c_3^3 \\ + 122b_1c_2c_3^5 + 3c_1^7 + 4c_2^7 + 356b_1c_2^3c_3^3 + 124b_1c_2^5c_3$$



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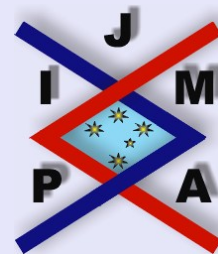
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$$\begin{aligned}
& + 276 b_1 c_2^4 c_3^2 + 280 c_1^2 c_2 c_3^4 + 534 c_1^2 c_2^2 c_3^3 + 24 b_1 c_2^6 + 72 c_1^2 c_2^5 + 61 c_1^2 c_3^5 \\
& + 20 c_1^6 c_2 + 26 c_1 c_2^6 + 59 c_2^2 c_1^5 + 72 c_2^2 c_3^5 + 110 c_1^3 c_2^4 + 85 c_1^3 c_3^4 + 19 c_1^6 c_3 \\
& + 50 c_1^5 c_3^2 + 102 c_1^4 c_2^3 + 78 c_1^4 c_3^3 + 110 c_2^3 c_3^4 + 104 c_2^4 c_3^3 + 24 c_2^6 c_3 \\
& + 64 c_2^5 c_3^2 + 22 b_1 c_3^6 + 24 c_1 c_3^6 + 26 c_2 c_3^6 + 107 c_1^5 c_2 c_3 + 268 c_1^4 c_2^2 c_3 \\
& + 242 c_1^4 c_2 c_3^2 + 133 c_1 c_2 c_3^5 + 380 c_1^3 c_2^3 c_3 + 508 c_1^3 c_2^2 c_3^2 + 310 c_2^4 c_1^2 c_3 \\
& + 552 c_2^3 c_1^2 c_3^2 + 280 c_2^2 b_1 c_3^4 + 46 c_4 b_1^2 c_1^4 + 84 c_4 b_1^2 c_2^4 + 78 c_4 b_1^2 c_3^4 \\
& + 6 c_4 b_1^4 c_1^2 + 8 c_4 b_1^4 c_2^2 + 6 c_4 b_1^4 c_3^2 + 56 c_4 c_1 c_2^5 + 42 c_4 c_1 c_3^5 + 28 c_4 c_1^3 b_1^3 \\
& + 48 c_4 c_2 c_1^5 + 45 c_4 c_2 c_3^5 + 192 c_4 c_1 b_1^2 c_3^3 + 326 c_1^3 c_2 c_3^3 + 305 c_1 c_2^2 c_3^4 \\
& + 386 c_1 c_2^3 c_3^3 + 134 c_1 c_2^5 c_3 + 298 c_1 c_2^4 c_3^2 + 44 c_4 b_1^3 c_3^3 + 33 c_4 b_1 c_1^5 \\
& + 52 c_4 b_1 c_2^5 + 39 c_4 b_1 c_3^5 + 168 c_4 b_1^2 c_1^3 c_2 + 8 c_4 b_1^4 c_1 c_2 + 4 c_4 b_1^4 c_1 c_3 \\
& + 8 c_4 b_1^4 c_2 c_3 + 72 c_4 b_1^3 c_1^2 c_2 + 60 c_4 b_1^3 c_1^2 c_3 + 88 c_4 b_1^3 c_1 c_2^2 \\
& + 60 c_4 b_1^3 c_1 c_3^2 + 104 c_4 b_1^3 c_2^2 c_3 + 96 c_4 b_1^3 c_2 c_3^2 + 150 c_4 c_1^4 b_1 c_2 \\
& + 152 c_4 c_1^3 b_1^2 c_3 + 276 c_4 c_1^2 b_1^2 c_2^2 + 208 c_4 c_1^2 b_1^2 c_3^2 + 187 c_4 c_1^4 c_2 c_3 \\
& + 376 c_4 c_1^3 c_2^2 c_3 + 320 c_4 c_1^3 c_2 c_3^2 + 214 c_4 c_1 b_1 c_2^4 + 169 c_4 c_1 b_1 c_3^4 \\
& + 252 c_4 c_2 b_1^2 c_3^3 + 306 c_4 c_2^2 b_1 c_1^3 + 240 c_4 c_2^3 b_1^2 c_1 + 264 c_4 c_2^3 b_1^2 c_3 \\
& + 352 c_4 c_2^2 b_1^2 c_3^2 + 404 c_4 c_2^3 c_1^2 c_3 + 516 c_4 c_2^2 c_1^2 c_3^2 + 182 c_4 c_2 b_1 c_3^4 \\
& + 436 c_4 c_1^2 b_1^2 c_2 c_3 + 552 c_4 c_1 b_1^2 c_2^2 c_3 + 496 c_4 c_1 b_1^2 c_2 c_3^2 + 48 c_4 c_2^3 b_1^3 \\
& + 130 c_4 c_1^2 c_2^4 + 91 c_4 c_1^2 c_3^4 + 42 c_4 c_1^5 c_3 + 115 c_4 c_1^4 c_2^2 + 82 c_4 c_1^4 c_3^2 \\
& + 160 c_4 c_1^3 c_2^3 + 104 c_4 c_1^3 c_3^3 + 105 c_4 c_2^2 c_3^4 + 130 c_4 c_2^3 c_3^3 + 46 c_4 c_2^5 c_3 \\
& + 98 c_4 c_2^4 c_3^2 + 135 c_4 b_1 c_1^4 c_3 + 228 c_4 b_1 c_1^3 c_3^2 + 352 c_4 b_1 c_1^2 c_2^3 \\
& + 252 c_4 b_1 c_1^2 c_3^3 + 338 c_4 b_1 c_2^2 c_3^3 + 202 c_4 b_1 c_2^4 c_3 + 344 c_4 b_1 c_2^3 c_3^2
\end{aligned}$$



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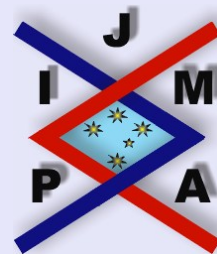
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$$\begin{aligned}
& + 196 c_4 c_1 c_2 c_3^4 + 364 c_4 c_1 c_2^2 c_3^3 + 216 c_4 c_1 c_2^4 c_3 + 368 c_4 c_1 c_2^3 c_3^2 \\
& + 338 c_4 c_1^2 c_2 c_3^3 + 590 c_4 b_1 c_1 c_2 c_3^3 + 668 c_4 b_1 c_1 c_2^3 c_3 + 868 c_4 b_1 c_1 c_2^2 c_3^2 \\
& + 16 c_4^2 b_1^3 c_1 c_2 + 8 c_4^2 b_1^3 c_1 c_3 + 80 c_4^2 b_1^2 c_1^2 c_2 + 58 c_4^2 b_1^2 c_1^2 c_3 \\
& + 96 c_4^2 b_1^2 c_1 c_2^2 + 62 c_4^2 b_1^2 c_1 c_3^2 + 16 c_4^2 b_1^3 c_2 c_3 + 96 c_4^2 b_1^2 c_2^2 c_3 \\
& + 88 c_4^2 b_1^2 c_2 c_3^2 + 12 c_4^2 b_1^3 c_1^2 + 34 c_4^2 b_1^2 c_1^3 + 16 c_4^2 b_1^3 c_2^2 \\
& + 48 c_4^2 b_1^2 c_2^3 + 12 c_4^2 b_1^3 c_3^2 + 96 b_1 c_4^2 c_1^3 c_2 + 148 b_1 c_4^2 c_1^2 c_2^2 \\
& + 120 b_1 c_4^2 c_1 c_2^3 + 72 b_1 c_4^2 c_1^3 c_3 + 88 b_1 c_4^2 c_1^2 c_3^2 + 196 b_1 c_4^2 c_1^2 c_2 c_3 \\
& + 240 b_1 c_4^2 c_1 c_2^2 c_3 + 208 b_1 c_4^2 c_1 c_2 c_3^2 + 9 c_4^2 c_1^5 + 8 c_4^2 c_2^5 \\
& + 36 c_4^2 b_1 c_2^4 + 38 c_4^2 c_1 c_2^4 + 36 c_4^2 c_1^4 c_2 + 68 c_4^2 c_1^3 c_2^2 + 72 c_4^2 c_1^2 c_2^3 \\
& + 8 c_3^6 c_4 + 5 c_4^2 c_3^5 + 4 c_3^7 + 38 c_4^2 b_1^2 c_3^3 + 22 c_4^2 b_1 c_3^4 + 92 c_4^2 c_1^3 c_2 c_3 \\
& + 144 c_4^2 c_1^2 c_2^2 c_3 + 120 c_4^2 c_1^2 c_2 c_3^2 + 23 c_4^2 c_1 c_3^4 + 27 c_4^2 c_1^4 c_3 + 38 c_4^2 c_1^3 c_3^2 \\
& + 42 c_4^2 c_1^2 c_3^3 + 24 c_4^2 c_2 c_3^4 + 46 c_4^2 c_2^2 c_3^3 + 26 c_4^2 c_2^4 c_3 + 44 c_4^2 c_2^3 c_3^2 \\
& + 80 c_4^2 b_1 c_1 c_3^3 + 84 c_4^2 b_1 c_2 c_3^3 + 96 c_4^2 b_1 c_2^3 c_3 + 120 c_4^2 b_1 c_2^2 c_3^2 \\
& + 88 c_4^2 c_1 c_2 c_3^3 + 100 c_4^2 c_1 c_2^3 c_3 + 126 c_4^2 c_1 c_2^2 c_3^2 + 3 c_4^3 c_1^4 + 6 c_4^3 b_1^2 c_1^2 \\
& + 8 c_4^3 b_1 c_1^3 + 2 c_4^3 c_2^4 + c_4^3 c_3^4 + 8 c_4^3 b_1^2 c_1 c_2 + 4 c_4^3 b_1^2 c_1 c_3 \\
& + 8 c_4^3 b_1^2 c_2 c_3 + 12 c_4^3 c_1^2 c_2 c_3 + 16 c_4^3 b_1 c_1^2 c_2 + 8 c_4^3 b_1 c_1^2 c_3 \\
& + 20 c_4^3 b_1 c_1 c_2^2 + 12 c_4^3 b_1 c_1 c_3^2 + 12 c_4^3 b_1 c_2^2 c_3 + 12 c_4^3 b_1 c_2 c_3^2 \\
& + 12 c_4^3 c_1 c_2^2 c_3 + 12 c_4^3 c_1 c_2 c_3^2 + 20 c_4^3 b_1 c_1 c_2 c_3 + 8 c_4^3 b_1^2 c_2^2 \\
& + 6 c_4^3 b_1^2 c_3^2 + 8 c_4^3 b_1 c_2^3 + 4 c_4^3 b_1 c_3^3 + 8 c_4^3 c_1^3 c_2 \\
& + 4 c_4^3 c_1^3 c_3 + 12 c_4^3 c_1^2 c_2^2 + 6 c_4^3 c_1^2 c_3^2 + 8 c_4^3 c_1 c_2^3 \\
& + 4 c_4^3 c_1 c_3^3 + 4 c_4^3 c_2 c_3^3 + 4 c_4^3 c_2^3 c_3 + 6 c_4^3 c_2^2 c_3^2 \geq 0.
\end{aligned}$$



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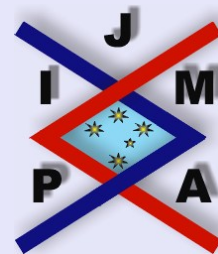
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## References

- [1] HORST ALZER, Problem 10337, *Amer. Math. Monthly*, **100**(8) (1993), 798.
- [2] JI-CHANG KUANG. *Applied Inequalities* (in Chinese), Shandong Science and Technology Press, 3rd. Ed., 2004, p. 156.
- [3] M. VOWE, An unsettled problem, *Amer. Math. Monthly*, **102**(7) (1995), 659–660.
- [4] L. YANG, Solving harder problems with lesser mathematics, *Proceedings of the 10th Asian Technology Conference in Mathematics*, December 12-16, 2005, Cheong-Ju, South Korea.
- [5] BAO-QIAN LIU, The generating operation and its application in the proof of the symmetrical inequality in  $n$  variables, *Journal of Guangdong Education Institute* (in Chinese), **25**(3) (2005), 10–14.



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