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ON ONE OF H. ALZER'S PROBLEMS

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Abstract

In this short note, the author solves an inequality problem which was posed by H. Alzer with difference substitution.

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Contents

1. The Problem

In 1993, H. Alzer posed the following inequality problem in [1]. In 2004, Ji-Chang Kuang reposed the problem in his monograph [2].

Problem 1. *Let* $a_1, \dots, a_n$ $(n \in N^*)$ *be real numbers with* $a_i \in (0, \frac{1}{2}]$*, then prove or disprove*

$$\prod_{k=1}^{n} \left(\frac{a_k}{1 - a_k} \right) \le \frac{\sum_{k=1}^{n} a_k^n}{\sum_{k=1}^{n} (1 - a_k)^n} \tag{1.1}$$

where $n = 4$ *or* $n = 5$*.*

In 1995, Michael Vowe pointed out that the inequality (1.1) holds when $n \le 3$ $(n \in N^*)$ and does not hold when $n \ge 6$ $(n \in N^*)$ in [3].

2. Solution of The Problem

In this section, we show the reader the proof of the inequality (1.1) while $n = 4$.

Proof. We set $a_k = \frac{b_k}{u}$ $(k = 1, 2, 3, 4)$, where $u > 0$ and $0 < a_k \le \frac{1}{2}$ $(k = 1, 2, 3, 4)$, then $0 < b_k \le \frac{1}{2}u$ $(k = 1, 2, 3, 4)$. So the inequality (1.1) is equivalent to the inequality as follows.

$$\prod_{k=1}^{4}\left(\frac{b_k}{u-b_k}\right) \le \frac{\sum_{k=1}^{4} b_k^4}{\sum_{k=1}^{4}(u-b_k)^4}. \tag{2.1}$$

Inequality (2.1) is equivalent to the following inequality.

$$\begin{aligned}
&({b_3}^4 + {b_4}^4 + {b_2}^4 + {b_1}^4 - 4\,b_1b_2b_3b_4)u^3 + (-b_1{b_3}^4 - {b_1}^5 + 4\,b_1b_2b_3{b_4}^2 \\
&- b_2{b_3}^4 - b_4{b_3}^4 - b_4{b_1}^4 - b_3{b_4}^4 - {b_4}^5 - b_4{b_2}^4 - b_1{b_4}^4 - {b_3}^5 \\
&- b_2{b_4}^4 - b_1{b_2}^4 - b_2{b_1}^4 - b_3{b_1}^4 + 4\,b_1b_2{b_3}^2b_4 - b_3{b_2}^4 - {b_2}^5 \\
&+ 4\,{b_1}^2b_2b_3b_4 + 4\,b_1{b_2}^2b_3b_4)u^2 + (b_2{b_3}^5 + b_1{b_3}^5 + {b_1}^5b_3 + b_1{b_4}^5 \\
&+ {b_1}^5b_4 + {b_2}^5b_3 - 6\,b_1b_2b_3{b_4}^3 - 6\,b_1b_2{b_3}^3b_4 - 6\,b_1{b_2}^3b_3b_4 \\
&- 6\,{b_1}^3b_2b_3b_4 + b_2b_4{b_1}^4 + b_3b_4{b_2}^4 + b_3b_4{b_1}^4 + b_1b_2{b_4}^4 \\
&+ b_1b_2{b_3}^4 + b_1b_3{b_4}^4 + b_1b_3{b_2}^4 + b_1b_4{b_3}^4 + b_1b_4{b_2}^4 + b_2b_3{b_4}^4 \\
&+ b_2b_3{b_1}^4 + b_2b_4{b_3}^4 + b_2{b_4}^5 + {b_2}^5b_4 + b_3{b_4}^5 + {b_3}^5b_4 + {b_1}^5b_2 + b_1{b_2}^5)u \\
&- b_2b_3{b_4}^5 + 3\,b_1b_2b_4{b_3}^4 - b_1b_3{b_4}^5 - b_1{b_3}^5b_4 + 3\,b_1b_3b_4{b_2}^4 \\
&+ 3\,b_1b_2b_3{b_4}^4 - {b_1}^5b_3b_4 - b_2{b_3}^5b_4 - {b_2}^5b_3b_4 - b_1{b_2}^5b_4 + 3\,b_2b_3b_4{b_1}^4 \\
&- {b_1}^5b_2b_3 - b_1b_2{b_4}^5 - b_1b_2{b_3}^5 - {b_1}^5b_2b_4 - b_1{b_2}^5b_3 \ge 0.
\end{aligned} \tag{2.2}$$

Inequality (2.2) is symmetrical for b_k $(1 \leq k \leq 4,\ k \in N^*)$, so there is no harm in supposing that $b_1 \leq b_2 \leq b_3 \leq b_4$. Then we can set

$$\begin{cases} b_2 = b_1 + c_1; \\ b_3 = b_1 + c_1 + c_2; \\ b_4 = b_1 + c_1 + c_2 + c_3; \\ u = 2(b_1 + c_1 + c_2 + c_3) + c_4, \end{cases} \tag{2.3}$$

where $c_i \geq 0\ (1 \leq i \leq 4, i \in N^*)$.

The substitution (2.3) was called a difference substitution in [4] (see also [5]). Substituting (2.3) in (2.2), we obtain the result (3.1) (see Appendix). Since every monomial on the left of (3.1) is nonnegative, the last inequality obviously holds, then the inequality (1.1) holds when $n = 4$.

Thus, the proof of the inequality (1.1) $(n = 4)$ is completed. □

3. Remarks

Remark 1. *In the same manner, we can also prove the inequality* (1.1) *holds when* $n = 5$.

Remark 2. *The operations in this paper were implemented using mathematics software Maple 9.0.*

Appendix

$$30b_1{c_4}^2{c_1}^4 + 10\,b_1{c_1}^6 + 120\,c_4{b_1}^3c_1c_2c_3 + 720\,c_4b_1{c_1}^2c_2{c_3}^2$$
$$+ 9\,c_4{c_1}^6 + 824\,c_4b_1{c_1}^2{c_2}^2c_3 + 10\,c_4{c_2}^6 + 498\,c_4b_1{c_1}^3c_2c_3$$
$$+ 13\,{b_1}^2{c_1}^5 + 51\,{b_1}^2{c_3}^5 + 44\,{b_1}^2{c_2}^5 + 8\,{b_1}^3{c_1}^4 + 40\,{b_1}^3{c_3}^4$$
$$+ 32\,{b_1}^3{c_2}^4 + 120\,{c_4}^2{b_1}^2c_1c_2c_3 + 10\,{b_1}^4{c_3}^3 + 20\,{b_1}^4c_1c_2c_3$$
$$+ 152\,{b_1}^3{c_1}^2c_2c_3 + 208\,{b_1}^3c_1{c_2}^2c_3 + 200\,{b_1}^3c_1c_2{c_3}^2 + 346\,{b_1}^2{c_1}^3c_2c_3$$
$$+ 624\,{b_1}^2{c_1}^2{c_2}^2c_3 + 600\,{b_1}^2{c_1}^2c_2{c_3}^2 + 550\,{b_1}^2c_1c_2{c_3}^3$$
$$+ 556\,{b_1}^2c_1{c_2}^3c_3 + 780\,{b_1}^2c_1{c_2}^2{c_3}^2 + 904\,b_1c_1{c_2}^2{c_3}^3 + 318\,b_1{c_1}^4c_2c_3$$
$$+ 696\,b_1{c_1}^3{c_2}^2c_3 + 652\,b_1{c_1}^3c_2{c_3}^2 + 756\,b_1{c_1}^2c_2{c_3}^3 + 824\,b_1{c_1}^2{c_2}^3c_3$$
$$+ 1128\,b_1{c_1}^2{c_2}^2{c_3}^2 + 490\,b_1c_1c_2{c_3}^4 + 504\,b_1c_1{c_2}^4c_3 + 912\,b_1c_1{c_2}^3{c_3}^2$$
$$+ 2\,{b_1}^4{c_1}^3 + 8\,{b_1}^4{c_2}^3 + 74\,{b_1}^2{c_1}^4c_2 + 40\,{b_1}^3{c_1}^3c_2 + 8\,{b_1}^4{c_1}^2c_2$$
$$+ 10\,{b_1}^4{c_1}^2c_3 + 12\,{b_1}^4c_1{c_2}^2 + 10\,{b_1}^4c_1{c_3}^2 + 20\,{b_1}^4{c_2}^2c_3 + 20\,{b_1}^4c_2{c_3}^2$$
$$+ 48\,{b_1}^3{c_1}^3c_3 + 80\,{b_1}^3{c_1}^2{c_2}^2 + 80\,{b_1}^3{c_1}^2{c_3}^2 + 80\,{b_1}^3c_1{c_2}^3$$
$$+ 80\,{b_1}^3c_1{c_3}^3 + 120\,{b_1}^3c_2{c_3}^3 + 112\,{b_1}^3{c_2}^3c_3 + 160\,{b_1}^3{c_2}^2{c_3}^2$$
$$+ 158\,{b_1}^2c_1{c_2}^4 + 165\,{b_1}^2c_1{c_3}^4 + 83\,{b_1}^2{c_1}^4c_3 + 178\,{b_1}^2{c_1}^3{c_2}^2$$
$$+ 180\,{b_1}^2{c_1}^3{c_3}^2 + 232\,{b_1}^2{c_1}^2{c_2}^3 + 220\,{b_1}^2{c_1}^2{c_3}^3 + 210\,{b_1}^2c_2{c_3}^4$$
$$+ 370\,{b_1}^2{c_2}^2{c_3}^3 + 194\,{b_1}^2{c_2}^4c_3 + 360\,{b_1}^2{c_2}^3{c_3}^2 + 112\,b_1c_1{c_3}^5$$
$$+ 116\,b_1c_1{c_2}^5 + 236\,b_1{c_1}^2{c_2}^4 + 210\,b_1{c_1}^2{c_3}^4 + 62\,b_1{c_1}^5c_2 + 64\,b_1{c_1}^5c_3$$
$$+ 168\,b_1{c_1}^4{c_2}^2 + 158\,b_1{c_1}^4{c_3}^2 + 260\,b_1{c_1}^3{c_2}^3 + 224\,b_1{c_1}^3{c_3}^3$$
$$+ 122\,b_1c_2{c_3}^5 + 3\,{c_1}^7 + 4\,{c_2}^7 + 356\,b_1{c_2}^3{c_3}^3 + 124\,b_1{c_2}^5c_3 \tag{3.1}$$

$$+ 276\, b_1 {c_2}^4 {c_3}^2 + 280\, {c_1}^2 c_2 {c_3}^4 + 534\, {c_1}^2 {c_2}^2 {c_3}^3 + 24\, b_1 {c_2}^6 + 72\, {c_1}^2 {c_2}^5 + 61\, {c_1}^2 {c_3}^5$$
$$+ 20\, {c_1}^6 c_2 + 26\, c_1 {c_2}^6 + 59\, {c_2}^2 {c_1}^5 + 72\, {c_2}^2 {c_3}^5 + 110\, {c_1}^3 {c_2}^4 + 85\, {c_1}^3 {c_3}^4 + 19\, {c_1}^6 c_3$$
$$+ 50\, {c_1}^5 {c_3}^2 + 102\, {c_1}^4 {c_2}^3 + 78\, {c_1}^4 {c_3}^3 + 110\, {c_2}^3 {c_3}^4 + 104\, {c_2}^4 {c_3}^3 + 24\, {c_2}^6 c_3$$
$$+ 64\, {c_2}^5 {c_3}^2 + 22\, b_1 {c_3}^6 + 24\, c_1 {c_3}^6 + 26\, c_2 {c_3}^6 + 107\, {c_1}^5 c_2 c_3 + 268\, {c_1}^4 {c_2}^2 c_3$$
$$+ 242\, {c_1}^4 c_2 {c_3}^2 + 133\, c_1 c_2 {c_3}^5 + 380\, {c_1}^3 {c_2}^3 c_3 + 508\, {c_1}^3 {c_2}^2 {c_3}^2 + 310\, {c_2}^4 {c_1}^2 c_3$$
$$+ 552\, {c_2}^3 {c_1}^2 {c_3}^2 + 280\, {c_2}^2 b_1 {c_3}^4 + 46\, c_4 {b_1}^2 {c_1}^4 + 84\, c_4 {b_1}^2 {c_2}^4 + 78\, c_4 {b_1}^2 {c_3}^4$$
$$+ 6\, c_4 {b_1}^4 {c_1}^2 + 8\, c_4 {b_1}^4 {c_2}^2 + 6\, c_4 {b_1}^4 {c_3}^2 + 56\, c_4 c_1 {c_2}^5 + 42\, c_4 c_1 {c_3}^5 + 28\, c_4 {c_1}^3 {b_1}^3$$
$$+ 48\, c_4 c_2 {c_1}^5 + 45\, c_4 c_2 {c_3}^5 + 192\, c_4 c_1 {b_1}^2 {c_3}^3 + 326\, {c_1}^3 c_2 {c_3}^3 + 305\, c_1 {c_2}^2 {c_3}^4$$
$$+ 386\, c_1 {c_2}^3 {c_3}^3 + 134\, c_1 {c_2}^5 c_3 + 298\, c_1 {c_2}^4 {c_3}^2 + 44\, c_4 {b_1}^3 {c_3}^3 + 33\, c_4 b_1 {c_1}^5$$
$$+ 52\, c_4 b_1 {c_2}^5 + 39\, c_4 b_1 {c_3}^5 + 168\, c_4 {b_1}^2 {c_1}^3 c_2 + 8\, c_4 {b_1}^4 c_1 c_2 + 4\, c_4 {b_1}^4 c_1 c_3$$
$$+ 8\, c_4 {b_1}^4 c_2 c_3 + 72\, c_4 {b_1}^3 {c_1}^2 c_2 + 60\, c_4 {b_1}^3 {c_1}^2 c_3 + 88\, c_4 {b_1}^3 c_1 {c_2}^2$$
$$+ 60\, c_4 {b_1}^3 c_1 {c_3}^2 + 104\, c_4 {b_1}^3 {c_2}^2 c_3 + 96\, c_4 {b_1}^3 c_2 {c_3}^2 + 150\, c_4 {c_1}^4 b_1 c_2$$
$$+ 152\, c_4 {c_1}^3 {b_1}^2 c_3 + 276\, c_4 {c_1}^2 {b_1}^2 {c_2}^2 + 208\, c_4 {c_1}^2 {b_1}^2 {c_3}^2 + 187\, c_4 {c_1}^4 c_2 c_3$$
$$+ 376\, c_4 {c_1}^3 {c_2}^2 c_3 + 320\, c_4 {c_1}^3 c_2 {c_3}^2 + 214\, c_4 c_1 b_1 {c_2}^4 + 169\, c_4 c_1 b_1 {c_3}^4$$
$$+ 252\, c_4 c_2 {b_1}^2 {c_3}^3 + 306\, c_4 {c_2}^2 b_1 {c_1}^3 + 240\, c_4 {c_2}^3 {b_1}^2 c_1 + 264\, c_4 {c_2}^3 {b_1}^2 c_3$$
$$+ 352\, c_4 {c_2}^2 {b_1}^2 {c_3}^2 + 404\, c_4 {c_2}^3 {c_1}^2 c_3 + 516\, c_4 {c_2}^2 {c_1}^2 {c_3}^2 + 182\, c_4 c_2 b_1 {c_3}^4$$
$$+ 436\, c_4 {c_1}^2 {b_1}^2 c_2 c_3 + 552\, c_4 c_1 {b_1}^2 {c_2}^2 c_3 + 496\, c_4 c_1 {b_1}^2 c_2 {c_3}^2 + 48\, c_4 {c_2}^3 {b_1}^3$$
$$+ 130\, c_4 {c_1}^2 {c_2}^4 + 91\, c_4 {c_1}^2 {c_3}^4 + 42\, c_4 {c_1}^5 c_3 + 115\, c_4 {c_1}^4 {c_2}^2 + 82\, c_4 {c_1}^4 {c_3}^2$$
$$+ 160\, c_4 {c_1}^3 {c_2}^3 + 104\, c_4 {c_1}^3 {c_3}^3 + 105\, c_4 {c_2}^2 {c_3}^4 + 130\, c_4 {c_2}^3 {c_3}^3 + 46\, c_4 {c_2}^5 c_3$$
$$+ 98\, c_4 {c_2}^4 {c_3}^2 + 135\, c_4 b_1 {c_1}^4 c_3 + 228\, c_4 b_1 {c_1}^3 {c_3}^2 + 352\, c_4 b_1 {c_1}^2 {c_2}^3$$
$$+ 252\, c_4 b_1 {c_1}^2 {c_3}^3 + 338\, c_4 b_1 {c_2}^2 {c_3}^3 + 202\, c_4 b_1 {c_2}^4 c_3 + 344\, c_4 b_1 {c_2}^3 {c_3}^2$$

$$
\begin{aligned}
&+196\,c_4c_1c_2c_3^4+364\,c_4c_1c_2^2c_3^3+216\,c_4c_1c_2^4c_3+368\,c_4c_1c_2^3c_3^2\\
&+338\,c_4c_1^2c_2c_3^3+590\,c_4b_1c_1c_2c_3^3+668\,c_4b_1c_1c_2^3c_3+868\,c_4b_1c_1c_2^2c_3^2\\
&+16\,c_4^2b_1^3c_1c_2+8\,c_4^2b_1^3c_1c_3+80\,c_4^2b_1^2c_1^2c_2+58\,c_4^2b_1^2c_1^2c_3\\
&+96\,c_4^2b_1^2c_1c_2^2+62\,c_4^2b_1^2c_1c_3^2+16\,c_4^2b_1^3c_2c_3+96\,c_4^2b_1^2c_2^2c_3\\
&+88\,c_4^2b_1^2c_2c_3^2+12\,c_4^2b_1^3c_1^2+34\,c_4^2b_1^2c_1^3+16\,c_4^2b_1^3c_2^2\\
&+48\,c_4^2b_1^2c_2^3+12\,c_4^2b_1^3c_3^2+96\,b_1c_4^2c_1^3c_2+148\,b_1c_4^2c_1^2c_2^2\\
&+120\,b_1c_4^2c_1c_2^3+72\,b_1c_4^2c_1^3c_3+88\,b_1c_4^2c_1^2c_3^2+196\,b_1c_4^2c_1^2c_2c_3\\
&+240\,b_1c_4^2c_1c_2^2c_3+208\,b_1c_4^2c_1c_2c_3^2+9\,c_4^2c_1^5+8\,c_4^2c_2^5\\
&+36\,c_4^2b_1c_2^4+38\,c_4^2c_1c_2^4+36\,c_4^2c_1^4c_2+68\,c_4^2c_1^3c_2^2+72\,c_4^2c_1^2c_2^3\\
&+8\,c_3^6c_4+5\,c_4^2c_3^5+4\,c_3^7+38\,c_4^2b_1^2c_3^3+22\,c_4^2b_1c_3^4+92\,c_4^2c_1^3c_2c_3\\
&+144\,c_4^2c_1^2c_2^2c_3+120\,c_4^2c_1^2c_2c_3^2+23\,c_4^2c_1c_3^4+27\,c_4^2c_1^4c_3+38\,c_4^2c_1^3c_3^2\\
&+42\,c_4^2c_1^2c_3^3+24\,c_4^2c_2c_3^4+46\,c_4^2c_2^2c_3^3+26\,c_4^2c_2^4c_3+44\,c_4^2c_2^3c_3^2\\
&+80\,c_4^2b_1c_1c_3^3+84\,c_4^2b_1c_2c_3^3+96\,c_4^2b_1c_2^3c_3+120\,c_4^2b_1c_2^2c_3^2\\
&+88\,c_4^2c_1c_2c_3^3+100\,c_4^2c_1c_2^3c_3+126\,c_4^2c_1c_2^2c_3^2+3\,c_4^3c_1^4+6\,c_4^3b_1^2c_1^2\\
&+8\,c_4^3b_1c_1^3+2\,c_4^3c_2^4+c_4^3c_3^4+8\,c_4^3b_1^2c_1c_2+4\,c_4^3b_1^2c_1c_3\\
&+8\,c_4^3b_1^2c_2c_3+12\,c_4^3c_1^2c_2c_3+16\,c_4^3b_1c_1^2c_2+8\,c_4^3b_1c_1^2c_3\\
&+20\,c_4^3b_1c_1c_2^2+12\,c_4^3b_1c_1c_3^2+12\,c_4^3b_1c_2^2c_3+12\,c_4^3b_1c_2c_3^2\\
&+12\,c_4^3c_1c_2^2c_3+12\,c_4^3c_1c_2c_3^2+20\,c_4^3b_1c_1c_2c_3+8\,c_4^3b_1^2c_2^2\\
&+6\,c_4^3b_1^2c_3^2+8\,c_4^3b_1c_2^3+4\,c_4^3b_1c_3^3+8\,c_4^3c_1^3c_2\\
&+4\,c_4^3c_1^3c_3+12\,c_4^3c_1^2c_2^2+6\,c_4^3c_1^2c_3^2+8\,c_4^3c_1c_2^3\\
&+4\,c_4^3c_1c_3^3+4\,c_4^3c_2c_3^3+4\,c_4^3c_2^3c_3+6\,c_4^3c_2^2c_3^2\ge 0.
\end{aligned}
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