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NOTES ON AN INEQUALITY

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ABSTRACT. In this note we prove a generalized version of an inequality which was first introduced by A. Q. Ngo, *et al.* and later generalized and proved by W. J. Liu, *et al.* in the paper: "On an open problem concerning an integral inequality", *J. Inequal. Pure & Appl. Math.*, **8**(3) 2007.

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1. INTRODUCTION

In [2] the following result was proved: If $f \geq 0$ is a continuous function on $[0,1]$ such that

$$\int_x^1 f(t)dt \geq \int_x^1 t dt, \quad \forall x \in [0,1], \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx, \quad \forall \alpha > 0.$$

The following question was raised in [2]: If f satisfies the above assumptions, under what additional assumptions can one claim that:

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \quad \forall \alpha, \beta > 0?$$

It was proved in [1] that if $f \geq 0$ is a continuous function on $[0,1]$ satisfying

$$\int_x^b f^\alpha(t)dt \geq \int_x^b t^\alpha dt, \quad \alpha, b > 0, \ \forall x \in [0,b],$$

then

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \quad \forall \beta > 0.$$

In this paper, we prove more general results, namely, Theorems 2.4 and 2.5 below.

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2. RESULTS AND PROOFS

Let us recall the following result:

Lemma 2.1 (Young's inequality)**.** *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we have:*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}.$$

Throughout the paper, $[a, b]$ denotes a bounded interval and all functions are real-valued. Let us prove the following lemma:

Lemma 2.2. *Let $f \in L^1[a, b]$, $g \in C^1[a, b]$. Suppose $f \geq 0$, $g > 0$ is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b],$$

then $\forall \alpha > 0$ the following inqualities hold

$$\int_a^b g^{\alpha}(x) f(x) dx \geq \int_a^b g^{\alpha+1}(x) dx, \tag{2.1}$$

$$\int_a^b f^{\alpha+1}(x) dx \geq \int_a^b f^{\alpha}(x) g(x) dx, \tag{2.2}$$

$$\int_a^b f^{\alpha+1}(x) dx \geq \int_a^b f(x) g^{\alpha}(x) dx. \tag{2.3}$$

Proof. First, let us prove (2.1). Let A, A^* denote

$$Af(x) := \int_a^x f(t)dt, \quad A^*f(x) := \int_x^b f(t)dt, \quad x \in [a, b],\ f \in L^1[a, b].$$

Note that these are continuous functions. From the assumption one has

$$A^*f(x) \geq A^*g(x), \quad \forall x \in [a, b].$$

This means

$$(A^*f - A^*g)(x) \geq 0, \quad \forall x \in [a, b].$$

Then $\forall h \in L^1[a, b],\ h \geq 0$, one obtains

$$\langle A^*f - A^*g, h \rangle := \int_a^b (A^*f - A^*g)(x) h(x) dx \geq 0. \tag{2.4}$$

Note that the left-hand side of (2.4) is finite since A^*f, A^*g are bounded and $h \in L^1[a, b]$. Thus, by Fubini's Theorem, one has

$$\langle f - g, Ah \rangle = \langle A^*f - A^*g, h \rangle \geq 0, \quad \forall h \geq 0, h \in L^1[a, b]. \tag{2.5}$$

Denote $h(x) = \alpha g(x)^{\alpha-1} g'(x)$. One has

$$Ah(x) = \int_a^x h(t)dt = g^{\alpha}(x) - g^{\alpha}(a), \quad \forall x \in [a, b].$$

By the assumption,

$$\langle f - g, g^{\alpha}(a) \rangle = g^{\alpha}(a) \int_a^b (f(x) - g(x)) dx \geq 0. \tag{2.6}$$

Since $h \geq 0$, from (2.5) and (2.6) one gets

$$\langle f - g, g^{\alpha} \rangle = \langle f - g, Ah \rangle + \langle f - g, g^{\alpha}(a) \rangle \geq 0, \quad \forall \alpha \geq 0. \tag{2.7}$$

Hence, (2.1) is obtained.

Since

$$(f(x)-g(x))(f^\alpha(x)-g^\alpha(x)) \ge 0, \quad \forall x \in [a,b], \quad \forall \alpha \ge 0,$$

one gets

$$\langle f-g, f^\alpha - g^\alpha \rangle \ge 0, \quad \forall \alpha \ge 0. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply

$$\langle f-g, f^\alpha \rangle = \langle f-g, f^\alpha - g^\alpha \rangle + \langle f-g, g^\alpha \rangle \ge 0, \quad \forall \alpha > 0.$$

Thus, (2.2) holds.

By Lemma 2.1,

$$\frac{1}{\alpha+1} f^{\alpha+1}(x) + \frac{\alpha}{\alpha+1} g^{\alpha+1}(x) \ge g^\alpha(x) f(x), \quad \forall x \in [a,b].$$

Thus,

$$\frac{1}{\alpha+1}\int_a^b f^{\alpha+1}(x)dx + \frac{\alpha}{\alpha+1}\int_a^b g^{\alpha+1}(x)dx \ge \int_a^b g^\alpha(x) f(x)dx, \quad \forall \alpha > 0. \tag{2.9}$$

From (2.1) and (2.9) one obtains

$$\int_a^b f^{\alpha+1}(x)dx \ge \int_a^b g^\alpha(x) f(x)dx, \quad \forall \alpha \ge 0.$$

The proof is complete. □

In particular, one has the following result

Corollary 2.3. *Suppose* $f \in L^1[a,b]$, $g \in C^1[a,b]$ $f, g \ge 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt \ge \int_x^b g(t)dt, \quad \forall x \in [a,b]$$

then the following inequality holds

$$\int_a^b f^\beta(x)dx \ge \int_a^b g^\beta(x)dx, \quad \forall \beta \ge 1. \tag{2.10}$$

Proof. Denote $f_\epsilon := f + \epsilon$, $g_\epsilon := g + \epsilon$ where $\epsilon > 0$. It is clear that $g_\epsilon > 0$ and

$$\int_x^b f_\epsilon(t)dt \ge \int_x^b g_\epsilon(t)dt, \quad \forall x \in [a,b].$$

By (2.1) and (2.3) in Lemma 2.2 one has

$$\int_a^b f_\epsilon^\beta(x)dx \ge \int_a^b g_\epsilon^\beta(x)dx, \quad \forall \beta \ge 1. \tag{2.11}$$

Inequality (2.10) is obtained from (2.11) by letting $\epsilon \to 0$. □

Theorem 2.4. *Suppose* $f \in L^1[a,b]$, $g \in C^1[a,b]$, $f, g \ge 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt \ge \int_x^b g(t)dt, \quad \forall x \in [a,b],$$

then $\forall \alpha, \beta \ge 0$, $\alpha + \beta \ge 1$, *the following inequality holds*

$$\int_a^b f^{\alpha+\beta}(x)dx \ge \int_a^b f^\alpha(x) g^\beta(x)dx. \tag{2.12}$$

Proof. Lemma 2.1 shows that

$$\frac{\alpha}{\alpha+\beta}f(x)^{\alpha+\beta}+\frac{\beta}{\alpha+\beta}g(x)^{\alpha+\beta}\ge f^{\alpha}(x)g^{\beta}(x),\quad \forall x\in[a,b],\,\forall\alpha,\beta>0.$$

Therefore, $\forall\alpha,\beta>0$ one has

$$\frac{\alpha}{\alpha+\beta}\int_a^b f(x)^{\alpha+\beta}dx+\frac{\beta}{\alpha+\beta}\int_a^b g(x)^{\alpha+\beta}dx\ge\int_a^b f^{\alpha}(x)g^{\beta}(x)dx. \tag{2.13}$$

Corollary 2.3 implies

$$\int_a^b f(x)^{\alpha+\beta}dx\ge\int_a^b g(x)^{\alpha+\beta}dx,\quad \forall\alpha,\beta\ge 0,\ \alpha+\beta\ge 1. \tag{2.14}$$

Inequality (2.12) is obtained from (2.13) and (2.14). □

Remark 1. Theorem 2.4 is not true if we drop the assumption $\alpha+\beta\ge 1$. Indeed, take $g\equiv 1$, $[a,b]=[0,1]$, and define

$$f(x)=c(1-x)^{c-1},\quad 0\le x\le 1,$$

where $c\in(0,1)$. One has

$$(1-x)^c=\int_x^1 f(t)dt\ge\int_x^1 g(t)dt=(1-x),\quad \forall x\in[0,1],\,c\in(0,1),$$

but

$$\frac{2\sqrt{c}}{c+1}=\int_0^1\sqrt{f(t)}dt<\int_0^1\sqrt{g(t)}dt=1,\quad \forall c\in(0,1).$$

Assuming that the condition $g\in C^1[a,b]$ can be dropped and replaced by $g\in L^1[a,b]$, we have the following result:

Theorem 2.5. *Suppose* $f,g\in L^1[a,b]$, $f,g\ge 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt\ge\int_x^b g(t)dt,\quad \forall x\in[a,b], \tag{2.15}$$

then

$$\int_a^b f^{\alpha+\beta}(x)dx\ge\int_a^b f^{\alpha}(x)g^{\beta}(x)dx,\quad \forall\alpha,\beta\ge 0,\ \alpha+\beta\ge 1. \tag{2.16}$$

Proof. Since $C^1[a,b]$ is dense in L^1, there exists a sequence $(g_n)_{n=1}^{\infty}\in C^1[a,b]$ such that g_n is nondecreasing, $g_n\nearrow g$ a.e. Since $g_n\nearrow g$ a.e.,

$$\int_x^b g(t)dt\ge\int_x^b g_n(t)dt,\quad \forall x\in[a,b],\,\forall n. \tag{2.17}$$

Inequalities (2.15), (2.17) and Theorem 2.4 imply

$$\int_a^b f^{\alpha+\beta}(x)dx\ge\int_a^b f^{\alpha}(x)g_n^{\beta}(x)dx,\quad \forall n,\,\forall\alpha,\beta\ge 0,\ \alpha+\beta\ge 1. \tag{2.18}$$

Since $f^{\alpha}g_n^{\beta}\nearrow f^{\alpha}g^{\beta}$ a.e., $f^{\alpha}g_n^{\beta}\ge 0$ is measurable satisfying (2.18), by the Monotone convergence theorem (see [3, 4]) $\|f^{\alpha}g_n^{\beta}\to f^{\alpha}g^{\beta}\|_{L^1}\to 0$ as $n\to\infty$. Hence,

$$\int_a^b f^{\alpha+\beta}(x)dx\ge\int_a^b f^{\alpha}(x)g^{\beta}(x)dx,\quad \forall\alpha,\beta\ge 0,\ \alpha+\beta\ge 1.$$

The proof is complete. □

Remark 2. One may wish to extend Theorem 2.5 to the case where $[a, b]$ is unbounded. Note that the case $b = \infty$ is not meaningful. It is because if $g \neq 0$ a.e., then both sides of (2.15) are infinite. If $b < \infty$ and $a = -\infty$ and inequality (2.15) holds for $a = \infty$, then it holds as well for all finite $a < 0$. Hence, inequality (2.16) holds for all $a < 0$. Thus, by letting $a \to -\infty$ in Theorem 2.5, one gets the result of Theorem 2.5 in the case $a = -\infty$.

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