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REFINEMENTS OF INEQUALITIES BETWEEN THE SUM OF SQUARES AND THE EXPONENTIAL OF SUM OF A NONNEGATIVE SEQUENCE

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ABSTRACT. Using probability theory methods, the following sharp inequality is established:

$$\frac{e^k}{k^k}\left(\sum_{i=1}^n x_i\right)^k \le \exp\left(\sum_{i=1}^n x_i\right),$$

where $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $x_i \ge 0$ for $1 \le i \le n$. Upon taking $k = 2$ in the above inequality, the inequalities obtained in [F. Qi, *Inequalities between the sum of squares and the exponential of sum of a nonnegative sequence*, J. Inequal. Pure Appl. Math. **8**(3) (2007), Art. 78] are refined.

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1. INTRODUCTION

In [1], the following two inequalities were found.

Theorem A. *For* $(x_1, x_2, \ldots, x_n) \in [0,\infty)^n$ *and* $n \ge 2$, *the inequality*

$$\frac{e^2}{4}\sum_{i=1}^n x_i^2 \le \exp\left(\sum_{i=1}^n x_i\right) \tag{1.1}$$

is valid. Equality in (1.1) *holds if* $x_i = 2$ *for some given* $1 \le i \le n$ *and* $x_j = 0$ *for all* $1 \le j \le n$ *with* $j \ne i$. *Thus, the constant* $\frac{e^2}{4}$ *in* (1.1) *is the best possible.*

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Theorem B. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then*

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \le \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.2}$$

Equality in (1.2) *holds if $x_i = 2$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in* (1.2) *is the best possible.*

In this note, by using some inequalities of normal random variables in probability theory, we will establish the following two inequalities whose special cases refine inequalities (1.1) and (1.2).

Theorem 1.1. *For $(x_1, x_2, \dots, x_n) \in [0,\infty)^n$ and $n \ge 1$, the inequality*

$$\frac{e^k}{k^k}\left(\sum_{i=1}^{n} x_i\right)^k \le \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.3}$$

holds for all $k \in \mathbb{N}$. Equality in (1.3) *holds if $\sum_{i=1}^{n} x_i = k$.*

Theorem 1.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then the inequality*

$$\frac{e^k}{k^k}\left(\sum_{i=1}^{\infty} x_i\right)^k \le \exp\left(\sum_{i=1}^{\infty} x_i\right) \tag{1.4}$$

is valid for all $k \in \mathbb{N}$. Equality in (1.4) *holds if $\sum_{i=1}^{\infty} x_i = k$.*

Our original ideas stem from probability theory, so we will prove the above theorems by using some normal inequalities. In fact, from the proofs of the above theorems in the next section, one will notice that there may be simpler proofs of them, by which we will obtain more the general results of Section 3.

2. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

In order to prove Theorem 1.1 and Theorem 1.2, the following two lemmas are necessary.

Lemma 2.1. *Let S be a normal random variable with mean μ and variance σ^2. Then*

$$\mathbb{E}\big(\exp\{tS\}\big) = \exp\left\{\mu t + \frac{t^2\sigma^2}{2}\right\}, \quad t \in \mathbb{R} \tag{2.1}$$

and

$$\mathbb{E}(S-\mu)^{2k} = \sigma^{2k}(2k-1)!!, \quad k \in \mathbb{N}. \tag{2.2}$$

Proof. The proof is straightforward. □

Lemma 2.2. *Let S be a normal random variable with mean 0 and variance σ^2. Then*

$$\frac{e^k}{(2k)^k(2k-1)!!}\mathbb{E}(S^{2k}) \le \mathbb{E}\big(e^S\big), \quad k \in \mathbb{N}. \tag{2.3}$$

Proof. Putting

$$f(x) = k\log x - \frac{x}{2}, \quad x \in (0,\infty),$$

it is easy to check that $f(x)$ takes the maximum value at $x = 2k$. By Lemma 2.1, we know that

$$\mathbb{E}(S^{2k}) = \sigma^{2k}(2k-1)!!, \quad \text{and} \quad \mathbb{E}\big(e^S\big) = e^{\sigma^2/2}.$$

Therefore, for the function

$$g(\sigma^2) = \frac{e^k}{(2k)^k(2k-1)!!}\sigma^{2k}(2k-1)!! - e^{\sigma^2/2} = \frac{e^k}{(2k)^k(2k-1)!!}\mathbb{E}(S^{2k}) - \mathbb{E}\big(e^S\big),$$

it is easy to check that $g(\sigma^2) \le 0$ and at the point $\sigma^2 = 2k$, the equality holds. $\square$

Now we are in a position to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $\{\xi_i\}_{1\le i\le n}$ be a sequence of independent normal random variables with mean zero and variance $\sigma_i^2 = 2x_i$ for all $i = 1, \dots, n$. Furthermore, let $S_n = \sum_{i=1}^n \xi_i$, and it is well known that S_n is a normal random variable with mean zero and variance $\sigma^2 = 2\sum_{i=1}^n x_i$. Therefore, we have

$$\mathbb{E}e^{S_n} = e^{\sigma^2/2} = \exp\left(\sum_{i=1}^n x_i\right)$$

and

$$\mathbb{E}(S_n^2) = \sigma^2 = 2\sum_{i=1}^n x_i.$$

From Lemma 2.2, we have

$$\frac{e^k}{(2k)^k(2k-1)!!}\mathbb{E}(S_n^{2k}) \le \mathbb{E}\big(e^{S_n}\big),$$

that is,

$$\frac{e^k}{k^k}\left(\sum_{i=1}^n x_i\right)^k \le \exp\left(\sum_{i=1}^n x_i\right), \tag{2.4}$$

where the equality holds in (2.4) if $\sum_{i=1}^n x_i = k$. $\square$

Proof of Theorem 1.2. This can be concluded by letting $n \to \infty$ in Theorem 1.1. $\square$

3. Further Discussion

In this section, we will give the general results of Theorem 1.1 and Theorem 1.2 by a simpler proof.

Theorem 3.1. *For $(x_1, x_2, \dots, x_n) \in [0,\infty)^n$ and $n \ge 1$, the inequality*

$$\frac{e^k}{k^k}\left(\sum_{i=1}^n x_i\right)^k \le \exp\left(\sum_{i=1}^n x_i\right) \tag{3.1}$$

holds for all $k \in (0,\infty)$. Equality in (3.1) *holds if $\sum_{i=1}^n x_i = k$. For $(x_1, x_2, \dots, x_n) \in (-\infty, 0]^n$ and $n \ge 1$, the inequality*

$$\frac{e^k}{|k|^k}\left(-\sum_{i=1}^n x_i\right)^k \ge \exp\left(\sum_{i=1}^n x_i\right) \tag{3.2}$$

holds for all $k \in (-\infty, 0)$. Equality in (3.2) *holds if $\sum_{i=1}^n x_i = k$.*

Proof. For all $C > 0$, $s > 0$ and $k > 0$, let $f(s) = \log C + k\log s - s$. It is easy to see that the function $f(s)$ takes its maximum at the point $s = k$. If we let $s = k$, then we can obtain $C = \frac{e^k}{k^k}$. The remainder of the proof is easy and thus omitted. $\square$

By similar arguments to those above, we can further obtain the following result.

Theorem 3.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then the inequality*

$$\frac{e^k}{k^k}\left(\sum_{i=1}^{\infty} x_i\right)^k \le \exp\left(\sum_{i=1}^{\infty} x_i\right) \tag{3.3}$$

is valid for all $k \in (0,\infty)$. Equality in (3.3) *holds if $\sum_{i=1}^{\infty} x_i = k$. In addition, let $\{x_i\}_{i=1}^{\infty}$ be a non-positive sequence such that $\sum_{i=1}^{\infty} x_i > -\infty$. Then the inequality*

$$\frac{e^k}{|k|^k}\left(-\sum_{i=1}^{\infty} x_i\right)^k \ge \exp\left(\sum_{i=1}^{\infty} x_i\right) \tag{3.4}$$

is valid for all $k \in (-\infty,0)$. Equality in (3.4) *holds if $\sum_{i=1}^{\infty} x_i = k$.*

4. Remarks

After proving Theorem 1.1 and Theorem 1.2, we would like to state several remarks and an open problem posed in [1].

Remark 1. If we take $k=2$ in inequality (1.3), then

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \le \frac{e^2}{4}\left(\sum_{i=1}^{n} x_i^2 + 2\sum_{1\le i<j\le n} x_i x_j\right) = \frac{e^2}{4}\left(\sum_{i=1}^{n} x_i\right)^2 \le \exp\left(\sum_{i=1}^{n} x_i\right) \tag{4.1}$$

which means that inequality (1.3) refines inequality (1.1).

Remark 2. If we let $k=2$ in inequality (1.4), then

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \le \frac{e^2}{4}\left(\sum_{i=1}^{\infty} x_i^2 + 2\sum_{j>i\ge 1} x_i x_j\right) = \frac{e^2}{4}\left(\sum_{i=1}^{\infty} x_i\right)^2 \le \exp\left(\sum_{i=1}^{\infty} x_i\right), \tag{4.2}$$

which means that inequality (1.4) refines inequality (1.2).

Remark 3. If we let $\sum_{i=1}^{n} x_i = y \ge 0$ in (1.3) or $\sum_{i=1}^{\infty} x_i = y \ge 0$ in (1.4), then inequalities (1.3) and (1.4) can be rewritten as

$$\frac{e^k}{k^k} y^k \le e^y \tag{4.3}$$

which is equivalent to

$$\left(\frac{y}{k}\right)^k \le e^{y-k} \quad \text{and} \quad \frac{y}{k} \le e^{\frac{y}{k}-1}. \tag{4.4}$$

Taking $\frac{y}{k} = s$ in the above inequality yields

$$s \le e^{s-1}. \tag{4.5}$$

It is clear that inequality (4.5) is valid for all $s \in \mathbb{R}$ and the equality in it holds if and only if $s=1$. This implies that inequalities (1.3) and (1.4) hold for $k \in (0,\infty)$ and $x_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} x_i \ge 0$ for $n \in \mathbb{N}$ or $\sum_{i=1}^{\infty} x_i \ge 0$ respectively and that the equalities in (1.3) and (1.4) hold if and only if $\sum_{i=1}^{n} x_i = k$ or $\sum_{i=1}^{\infty} x_i = k$ respectively.

Remark 4. In [1], Open Problem 1, was posed: For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, determine the best possible constants $\alpha_n, \lambda_n \in \mathbb{R}$ and $0 < \beta_n, \mu_n < \infty$ such that

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{4.6}$$

Recently, in a private communication with the third author, Huan-Nan Shi proved using majorization that the best constant in the left hand side of (4.6) is

$$\beta_n = \frac{e^{\alpha_n}}{\alpha_n^{\alpha_n}} \tag{4.7}$$

if $\alpha_n \geq 1$. This means that the inequality

$$\exp\left(2\sum_{i=1}^{n} x_i\right) \leq n^{1-\lambda_n}\left(\sum_{i=1}^{n} x_i\right)^{\lambda_n} \sum_{i=1}^{n} x_i^{\lambda_n} \tag{4.8}$$

holds for $\lambda_n \in \mathbb{R}$. If $x_i = 1$ for $1 \leq i \leq n$, then the above inequality becomes

$$e^{2n} \leq n^{1-\lambda_n} n^{\lambda_n} n = n^2 \tag{4.9}$$

which is not valid. This prompts us to check the validity of the right-hand inequality in (4.6): If the right-hand inequality in (4.6) is valid, then it is clear that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n} \leq \mu_n \left(\sum_{i=1}^{n} x_i\right)^{\lambda_n}, \tag{4.10}$$

which is equivalent to

$$e^x \leq \mu_n x^{\lambda_n} \tag{4.11}$$

for $x \geq 0$ and two constants $\lambda_n \in \mathbb{R}$ and $0 < \mu_n < \infty$. This must lead to a contradiction. Therefore, Open Problem 1 in [1] should and can be modified as follows.

Open Problem 1. *For $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $n \in \mathbb{N}$, determine the best possible constants $\alpha_n, \lambda_n \in \mathbb{R}$ and $0 < \beta_n, \mu_n < \infty$ such that*

$$\beta_n \sum_{i=1}^{n} |x_i|^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} |x_i|^{\lambda_n}. \tag{4.12}$$

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